Chapter 3: Ordinal and Cardinal Numbers

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Introduction

The natural numbers are often viewed as abstractions of finite sets. For instance, 7 is an abstraction of the set of weekdays or the set of mortal sins. But natural numbers can also be used to describe the *position* of an element in a sequence or a chain. For instance, July is the 7th month in the total order of all months ordered in the standard manner

$$\text{January} < \text{February} < \text{March} < \text{April} < \ldots < \text{December}.$$

Actually instead of saying that 7 is the position of July, we may view it as the length of the ascending chain which includes July and all its predecessors

$$\text{January} < \text{February} < \text{March} < \text{April} < \text{May} < \text{June} < \text{July}.$$

In the very first example 7 denotes the equivalence class of all sets of size 7, in which case we talk of the *cardinal* number 7. In the second case 7 denotes the equivalence class of ascending chains of length 7. In this case we refer to 7 as an *ordinal number*, or simply *ordinal*. Hence, ordinal numbers carry more information than cardinal numbers, namely order in addition to size.

Definition 1. *(Informal, first attempt) An* ordinal number *is a mathematical object which uniquely describes a total order.*

Example 1. *Every natural number $n \in \mathbb{N}$ is, intuitively, an ordinal number since n describes the chain $0 < 1 < 2 \ldots < n-1 < n$.*

In fact, any total order over a finite set can be described as a chain $0 < 1 < 2 \ldots < n-1 < n$ by renaming elements in a suitable way.

Example 2. *Consider a strict total order $(\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}, <)$ with seven elements. In principle, the elements $a_1, \ldots, a_7$ can be ordered in many[1] different ways. For example, $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7$ would constitute one ordering, and $a_2 < a_4 < a_5 < a_7 < a_1 < a_3 < a_6$ another. However, conceptually, there is no relevant distinction between these two orderings and we can easily translate one order to the other simply by renaming the elements. This holds more generally: for any finite set $A = \{a_1, \ldots, a_k\}$ with k elements there is, up to renaming of elements, only one strict total order, namely $a_1 < a_2 \ldots a_{k-1} < a_k$.*

[1] 504, to be precise.

For infinite sets and chains the situation is more complicated. Consider the following strict chains (in this chapter we usually employ the notation $0, 1, 2, \ldots$ when writing a chain $0 < 1 < 2 < \ldots$):

1. $\mathbb{Z}$ ordered as $\ldots, -2, -1, 0, 1, 2, \ldots$.
2. $\mathbb{Z}$ ordered as $0, -1, 1, -2, 2, \ldots$
3. $\mathbb{N}$ ordered as $0, 1, 2, 3, 4, \ldots$
4. $\mathbb{N}$ ordered as $1, 2, 3, 4, \ldots, 0$

Abstracting away from order it is evident that all four orders have the same (countable) cardinality. However, structurally they are pairwise different with the exception of (2) and (3). To explain why we need to introduce some additional notions.

Definition 2. *A function f from $(A, \leq)$ to $(B, \preceq)$ is called monotonic (isotone, order-preserving) iff $x \leq y$ implies $f(x) \preceq f(y)$ for all $x, y \in A$.*

We sometimes say that f is an order-homomorphism[2] (or order-morphism) from $(A, \leq)$ into $(B, \preceq)$ when f is monotonic.

[2] homo $\approx$ similar, morphism $\approx$ shape, hence the term homomorphism suggests that the two orders have a similar shape.

Definition 3. *An order-homomorphism f from $(A, \leq)$ into $(B, \preceq)$ is called*

- *a monomorphism if f is injective, i.e., if $f(x) = f(y)$ then $x = y$;*
- *an epimorphism if f is surjective, i.e., for every $y \in B$ there exists an $x \in A$ such that $f(x) = y$; and*
- *an isomorphism if f is bijective (injective and surjective).*

Two ordered sets $\mathcal{A} := (A, \leq)$ and $\mathcal{B} := (B, \preceq)$ are said to be (order-)*isomorphic* if there exists an order-isomorphism $f\colon A \to B$. We write $\mathcal{A} \simeq \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Let us then return to the orders:

1. $\mathbb{Z}$ ordered as $\ldots, -2, -1, 0, 1, 2, \ldots$.
2. $\mathbb{Z}$ ordered as $0, -1, 1, -2, 2, \ldots$
3. $\mathbb{N}$ ordered as $0, 1, 2, 3, 4, \ldots$
4. $\mathbb{N}$ ordered as $1, 2, 3, 4, \ldots, 0$

The first order has neither a minimal nor a maximal element. Hence it cannot be isomorphic to a chain with a minimal (or maximal) element. The last order has both a minimal and a maximal element and is therefore not isomorphic to any of the other three; the two middle orderings have only a minimal element. In fact, they are isomorphic – there exists a bijective and order-preserving mapping from $\mathbb{Z}$ to $\mathbb{N}$ (and hence also in the other direction), namely

$$f(n) = \begin{cases} 2n \text{ if } n \geq 0, \\ -2n - 1 \text{ if } n < 0. \end{cases}$$

Concerning 4, it is not hard to see that the function $h(x) = x + 1$ is an injective order-homomorphism from 3 to 4 (if $x, y \in \mathbb{N}$ and x is smaller than y then $x + 1$ is smaller than $y + 1$). However, it is not an isomorphism since it is not surjective (no element is mapped to 0). Thus, intuitively:

1. 2 and 3 are different representations of the same ordinal number,

2. the ordinal number of 2 and 3 is smaller than the ordinal number corresponding to 4 since there exists an order-homomorphism from the former to the latter, but not vice versa, and

3. 1 cannot be (obviously) related to the other orderings at all.

One way to formalize the difference between (1) and the other ones is that the former has no minimal element, while the latter three all do. In fact, any subset of (2), (3) or (4) has a minimal element, i.e., they are well-orders. In the rest of this chapter we will exclusively concentrate on well-orders and the associated ordinals. This can be motivated as follows: if we only consider well-orders then the resulting notion of an ordinal is very natural and leads to a robust mathematical theory where all ordinals can be related to each other.

The road-map for the rest of this chapter is now that we want to understand (primarily infinite) well-orderings better and describe them up to isomorphism. Surprisingly, we will see that there exists a canonical sequence of well-orderings such that *any* well-ordering is isomorphic to an element in this sequence. After having properly defined ordinal numbers we will also see that they can be used to define cardinal numbers in an unambiguous way.

Definitions

It is tempting to define an ordinal number as the equivalence class of a well-order induced by $\simeq$. However, in standard set theory such an object is too large to constitute a set, so we investigate an alternative construction. Here, the idea is to (1) define an ordinal as a well-ordering which satisfies a relevant property, (2) use this property to construct a canonical sequence of ordinals, and (3) show that any well-ordering is isomorphic to exactly one element in this sequence. Without further ado we now present the basic definition of an ordinal[3]. The basic intuition is that we want to define an ordinal α as a set $\{\alpha_1, \alpha_2, \ldots\}$ where each α_i is a smaller ordinal than α_{i+1}.

[3] The definitions in this section essentially follow the lecture notes available at http://www2.math.uu.se/~vera/undervisning/mangdlara/12/set_and_model_theory.pdf

Definition 4. *An* ordinal *is a set α which:*

1. *is well-ordered by $\in$ (i.e., by set membership), and*

2. *is* transitive*: if $a \in \alpha$ then $a \subseteq \alpha$.*

Note that $\emptyset$ is by definition an ordinal which describes the well-ordering with 0 elements. Next, consider $\{\emptyset\}$, i.e., the set containing $\emptyset$. This is also an ordinal since (1) $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$, and (2) it is trivially well-ordered by $\in$ since it only contains one element. This ordinal represents a well-ordering of one element. More generally we can form an ordinal $\alpha \cup \{\alpha\}$ from *any* ordinal α, the so-called *successor ordinal* of α.

Definition 5. *Let α be an ordinal. The* successor ordinal *of α is defined as $S(\alpha) = \alpha \cup \{\alpha\}$.*

For reasons which will be made clear later we sometimes write $\alpha + \mathbf{1}$ for $S(\alpha)$. The intuition behind the successor ordinal $S(\alpha)$ is that it describes the ordering obtained by adding α itself to the ordering. We will soon see why this is relevant. It is then not difficult to prove that $S(\alpha)$ is indeed also an ordinal whenever α is an ordinal.

Lemma 1. *If α is an ordinal then $S(\alpha)$ is an ordinal.*

Proof. Let $a \in S(\alpha) = \alpha \cup \{\alpha\}$. If $a \in \alpha$ then $a \subseteq \alpha \subseteq \alpha \cup \{\alpha\}$ (since α is an ordinal). But if $a = \alpha$ then it trivially holds that $\alpha \subseteq \alpha \cup \{\alpha\}$. Next, we prove that $S(\alpha)$ is well-ordered by $\in$. It is easy to show that $\in$ is a strict total order so we only prove that every set $A \subseteq S(\alpha)$ contains a minimal element. Indeed:

1. if $A \subseteq \alpha$ then A must contain a minimal element since α is an ordinal, and

2. if $A \subseteq S(\alpha)$ and $\alpha \in A$ then every $a \in A$ distinct from α satisfies $a \in \alpha$, and the minimal element in $A \setminus \{\alpha\}$ is therefore the same as the minimal element in A.

$\square$

Since $\emptyset$ is (trivially) an ordinal this leads to the following canonical construction of ordinals[4] where we start with $\emptyset$ and then compute $S(\emptyset) = \{\emptyset\}$, $S(S(\emptyset)) = \{\emptyset, \{\emptyset\}\}$, and so on. Here, we denote the nth number in the sequence simply by $\mathbf{n}$ since it can be viewed as an alternative representation of the natural number n.

[4] Due to John Von Neumann.

Notation	Canonical representation
$\mathbf{0}$	$\emptyset$
$\mathbf{1}$	$\{\emptyset\} = \mathbf{0} \cup \{\mathbf{0}\} = \{\mathbf{0}\}$
$\mathbf{2}$	$\{\emptyset, \{\emptyset\}\} = \mathbf{1} \cup \{\mathbf{1}\} = \{\mathbf{0}, \mathbf{1}\}$
$\mathbf{3}$	$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \mathbf{2} \cup \{\mathbf{2}\} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$
$\mathbf{4}$	$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} = \mathbf{3} \cup \{\mathbf{3}\} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$
etc.	

These ordinal numbers can then be ordered in the following way: $\mathbf{n} < \mathbf{m}$ iff $\mathbf{n} \in \mathbf{m}$. For example, $\mathbf{1} < \mathbf{2}$ since $\mathbf{1} = \{\emptyset\} \in \mathbf{2} = \{\emptyset, \{\emptyset\}\}$. Are we then done? No, not even close: consider the set constructed by

$$\omega = \bigcup_{n \geq 0} \{\mathbf{n}\},$$

i.e., the limit of the sequence $\emptyset, S(\emptyset), S(S(\emptyset)), \ldots$. We claim (without proof) that ω is also an ordinal, the first so-called *limit ordinal*. The intuition behind this object is that it is the least ordinal which describes all final ordinals. More generally we define a limit ordinal as follows.

Definition 6. *An ordinal α is said to be a* limit ordinal *if (1) $\emptyset \in \alpha$ and (2) if $a \in \alpha$ then $S(a) \in \alpha$.*

In other words limit ordinals are closed under the successor operation. Now, are we done? No, again, even from a limit ordinal ω we can form a new successor ordinal $\omega \cup \{\omega\}$, and from this ordinal we can form $(\omega \cup \{\omega\}) \cup \{\omega \cup \{\omega\}\}$, and so on, and if we repeat this we can again construct a limit ordinal which is usually denoted $\omega + \omega$. It should be reasonably clear that we can construct new ordinals in this way, but the interesting and non-trivial implication is that *all* ordinals can be constructed in this way.

Lemma 2. *Every ordinal is either:*

1. *$\emptyset$,*
2. *a successor ordinal, or*
3. *a limit ordinal.*

Proof. We provide a proof sketch but omit certain technical steps. Let $\alpha \neq \emptyset$ be an ordinal which is not a limit ordinal. We will show that α must be a successor ordinal. Since $\alpha \neq \emptyset$ it follows that $\emptyset \subset \alpha$. But then $\emptyset \in \alpha$ since for *any* $a, b \in \alpha$ we have $a \in b$ or $b \in \alpha$ (remember that every ordinal is well-ordered by $\in$ which in particular implies that any two elements are comparable. Next, since α is not a limit ordinal there has to exist an ordinal $\beta \in \alpha$ such that $S(\beta) = \beta \cup \{\beta\} \notin \alpha$. However, one can then show that it must be the case that $\alpha = \beta \cup \{\beta\}$[5], i.e., $\alpha = S(\beta)$. □

[5] Why? For *any* ordinals α and β one can prove that either $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.

Even better, one can prove that *every* well-order is isomorphic to a unique ordinal.

Theorem 1. *Every well-order $(A, <)$ is isomorphic to a unique ordinal.*

Proof. The proof is not trivial and we only provide a brief sketch. The most important property is that one can prove that $(A, <)$ is isomorphic to a unique ordinal if $(\{x \in A \mid x < a\}, <)$ is isomorphic to a unique ordinal for every $a \in A$. This condition roughly provides a method for proving the required isomorphism by proving the existence of isomorphisms for the simpler objects $(\{x \in A \mid x < a\}, <)$.

Thus, assume that $(A, <)$ is not isomorphic to a unique ordinal. Then there has to exist a least element $a \in A$ such that $(\{x \in A \mid x < a\}, <)$ is not isomorphic to a unique ordinal (by the property stated above). But this set cannot be empty since then we would trivially have an isomorphism to the ordinal $\emptyset$. Since a was the least element with the stated property, any $(\{x \in A \mid x < b\}, <)$ must be isomorphic to a unique ordinal. But then $(\{x \in A \mid x < a\}, <\})$ must be isomorphic to a unique ordinal. □

Due to this we for a well-order $(A, <)$ write $\textsc{ord}((A, <))$ for the unique ordinal isomorphic to $(A, <)$.

Example 3. *For the last time we return to the orderings:*

1. $\mathbb{Z}$ *ordered as* $\ldots, -2, -1, 0, 1, 2, \ldots$.
2. $\mathbb{Z}$ *ordered as* $0, -1, 1, -2, 2, \ldots$
3. $\mathbb{N}$ *ordered as* $0, 1, 2, 3, 4, \ldots$
4. $\mathbb{N}$ *ordered as* $1, 2, 3, 4, \ldots, 0$

It follows that:

- *1 is not isomorphic to any ordinal since it is not a well-order.*
- $\textsc{ord}(2) = \textsc{ord}(3) = \omega$.
- $\textsc{ord}(4) = \omega + \mathbf{1}$.

Furthermore, $\omega < \omega + \mathbf{1}$.

Definitions by Transfinite Recursion

In this section we define some useful functions on ordinals. These definitions are defined in a familiar recursive style where we define (e.g). a function $f(\alpha)$ via $f(\beta)$ for some smaller ordinal β, but require an additional condition when α is a limit ordinal.

Ordinal Arithmetics

Recall that the successor ordinal $S(\alpha)$ (also written $\alpha + \mathbf{1}$) can be viewed as the ordinal obtained by extending α with exactly one new element, namely α itself. This idea can be generalized as follows.

Definition 7. *Let α and β be ordinals. We define $\alpha + \beta$ recursively*[6] *as:*

- $\alpha + \beta = \alpha$ *when* $\beta = \mathbf{0}$.
- $\alpha + \beta = (\alpha + (\beta - 1)) + 1$ *when β has an immediate predecessor, i.e.,* $\beta = \delta + \mathbf{1}$ *for some ordinal δ.*
- $\alpha + \beta = \bigcup_{\delta < \beta}(\alpha + \delta)$ *when β is a limit ordinal.*

[6] Sometimes called *transfinite recursion*. It can be proven (via *transfinite induction*) that functions defined in this style are unique and exists, the so-called *transfinite recursion theorem*.

For example, $\mathbf{3} + \mathbf{4} = \mathbf{7}$, and ordinal arithmetic can thus be viewed as an extension of integer arithmetic. However, the commutative law does *not* hold in general for addition of ordinals. Consider the ordinals

$$\omega = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots\} \text{ and } \mathbf{1} = \{\mathbf{0}\}.$$

Then one can show that $\omega + \mathbf{1} \neq \omega$ while $\mathbf{1} + \omega = \omega$. Thus, addition of ordinals does not always behave as addition over the natural numbers. Note also that the following holds.

Theorem 2. *If $\beta \neq \mathbf{0}$ then $\alpha < \alpha + \beta$ for all ordinals α, β.*

Ordinal arithmetic also provides a useful mechanism for combining two well-orders $(A, \leq)$ and $(B, \prec)$ by adding the associated ordinals. Intuitively, the resulting well-order is simply the result of taking the disjoint union of A and B[7] and ordering elements such that a precedes b whenever $a \in A$ and $b \in B$.

[7] It might be the case that $A \cap B$ is not empty, and in that case we need to rename elements.

Since we now can add two ordinals it is perhaps no great surprise that we can also multiply ordinals (recall that multiplication as we normally understand it is just repeated addition).

Definition 8. *Let α and β be ordinals. We define $\alpha \cdot \beta$ recursively as:*

- $\alpha \cdot \beta = \mathbf{0}$ *when* $\beta = \mathbf{0}$.
- $\alpha \cdot \beta = (\alpha \cdot (\beta - 1)) + \alpha$ *when β has an immediate predecessor, i.e.,* $\beta = \delta + 1$ *for some ordinal δ.*

- $\alpha \cdot \beta = \bigcup_{\delta<\beta}(\alpha \cdot \delta)$ *when* β *is a limit ordinal.*

It is not hard to prove that $\mathbf{2} \cdot \omega = \omega$, but that $\omega \cdot \mathbf{2} = \omega + \omega \neq \omega$.

More generally, the finite n-fold product of α is often written $\alpha^{\mathbf{n}}$, and it is also possible to define α^{β} for all ordinals α and β (see e.g. [8] for an extensive exposition of ordinal numbers and ordinal arithmetic).

[8] P. Halmos. *Naive Set Theory*. van Nostrand, 1961

Ordinal Powers of Functions

We next outline some notions useful for generalized definition of infinite and transfinite (beyond infinity) sets. Consider a function $f\colon A \to A$ on a complete lattice $(A, \leq)$. The (ascending) ordinal powers of f are defined as follows:[9]

[9] Most of this applies also to the case when $(A, \leq)$ is a ccpo.

$$
\begin{array}{lll}
f^0(x) & := & x \\
f^{\alpha+1}(x) & := & f(f^{\alpha}(x)) \text{ for successor ordinals } \alpha+1 \\
f^{\alpha}(x) & := & \bigvee_{\beta<\alpha} f^{\beta}(x) \text{ for limit ordinals } \alpha
\end{array}
$$

When x equals $\perp$ we simply write f^{α} instead of $f^{\alpha}(\perp)$. That is:

$$
\begin{array}{lll}
f^0 & := & \perp \\
f^{\alpha+1} & := & f(f^{\alpha}) \text{ for successor ordinals } \alpha+1 \\
f^{\alpha} & := & \bigvee_{\beta<\alpha} f^{\beta} \text{ for limit ordinals } \alpha
\end{array}
$$

The definition of $f^{\alpha}(x)$ also applies when A is a ccpo if f is monotonic and $x \leq f(x)$. For complete lattices we also have a corresponding dual notion of descending ordinal powers:

$$
\begin{array}{lll}
f^0(x) & := & x \\
f^{\alpha+1}(x) & := & f(f^{\alpha}(x)) \text{ for successor ordinals } \alpha+1 \\
f^{\alpha}(x) & := & \bigwedge_{\beta<\alpha} f^{\beta}(x) \text{ for limit ordinals } \alpha
\end{array}
$$

Example 4. *Consider a transition system $(C, \to, I)$ with initial configurations I; i.e. $I \subseteq C$. Let* $\textsc{step}\colon 2^C \to 2^C$ *be a step function*

$$\textsc{step}(x) := \{c \in C \mid \exists c' \in x \text{ such that } c' \to c\}$$

Then

$$
\begin{array}{lll}
\textsc{step}^0(I) & = & I \\
\textsc{step}^1(I) & = & \textsc{step}(I) \\
\textsc{step}^2(I) & = & \textsc{step}(\textsc{step}(I)).
\end{array}
$$

That is, for finite ordinals n, $\textsc{step}^n(I)$ is the set of all configurations reachable in n steps from some initial configuration. Moreover, the least infinite ordinal power, corresponding to the limit ordinal ω is

$$\textsc{step}^{\omega}(I) = \bigcup_{n<\omega} \textsc{step}^n(I).$$

The limit ordinal is the set of all configurations reachable in a finite (but unbounded) number of steps from an initial configuration.

Example 5. *Consider a relation $R \subseteq A \times A$. Let $f_{R\circ}\colon 2^{A\times A} \to 2^{A\times A}$ be defined as $R \circ S$, i.e. the function which composes the relation R with the relation $S \subseteq A \times A$. The ordinal powers of $f_{R\circ}$ are as follows*

$$
\begin{array}{lcl}
f^{0}_{R\circ}(S) & = & S \\
f^{\alpha+1}_{R\circ}(S) & = & f_{R\circ}((R\circ)^{\alpha}(S)) = R \circ f^{\alpha}_{R\circ}(S) \\
f^{\alpha}_{R\circ}(S) & = & \bigcup_{\beta<\alpha} f^{\beta}_{R\circ}(S) \text{ for limit ordinals } \alpha.
\end{array}
$$

And

$$
\begin{array}{lcl}
f^{0}_{R\circ}(\mathrm{ID}_A) & = & \mathrm{ID}_A \\
f^{1}_{R\circ}(\mathrm{ID}_A) & = & R \circ \mathrm{ID}_A = R \\
f^{2}_{R\circ}(\mathrm{ID}_A) & = & R \circ R \circ \mathrm{ID}_A = R \circ R \\
f^{3}_{R\circ}(\mathrm{ID}_A) & = & R \circ R \circ R \\
\text{etc.} & &
\end{array}
$$

It follows that $f^{\omega}_{R\circ}(\mathrm{ID}_A)$ is the reflexive and transitive closure of R. Similarly $f^{\omega}_{R\circ}(R)$ is the transitive closure of R.

Cardinal Numbers

We now illustrate how one can formally define the concept of a cardinal number using ordinal numbers. For cardinal numbers we say that two (possibly infinite) sets A and B have the same cardinality iff there exists a bijective mapping $f\colon A \to B$ (and hence a bijective mapping $f^{-1}\colon B \to A$). We write $A \sim B$ if there exists a bijection from A to B. We know for instance that $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$ but $\mathbb{N} \not\sim \mathbb{R}$. It is easy to prove that $\sim$ is an equivalence relation, and, intuitively, each equivalence class of this relation is a cardinal number: the least cardinal number, written $\mathbf{0}$, is the equivalence class that contains $\varnothing$ (and nothing else); the next cardinal number, written $\mathbf{1}$, is the equivalence class of all singleton sets, etc. The least infinite cardinal number is written $\aleph_0$ and contains the set of all natural numbers, as well as all other infinite, enumerable sets.

However, this point of view is oversimplified. Strictly speaking, $\sim$ is not a relation since it should be a subset of the Cartesian product of the set of all sets, and such a set does not exist in standard set theory. However, we can circumvent this difficulty by associating the cardinality of a set with an ordinal number.

Definition 9. *The* cardinality *of a set A (written $|A|$) is the least ordinal α such that there exists a bijection between A and α*[10].

[10] Additionally, one can prove that a unique ordinal α always exists, but this requires the *well-ordering theorem* which we have not introduced.

Sets of the form $|A|$ are then naturally referred to as *cardinal numbers*. For a finite set $|A|$ the cardinality is simply the same as the ordinal $|A| = \mathbf{k}$, i.e., the number of elements, but for infinite sets the two notions differ. For example, $|\mathbb{N}| = \omega = \aleph_0$, and $|\mathbb{N} \cup \{\omega\}| = \aleph_0 = \omega$

even though ω and $\omega \cup \{\omega\}$ are different ordinals. We define the following basic relations and operations on cardinal numbers.

Definition 10. *Let A and B be two sets.*

1. $|A| \leq |B|$ *if there exists an injective function from A to B.*

2. $|A| + |B| = |A \cup B|$ *(cardinal addition).*

3. $|A| \cdot |B| = |A \times B|$ *(cardinal multiplication).*

4. $2^{|A|} = |2^A|$ *(cardinality of powerset, recall that 2^A is the powerset of A).*

5. $|A|^{|B|} = |A^B|$ *(cardinal exponentiation, recall that A^B is the set of all functions from B to A).*

The relation $\leq$ over cardinals is antisymmetric: if $|A| \leq |B|$ and $|B| \leq |A|$ then there exists a bijection between A and B and $|A| = |B|$ (the *Cantor-Schröder–Bernstein theorem*). Any set A where $|A| = \aleph_0$ is said to be *countable*, and if $\aleph_0 < |A|$ then A is said to be *uncountable.* It is known (and not difficult to prove via diagonalization) that $\aleph_0 < |\mathcal{R}| = 2^{|\mathbb{N}|}$[11] It is believed that there is no cardinal number between $\aleph_0$ and $|\mathcal{R}|$, the so-called *continuum hypothesis*. However, Gödel proved that this statement is independent of the axioms of standard set theory (ZFC), meaning that the statement can neither be proven or disproven within ZFC.

[11] I.e., the cardinality of the natural numbers is smaller than the cardinality of the real numbers, which is sometimes said to be of *continuum cardinality*.

Exercises

3.1 Give an example of a well-order of $\mathbb{N}$ with the ordinal number $\omega + \omega + \omega$ (or $\omega \cdot 3$).

3.2 Give an example of a well-order of $\mathbb{N}$ with the ordinal number $\omega \cdot \omega$ (i.e. intuitively an infinite sequence of infinite sequences of natural numbers).

References

P. Halmos. *Naive Set Theory*. van Nostrand, 1961.