Chapter 5: Fixed Points

Victor Lagerkvist (based on lecture notes by Ulf Nilsson & Wlodek Drabent)

Introduction

In this chapter we consider the problem of finding solutions to equations of the form

$$f(x) = x$$

where $f\colon A \to A$. An element $a \in A$ such that $f(a) = a$ is called a *fixed point* of f. Note that our problem is more far-reaching than this. If we want to solve an equation $f(x) = 0$, this can be reformulated as the problem of solving the equation $f(x) + x = x$; hence, if we define $g(x) := f(x) + x$ we have again the problem $g(x) = x$. Of course, our function may not have any fixed points (e.g. $f(n) := n + 1$), or if may have more than one fixed point (e.g. $f(n) := n$). The problem of determining if a function has a fixed point is undecidable in general and in this chapter we focus on two sufficient conditions under which we can guarantee the existence of fixed points, and in some cases even compute them (or at least approximate them arbitrarily well).

For a historic account of the use of fixed points in semantics of programming languages and logics, see Lassez et al. [1].

[1] J.-L. Lassez, V.L. Nguyen, and E.A. Sonenberg. Fixed point theorems and semantics: A folk tale. *Inform. Process. Lett.*, 14:112–116, 1982

Basic Notions

We summarize some basic properties of functions on ordered sets.

Definition 1. *Let $(A, \leq)$ be a poset. A function $f\colon A \to A$ is said to be*

- *monotone (order-preserving) iff $f(x) \leq f(y)$ whenever $x \leq y$,*
- *antimonotone iff $f(x) \geq f(y)$ whenever $x \leq y$,*
- *inflationary iff $x \leq f(x)$ for all $x \in A$, and*
- *idempotent iff $f(f(x)) = f(x)$ for all $x \in A$.*

A trivial case where we can always guarantee the existence of fixed points is when the map is idempotent.

Theorem 1. *Let $(A, \leq)$ be a non-empty ccpo and $f\colon A \to A$ idempotent. Then f has a (not necessarily unique) fixed point.*

We also introduce the notion of *continuous* maps. We first attempt a generic definition which is subsequently adapted to complete lattices and ccpo's.

Definition 2. *Let $(A, \leq)$ be a lattice. A function $f\colon A \to A$ is continuous if it preserves least upper bounds: if $B \subseteq A$ and $\bigvee B$ exists, then $\bigvee \{f(x) \mid x \in B\}$ exists and equals $f(\bigvee B)$.*

In the case of a complete partial order – which only guarantees the existence of suprema for ascending chains – we may specialize this as follows.

Definition 3. *Let $(A, \leq)$ be a ccpo. A function $f\colon A \to A$ is called (chain-) continuous if*

$$f(\bigvee\{x_0, x_1, x_2, \ldots\}) = \bigvee\{f(x_0), f(x_1), f(x_2), \ldots\}$$

for every ascending chain $x_0 < x_1 < x_2 < \ldots$ in A.

In the case of a complete lattice – where every subset has a supremum – we get:

Definition 4. *Let $(A, \leq)$ be a complete lattice. A function $f\colon A \to A$ is continuous if*

$$f(\bigvee B) = \bigvee\{f(x) \mid x \in B\}$$

for every $B \subseteq A$.

A continuous map is always monotone (prove this), but the converse is not true in general.

Knaster-Tarski's theorem

In this section we prove a classic fixed point theorem due to Knaster-Tarski (see Tarski [2]) which concerns the existence of (least and greatest) fixed points of monotone maps on complete lattices. We also give, without proof, a similar result concerning the existence of (least) fixed points of monotone maps on ccpo's.

[2] A. Tarski. A lattice-theoretic fixpoint theorem and it´s application. *Pacific J. Math.*, 5:285–309, 1955

Definition 5. *Let $(A, \leq)$ be a poset and consider a map $f\colon A \to A$. An $x \in A$ such that $f(x) \leq x$ is called a pre-fixed point of f. Similarly, $x \in A$ is called a post-fixed point of f iff $x \leq f(x)$.*

Example 1. *As defined previously a set $B \subseteq A$ is closed under the map $f\colon A \to A$ iff $f(x) \in B$ whenever $x \in B$. We may "lift" f to $F\colon 2^A \to 2^A$ as follows*

$$F(X) := \{f(x) \mid x \in X\}.$$

Then $B \subseteq A$ is closed under f iff B is a pre-fixed point of F, i.e. if $F(B) \subseteq B$. (In many cases the same symbol f is used for both functions, by abuse of notation.)

We now state and prove the Knaster-Tarski fixed point theorem.

Theorem 2. *Let $(A, \leq)$ be a complete lattice and $f\colon A \to A$ be a monotone function. Then*

$$\bigwedge\{x \in A \mid f(x) \leq x\}$$

is the least fixed point of f, and

$$\bigvee\{x \in A \mid x \leq f(x)\}$$

is the greatest fixed point of f.

Proof. We prove the first part of the theorem. The second result can be shown dually. Consider the set $\mathcal{S} := \{x \in A \mid f(x) \leq x\}$ of all pre-fixed points of f. The set $\mathcal{S}$ clearly is non-empty since at least $f(\top) \leq \top$. Now let $y := \bigwedge \mathcal{S}$. Then by definition $y \leq x$ for every $x \in \mathcal{S}$. By monotonicity of f, $f(y) \leq f(x) \leq x$, for every $x \in \mathcal{S}$. Hence $f(y) \leq x$ for every $x \in \mathcal{S}$, i.e. $f(y)$ is a lower bound of $\mathcal{S}$. But since y is the greatest lower bound we must have

$$f(y) \leq y. \tag{1}$$

Moreover, $f(f(y)) \leq f(y)$ (by monotonicity) which means that $f(y) \in \mathcal{S}$. Hence

$$y \leq f(y). \tag{2}$$

Our two inequalities imply not only that y is a fixed point but also that y is least, since all fixed points (including y) must be contained in $\mathcal{S}$. $\square$

That is, the least pre-fixed point of f is the least fixed point of f. Dually, the greatest post-fixed point of f is the greatest fixed point of f.

The Knaster-Tarski theorem concerns complete lattices, but the existence of a least fixed point holds also for monotone maps on ccpo's; however, the proof is more complicated, and we give the theorem without a proof.

Theorem 3. *Let $(A, \leq)$ be a ccpo and $f\colon A \to A$ monotone. Then f has a least fixed point.*

The least fixed point of f is denoted $\textsc{lfp}(f)$. Alternatively it is written

$$\mu x.f(x)$$

with reading: the least x such that $f(x) = x$. The greatest fixed point of f is often denoted $\textsc{gfp}(f)$, alternatively

$$\nu x.f(x).$$

Example 2. *Recall that the factorial function on* $\mathbb{N}$ *can be defined as* $\mathrm{fac}(0) = 1$ *and* $\mathrm{fac}(n) = \mathrm{fac}(n-1) \cdot n$ *otherwise. Via the machinery developed in this chapter we can give a formal account of the semantics of such recursive functions, where the basic idea is to understand a recursive function as a limit of a sequence of finite approximations (i.e., partial functions defined on a finite number of arguments). We can formalize this idea as follows.*

For a non-empty partial function $f\colon \mathbb{N} \to \mathbb{N}$ *let*

$$F(f) = f \cup \{(x+1, y \cdot (x+1)) \mid (x,y) \in f\}$$

and additionally let

$$F(\emptyset) = \{(0,1)\}.$$

Hence, the idea behind this function is that, given a partially defined factorial function, it returns a partially defined factorial function which is defined for at least one additional argument. We claim that F *is a monotone function: assume that* $f \subseteq g$ *for two partial functions* f *and* g*. Then* $F(f) \subseteq F(g)$ *must hold. Let* A *be the set of partial functions from* $\mathbb{N}$ *to* $\mathbb{N}$ *and recall that* $(A, \subseteq)$ *is a ccpo. It follows that the least fixed point of* F *exists and equals* $\bigcap\{f \in A \mid F(f) \leq f\}$*, i.e., the intersection of all partial functions* f *where* $F(f)$ *does not produce a new value*[3]*. In this example it is not hard to see that this least fixed point equals* fac*, but in general the Knaster-Tarski theorem only guarantees the existence of a least fixed point without any hints of how to practically compute it. We will soon revisit this issue in the context of Kleene's fixed point theorem.*

[3] What are those functions? Can there exist more than one?

Example 3. *Recall from Chapter 3 that two sets* A *and* B *have the same cardinality (*$|A| = |B|$*) if there exists an injective function* $f\colon A \to B$ *and an injective function* $g\colon B \to A$*. We claimed without proof that this also implies the existence of a bijective function* $h\colon A \to B$ *(the* Cantor-Schröder–Bernstein *theorem). Via the Knaster-Tarski fixed point theorem we are now in a position to give an elementary proof. First, for any* $X \subseteq A$ *we define the function*

$$\phi(X) = A \setminus g(B \setminus f(X)).$$

We claim that ϕ *is monotone. Indeed, assume that* $X \subseteq Y \subseteq A$*: then*

$$\phi(X) = A \setminus g(B \setminus f(X)) \subseteq \phi(Y) = A \setminus g(B \setminus f(Y))$$

since $f(X) \subseteq f(Y)$ *implies that* $g(B \setminus f(X)) \supseteq g(B \setminus f(Y))$*. Next, since* $(2^A, \subseteq)$ *is a complete lattice we (via Knaster-Tarski's fixed point theorem) conclude that* $\mu X.\phi(X)$ *for some set* $X \subseteq A$*. Then*

$$\phi(X) = A \setminus g(B \setminus f(X)) = X$$

and

$$A \setminus X = g(B \setminus f(X)).$$

Let $g_{|B \setminus f(X)}$ and $f_{|X}$ be g and f restricted to $B \setminus f(X)$, respectively X. It follows that these functions are bijections and we define the desired bijection $h\colon A \to B$ as

$$h(x) = f(x)$$

if $x \in X$ and

$$h(x) = (g_{B \setminus f(X)})^{-1}(x)$$

if $x \notin X$.

Kleene's Fixed Point Theorem

Knaster-Tarski's theorem concerns the existence of least and greatest fixed points. We now turn to the problem of computing, or at least approximating, fixed points. First, we recall the following theorem from Chapter 4.

Theorem 4. *Let $(A, \leq)$ be a complete lattice and assume that $f\colon A \to A$ is monotonic. Then $f^{\alpha} \leq f^{\alpha+\mathbf{1}}$ for all ordinals α.*

The following theorem, due to Kleene, provides a hint on how to compute the least fixed point, in case of a continuous map.

Theorem 5. *Let $(A, \leq)$ be a ccpo (or a complete lattice) and assume that $f\colon A \to A$ is continuous. Then $f^{\omega}(\bot)$ is the least fixed point of f.*

Proof. We begin by proving that $f^{\omega}(\bot)$ is a fixed point of f, and then show that it must be the least fixed point. First, Theorem 4 implies that $\bot = f^{\mathbf{0}}(\bot) \leq f^{\mathbf{1}}(\bot) \leq \ldots$ is an ascending chain, which means that the least upper bound of $\{f^n(\bot) \mid n < \omega\}$ exists and equals

$$f^{\omega}(\bot) = \bigvee_{n<\omega} f^n(\bot). \tag{3}$$

Since f is continuous we then have:

$$\begin{aligned} f(\bigvee_{n<\omega} f^n(\bot)) &= \bigvee_{n<\omega} f(f^n(\bot)) \\ &= \bigvee_{\mathbf{1}\leq n<\omega} f^n(\bot) \\ &= f^{\omega}(\bot) \text{ (since } f^0(\bot) = \bot \text{ and since } x \vee \bot = x \text{ for any } x). \end{aligned}$$

We finally demonstrate that $f^{\omega}(\bot)$ must be the least fixed point of f. Let x be an arbitrary fixed point of f. We show that x is an upper bound of $\{f^n(\bot) \mid n < \omega\}$ by proving that $f^n(\bot) \leq x$ for every $n < \omega$ via induction:

1. $n = \mathbf{0}$: then $\bot \leq x$ which due to monotonicity of f implies that $f(\bot) \leq x$.

2. Assume that $f^n(\bot) \leq x$ for some $n > \mathbf{0}$. Then $f^{n+\mathbf{1}}(\bot) \leq f(x) = x$ (monotonicity and since x is a fixed point of f).

Hence, x is an upper bound of $\{f^n(\bot) \mid n < \omega\}$. But since $f^\omega(\bot)$ is the *least* upper bound, $f^\omega(\bot) \leq x$, and $f^\omega(\bot)$ therefore is the least fixed point of f. $\square$

From Chapter 3 we know that

$$f^\omega(\bot) = \bigvee_{n<\omega} f^n(\bot).$$

That is, $\textsc{lfp}(f)$ is the least upper bound of the so-called *Kleene sequence*,

$$\bot, f(\bot), f^2(\bot), \ldots, f^n(\bot), \ldots$$

In fact, since f is monotonic this is an ascending chain, which means that $\textsc{lfp}(f) = \lim_{n\to\infty} f^n(\bot)$.

Example 4. *We continue with Example 2. First, we claim that F is continuous on the ccpo of partial functions from $\mathbb{N}$ to $\mathbb{N}$. Thus, let $f_1 \subset f_2 \subset \ldots$ be an ascending chain of partial functions. One can then show that $F(\bigcup\{f_1, f_2, \ldots\}) = \bigcup\{F(f) \mid f \in \{f_1, f_2, \ldots\}\}$. Hence, via Kleene's fixed point theorem we conclude that F^ω is the least fixed point of F and that our function in this case is the least upper bound of the sequence*

$$F^0, F^1, F^2, F^3, \ldots$$

i.e.,

$$\emptyset, \{(1,1)\}, \{(1,1), (2,2)\}, \{(1,1), (2,2), (3,6)\}, \ldots.$$

Example 5. *Consider the following equation over strings of some alphabet Σ with $1 \in \Sigma$*

$$w = 1w. \tag{4}$$

Such equations occur frequently when studying semantics of perpetual processes; for instance, in reactive, distributed and concurrent systems. The equation may for example model the behavior of a process w which sends a message 1 and then behaves like w again, in infinity.

It should be clear that no finite string satisfies (4) since the righthand side is always one character longer than the lefthand side if w is finite. That is, there are no solutions in Σ^*. Hence, consider instead the set of all possibly infinite strings $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. We may rewrite (4) as a fixed point equation

$$w = \textsc{one}(w) \tag{5}$$

where

$$\textsc{one}(x) := \{(0 \mapsto 1)\} \cup \{(n+1 \mapsto a) \mid (n \mapsto a) \in x\}.$$

Then

$$\begin{array}{rcl} \textsc{one}^0(\emptyset) & = & \emptyset \\ \textsc{one}^1(\emptyset) & = & \{0 \mapsto 1\} \\ \textsc{one}^2(\emptyset) & = & \{0 \mapsto 1, 1 \mapsto 1\} \\ \textsc{one}^3(\emptyset) & = & \{0 \mapsto 1, 1 \mapsto 1, 2 \mapsto 1\} \\ & \vdots & \end{array}$$

with the limit

$$\textsc{one}^\omega(\emptyset) = \bigcup_{n<\omega} \textsc{one}^n(\emptyset) = \{0 \mapsto 1, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, \ldots\}.$$

Now $\textsc{one}\colon \Sigma^\infty \to \Sigma^\infty$ *can be shown to be chain-continuous; for every chain* $w_0 < w_1 < w_2 < \ldots$ *we have that,*

$$\textsc{one}(\bigcup_{i\geq 0} w_i) = \bigcup_{i\geq 0} \textsc{one}(w_i) \tag{6}$$

Hence, $\textsc{one}^\omega(\emptyset)$ *is the (unique) least solution to (4).*

Exercises

5.1 Give an example of a function which is monotone but not inflationary and vice versa.

5.2 Prove that functional composition preserves monotonicity. That is, if $f\colon A \to A$ and $g\colon A \to A$ are monotone, then so is $f \circ g$. Note: $(f \circ g)(x) := f(g(x))$.

5.3 Prove that functional composition preserves continuity.

5.4 Define a map Φ such that $\Phi^\omega(\bot)$ is the Fibbonacci function,

$$\{0 \mapsto 1, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 5, 5 \mapsto 8, \ldots\}.$$

Show that the map is chain-continuous.

5.5 Give an example of a complete lattice $(A, \leq)$ and a monotonic map $f\colon A \to A$ such that $f^\omega(\bot)$ is not a fixed point of f.

5.6 Prove that a continuous map on a complete lattice (or a ccpo) is always monotone.

5.7 Show that if there exists an $n \in \mathbb{N}$ such that x is a unique fixed point of f^n (i.e. for every y such that $f^n(y) = y$ it holds that $x = y$), then x must be a fixed point of f.

5.8 Let $(A, \leq)$ be a complete lattice (or a ccpo) and $f\colon A \to A$ monotone. Show, using transfinite induction, that $f^\alpha(\bot) \leq \textsc{lfp}(f)$ for every ordinal α.

References

J.-L. Lassez, V.L. Nguyen, and E.A. Sonenberg. Fixed point theorems and semantics: A folk tale. *Inform. Process. Lett.*, 14:112–116, 1982.

A. Tarski. A lattice-theoretic fixpoint theorem and it´s application. *Pacific J. Math.*, 5:285–309, 1955.