Chapter 2: Algebraic Structures

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Introduction

In this chapter we study two types of partially ordered sets which are extensively used in many areas of computer science. We first consider *lattices* and *complete lattices*, followed by *chain complete partial orders* (ccpo's).

Lattices and Complete Lattices

We survey basic definitions and and fundamental properties of lattices. For more elaborate expositions, see Birkhoff [1] or Grätzer [2].

We first define the notions of down-sets, or order ideals, and the dual notions of up-sets, and order filters.

[1] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3rd edition, 1967

[2] G. Grätzer. *General Lattice Theory*. Academic Press, 1978

Definition 1. *Let $(A, \leq)$ be a poset. A set $B \subseteq A$ is called a down-set (or an order ideal) iff*

$$y \in B \text{ whenever } x \in B \text{ and } y \leq x.$$

A set $B \subseteq A$ induces a down-set, denoted $B\!\downarrow$,

$$B\!\downarrow := \{x \in A \mid \exists y \in B, x \leq y\}.$$

By $\mathcal{O}(A)$ we denote the set of all down-sets in A,

$$\{B\!\downarrow \mid B \subseteq A\}.$$

A notion of up-set, also called order filter, is defined dually.

Let us now introduce the central notions of (least) upper bound and (greatest) lower bound.

Definition 2. *Let $(A, \leq)$ be a poset and $B \subseteq A$. Then $x \in A$ is called an upper bound of B iff $y \leq x$ for all $y \in B$ (often written $B \leq x$ by abuse of notation). The notion of lower bound is defined dually.*

Note that the set of all lower bounds of $\{x\}$, or simply of x, is identical to $\{x\}\!\downarrow$, i.e. the down-set of x. More generally, the set of all lower bounds of $B \subseteq A$ equals

$$\bigcap_{x \in B} \{x\}\!\downarrow.$$

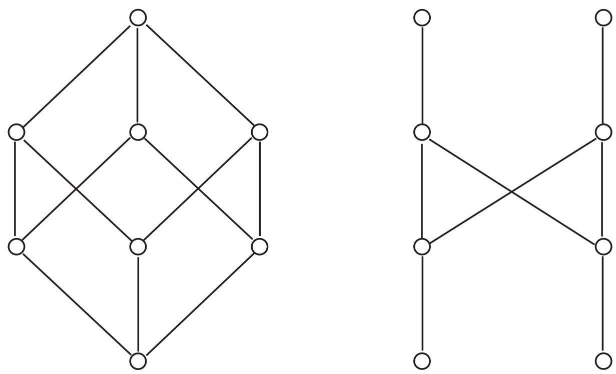


Figure 1: A lattice (left) and a poset which is not a lattice (right)

Definition 3. *Let $(A, \leq)$ be a poset and $B \subseteq A$. Then $x \in A$ is called a least upper bound of B iff $B \leq x$ and $x \leq y$ whenever $B \leq y$. The notion of greatest lower bound is defined dually.*

So a least upper bound of B is the least element of $\{ y \in A \mid B \leq y \}$, and a greatest lower bound of B is the greatest element of $\{ y \in A \mid B \geq y \}$. Thus least upper bounds (and greatest lower bounds) are unique, if they exist. So we say *the* least upper (greatest lower) bound.

Definition 4. *A lattice is a poset $(A, \leq)$ where every pair of elements $x, y \in A$ has a least upper bound, denoted $x \vee y$, and greatest lower bound, denoted $x \wedge y$.*

The least upper bound (abbreviated lub) $x \vee y$ of $\{x, y\}$ is sometimes called the *join* or *supremum* of x and y, and the greatest lower bound (abbreviated glb) $x \wedge y$ of $\{x, y\}$ is sometimes called the *meet* or *infimum* of x and y. Alternative notations are $\text{lub}(x, y)$, or $\sup(x, y)$, and $\text{glb}(x, y)$ or $\inf(x, y)$.

Figure 1 depicts two posets as Hasse diagrams. The leftmost poset is a lattice, while the rightmost is not (why?).

Example 1. *The following are examples of lattices*

- *The set 2^A under $\subseteq$ is a lattice with least upper bound $\cup$ and greatest lower bound $\cap$.*
- *The set $\mathbb{Z}$ under $\leq$ is a lattice with the function min as greatest lower bound, and the function max as least upper bound.*
- *The set of regular languages over some alphabet Σ, ordered by $\subseteq$, is a lattice with intersection as greatest lower bound and union as least upper bound (recall that regular languages are closed under both intersection and union).*

The set of all context-free languages is not a lattice under $\subseteq$ (when $|\Sigma| > 1$). Context-free languages are closed under union. However, they are not closed under intersection; for example, both $L_1 = \{a^i b^i c^j \mid i, j \geq 1\}$ and $L_2 = \{a^i b^j c^j \mid i, j \geq 1\}$ are context-free, but their intersection $L_1 \cap L_2 = \{a^i b^i c^i \mid i \geq 1\}$ is the standard example of a non context-free language. Let a context-free language $L' \subseteq \Sigma^*$ be a lower bound of $\{L_1, L_2\}$. Then $L' \subset L_1 \cap L_2$, and there exists a string $w \in (L_1 \cap L_2) \setminus L'$. As $L' \cup \{w\}$ is context-free, and $L' \cup \{w\} \not\subseteq L'$, L' is not the lub *of $\{L_1, L_2\}$. Hence no lower bound is the* lub*, so the* lub *does not exist.*

A *join semi-lattice* is a poset where the least upper bound exists for any pair of elements (but where the greatest lower bound might or might not exist). Similarly, a *meet semi-lattice* is a poset where the greatest lower bound exists for any pair of elements (and where the least upper bound might or might not exist).

Thus, a lattice involves a poset and two operations. Hence a lattice really is a structure $(A, \leq, \wedge, \vee)$. However, the two operations actually follow from $\leq$ and vice versa. That is, a lattice is given unambiguously either by the partial order or the two bounds (this will discussed in detail in the next section). As a consequence we sometimes say that $(A, \leq)$ is a lattice assuming implicitly the existence also of $\wedge$ and $\vee$; sometimes we say instead that $(A, \wedge, \vee)$ is a lattice assuming tacitly the ordering $\leq$.

Definition 5. *Let $(A, \leq)$ be a poset. An element $a \in A$ is said to cover an element $b \in A$ iff $a > b$ and there is no $c \in A$ such that $a > c > b$.*

Example 2. *The element $\{0, 1\}$ covers $\{0\}$ in $(2^{\{0,1,2\}}, \subseteq)$. But it is not the only element covering $\{0\}$, since $\{0\}$ is also covered by $\{0, 2\}$.*

Lemma 1. *Consider a poset $(A, \leq)$, sets $C \subseteq B \subseteq A$ and the induced poset $(B, \leq)$. If $a \in B$ is the lub of a set C in $(A, \leq)$ then a is the lub of C in $(B, \leq)$. The analogical property holds for glb's.*

If the lub a of C in A is not a member of B (but $C \subseteq B$) then C may have the lub in B (which is distinct from the lub in A), or it may have no lub in B. The same holds for the glb of C.

Example 3. *In Example 1 the set $C = \{L_1, L_2\}$ has a glb in the lattice $(A, \subseteq)$ of all languages, but the glb does not exist in the poset $(B, \subseteq)$ of context-free languages. Consider $B' = \{\emptyset, L_1, L_2, L_1 \cup L_2\} \subseteq B$. In the poset $(B', \subseteq)$ (which is a lattice), the set C has a glb, which is $\emptyset$ – distinct from the glb of C in A.*

Note that in a lattice there exist greatest lower and upper bounds of each finite subset (prove this), but not necessarily of an infinite one. We introduce complete lattices, in which such bounds exist.

Definition 6. *A complete lattice is a poset $(A, \leq)$ where every subset $B \subseteq A$ (finite or infinite) has a least upper bound $\bigvee B$ and a greatest lower bound $\bigwedge B$. The element $\bigvee A$ is called the top element and is usually denoted $\top$. The element $\bigwedge A$ is called the bottom element and is denoted $\perp$.*

Every complete lattice is a lattice since every pair of elements has a least upper and greatest lower bound, but the converse does not hold in general as illustrated by the following example.

Example 4. *The set of all natural numbers $\mathbb{N}$ under the standard non-strict ordering $\leq$ is a lattice; any pair of natural numbers has a least upper bound (namely the supremum of the two), and a greatest lower bound (namely the infimum of the two). However, it is not a complete lattice; any finite subset has a least upper, and greatest lower bound, but the set of all natural numbers does not have a least upper bound. (However, it does have a greatest lower bound.) On the other hand, if we add a top element $\top$ to the natural numbers we have a complete lattice.*

Example 5. *The powerset 2^A of any set A is a complete lattice under standard set inclusion $\subseteq$. Let $A_i \subseteq 2^A$ for each $i \in I$. Then we have the least upper bound*

$$\bigcup_{i \in I} A_i := \{a \mid a \text{ is a member of some } A_i\}.$$

The greatest lower bound is defined dually

$$\bigcap_{i \in I} A_i := \{a \mid a \text{ is a member of every } A_i\}.$$

Example 6. *The lattice of all regular languages over an alphabet Σ, let us call it $(\mathcal{R}, \subseteq)$ (see Example 1), is not a complete lattice; there is the least and the greatest element, namely $\emptyset$ and Σ^*, however the union (or intersection) of an infinite set of regular languages may be not regular, thus it is not an element of the lattice. Moreover, in the lattice there may not exist the lub (or glb) of such sets. For instance, all of the following languages are trivially regular (as each is finite)*

$$\begin{array}{lcl} L_0 & = & \{\epsilon\} \\ L_1 & = & \{ab\} \\ L_2 & = & \{aabb\} \\ L_3 & = & \{aaabbb\} \\ \textit{etc.} & & \end{array}$$

but their union $L = \bigcup\{L_0, L_1 \ldots\} = \{a^n b^n \mid n \geq 0\}$ is not a regular language, $L \notin \mathcal{R}$. No regular language is the least upper bound of $\{L_0, L_1 \ldots\}$ in $(\mathcal{R}, \subseteq)$. Note that any upper bound $M \in \mathcal{R}$ of $\{L_0, L_1 \ldots\}$ contains a string $x \notin L$, so $M \setminus \{x\} \in \mathcal{R}$ and $M \setminus \{x\} \subset M$, hence M is not the least upper bound.

We finally survey some special lattices that enjoy additional algebraic properties.

Definition 7. *Let $(A, \leq)$ be a lattice with $\perp$ and $\top$. We say that $a \in A$ is a complement of $b \in A$ iff $a \vee b = \top$ and $a \wedge b = \perp$.*

It follows that the complement of $\perp$ is $\top$, and vice versa (provided that $\perp, \top$ exist, of course).

Definition 8. *We say that a lattice is complemented if every element has a complement.*

The lattice of regular languages over some alphabet Σ, ordered by $\subseteq$ is a complemented lattice; in this particular case the complement of each regular language L is unique – it is $\Sigma^* \setminus L$. However, a complement of an element in a complemented lattice need not be unique (see exercises). If the complement of all elements x is unique, it is denoted x'; hence, $x \wedge x' = \perp$ and $x \vee x' = \top$.

Definition 9. *A lattice $(A, \leq)$ is said to be distributive iff $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$.*

It can be shown that $\vee$ distributes over $\wedge$ iff $\wedge$ distributes over $\vee$; hence, in a distributive lattice we also have that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (see exercises). We show that in a complemented, distributive lattice the complement of each element is unique. Suppose that both b and c are complements of a, then

$$b = b \wedge \top = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = \perp \vee (b \wedge c) = b \wedge c$$

Hence $b \leq c$. By an analogous argument $c \leq b$, in which case by necessity $b = c$, hence the complement of a must be unique.

Definition 10. *A lattice $(A, \leq)$ is said to be Boolean iff it is complemented and distributive.*

Example 7. *The set of all regular languages $\mathcal{R}$ over some alphabet Σ is a lattice with $\cap, \cup$ as the lattice operations. It has the top and bottom element (namely Σ^* and $\emptyset$), and it is complemented, as shown above. As $\cap$ distributes over $\cup$, the lattice is distributive, and hence Boolean.*

Example 8. *Not surprisingly, Boolean algebras and Boolean lattices coincide; that is, a Boolean algebra $(B, +, \cdot, ', 0, 1)$ is a Boolean lattice with least upper bound $+$, greatest lower bound $\cdot$, complement $'$, bottom element 0 and top element 1. Recall that a Boolean algebra is an algebraic structure satisfying the following laws,*

Commutative laws:	$a + b = b + a$	$a \cdot b = b \cdot a$
Distributive laws:	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
Identity laws:	$a + 0 = a$	$a \cdot 1 = a$
Inverse laws:	$a + a' = 1$	$a \cdot a' = 0$

for all $a, b, c \in B$.

Definition 11. *Let* A *be a set and* $B \subseteq 2^A$ *be closed w.r.t.* $\cap$ *and* $\cup$. *Then* $(B, \subseteq)$ *is a lattice in which the lub of* $a, b \in B$ *is* $a \cup b$, *and their glb is* $a \cap b$. *We refer to it as a lattice of sets.*

The fact that $(B, \subseteq)$ is a lattice with glb's and lub's given by $\wedge, \vee$ follows from Lemma 1.

Example 9. *Here we consider sets of sets, ordered by* $\subseteq$. *Example 1 showed a case in which* $(B, \subseteq)$ *is not a lattice (*B *is there the set of context-free languages), and in Example 6* $(\mathcal{R}, \subseteq)$ *is not a complete lattice. Now we show that the lub in* $(B, \subseteq)$ *may be distinct from the union of elements of* B. *Consider the set* B_0 *of finite languages of not more than 3 elements over an alphabet* Σ, *and let* $B = B_0 \cup \{\Sigma^*\}$. *Note that* B *is closed under* $\cap$, *but not under* $\cup$. *Now* $(B, \subseteq)$ *is a lattice in which the glb is given by* $\cap$, *but if* $|L \cup L'| > 3$ *then the lub of* L, L' *is* Σ^*.

Theorem 1. *We have the following results for lattices of sets:*

1. *Any lattice of sets is distributive.*
2. $(2^A, \subseteq)$ *is distributive, and Boolean.*

The proofs are left as exercises.

Lattices as Algebras

Our definition of lattice is based on partially ordered sets. However, there is an equivalent algebraic definition. Consider an algebra $(A, \wedge, \vee)$ with operators $\wedge \colon A \times A \to A$ and $\vee \colon A \times A \to A$. The algebraic structure $(A, \wedge, \vee)$ is a lattice if the operators satisfy the following laws, for all $a, b, c \in A$.

(L_1) Idempotency: $a \wedge a = a \vee a = a$

(L_2) Commutativity: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

(L_3) Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$

(L_4) Absorption: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Now let $a \leq b$ iff $a \wedge b = a$ (or $a \vee b = b$); then it follows that the ordered set $(A, \leq)$ is a lattice, i.e. every pair of elements $a, b \in A$ has a least upper bound (namely $a \vee b$) and a greatest lower bound (namely $a \wedge b$). The proof of this equivalence is left as an exercise.

Closure Operators and Lattices

A *closure operator* is a fundamental mathematical tool which describes a transformation of a set. We begin by defining this concept formally and then relate closure operators to complete lattices.

Definition 12. *A **closure operator** on a set A is a function $\mathcal{C} \colon \mathcal{P}(A) \to \mathcal{P}(A)$ that satisfies the following properties:*

1. ***Idempotence**: For all $X \subseteq A$, $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$.*
2. ***Monotonicity**: For all $X, Y \subseteq A$, if $X \subseteq Y$, then $\mathcal{C}(X) \subseteq \mathcal{C}(Y)$.*
3. ***Inclusion-preserving**: For all $X \subseteq A$, $X \subseteq \mathcal{C}(X)$.*

Example 10. *We explore an example of a closure operator where the basic objects are Boolean functions. If $f \colon \{0,1\}^n \to \{0,1\}$ is an n-ary Boolean function and $g_1, \ldots, g_n$ are m-ary Boolean functions then the* composition *of $f, g_1, \ldots, g_n$, written $f \circ g_1, \ldots, g_n$, is the m-ary Boolean function defined as*

$$f \circ g_1, \ldots, g_n(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

*for all $x_1, \ldots, x_m \in \{0,1\}$. For $n \geq 1$ and $1 \leq i \leq n$ let the i*th projection *be defined as $\pi^n_i(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_i, \ldots, x_n \in \{0,1\}$, i.e., an n-ary Boolean function which simply returns its ith argument unchanged.*

For a set of Boolean functions F let $[F]$ be the smallest set of Boolean functions which (1) contains all projection functions and (2) is closed under composition of Boolean functions. We verify that $[\cdot]$ is indeed a closure operator. Thus, let F and G be two sets of Boolean functions.

1. *Inclusion-preserving: $F \subseteq [F]$ follows trivially from the definition.*
2. *Monotonicity: if $F \subseteq G$ then $[F] \subseteq [G]$ follows from the observation that any function $f \circ g_1, \ldots, g_n \in [F]$ where $f, g_1, \ldots, g_n \in F$ is trivially included in $[G]$ since $F \subseteq G$.*
3. *Idempotence: $[F] = [[F]]$ since (1) $[F] \subseteq [[F]]$ by the inclusion-preservation property, and $[[F]] \subseteq [F]$ from the observation that any function $f \circ g_1, \ldots, g_n \in [[F]]$ where $f, g_1, \ldots, g_n \in [F]$ is contained in $[F]$, too, since it is closed under composition.*

The intuition is then that the set of functions $[F]$ represents everything that can be defined via a base set F, similarly, to how one in a vector space can represent each vector as a linear combination of vectors in a base set. For historical reasons, the sets $[F]$ are called clones*. For example, if $F = \{f_1, f_2\}$ where $f_1(x, y) = x \wedge y$ and $f_2(x) = 1 - x$ then $[F]$ contains every Boolean function and is the largest clone on the Boolean domain.*

We proceed by relating closure operators to complete lattices. First, assume that $\mathcal{C}\colon \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator on a set A. Consider the set $A_{\mathcal{C}} = \{\mathcal{C}(X) \mid X \subseteq A\}$ of all closed subsets of A. It is then easy to see that $(A_{\mathcal{C},\subseteq})$ is a complete lattice with:

1. least element $\mathcal{C} = \emptyset$,
2. greatest element $\mathcal{C}(A)$,
3. meet $\wedge$ defined as $X \wedge Y = X \cap Y$ for $X, Y \in A_{\mathcal{C}}$, and
4. join $\vee$ defined as $X \vee Y = \mathcal{C}(X \cup Y)$ for $X, Y \in A_{\mathcal{C}}$.

In the other direction, the closure operator corresponding to a complete lattice $(A, \leq)$ with meet $\wedge$ and join $\vee$ can be constructed as $\mathcal{C}_{\leq}(X) =\downarrow \vee X$, i.e., the set of all elements in X below the join of X.

Example 11. *Via the results in this section we can conclude that* $(C, \subseteq)$ *for* $C = \{[F] \mid F$ *is a set of Boolean functions*$\}$ *constitutes a complete lattice, the* lattice of Boolean clones, *or* Post's lattice, *eponymously named after Emil Post. The least element in this lattice is the set* $\{\pi_i^n \mid n \geq 1, 1 \leq i \leq n\}$ *of all Boolean projections, and the largest element is simply the set of all Boolean functions. If we again consider the two functions* $f_1(x, y) = x \wedge y$ *and* $f_2(x) = 1 - x$ *then neither* $[\{f_1\}]$ *nor* $[\{f_2\}]$ *equals the greatest element but their join* $[\{[\{f_1\}] \cup [\{f_2\}]] = [\{f_1, f_2\}]$ *does. This mirrors the well-known fact that a gate set consisting only of conjunction and negation is sufficient to implement any Boolean function.*

Chain-Complete Partial Orders

Lattices possess many appealing properties, but the requirement that any pair of elements, or any subset of elements, has both a lub and glb is often an unnecessarily strong requirement. A number of results in computer science rely only on the fact that all chains have least upper bounds.

Example 12. *A poset* $C = (2^{A \times B}, \subseteq)$ *of relations is a lattice. Its subset,* $(A \twoheadrightarrow B, \subseteq)$ *– the poset of partial functions from* A *to* B*, is a meet semi-lattice*[3]*. Note that total functions are maximal elements of this semi-lattice. The subset of total functions from* A *to* B*, i.e.* $(A \to B, \subseteq)$*, is an anti-chain.*

[3] Why is it not a join semi-lattice?

Definition 13. *A partial order* $(A, \leq)$ *is said to be chain complete (abbreviated* ccpo*) if each ascending chain*

$$a_0 < a_1 < a_2 < \ldots$$

has a least upper bound $\bigvee\{a_0, a_1, a_2, \ldots\} \in A$*.*

(It follows that A *has the least element* $\perp$*, as it is the lub of the empty chain.)*

The intuition is that the chain is a sequence of approximations, and the lub is the limit – the object approximated by the sequence.

Let us note that a more common definition of ccpo requires least upper bounds of every chain (and there exist chains which cannot be represented as ascending chains). Sometimes the ccpo's from the definition above are called ω-cpo's.

Example 13. *The set $\mathbb{N}$ of natural numbers under $\leq$ is not a ccpo. It does have a least element, namely 0, but no infinite ascending chain*

$$n_1 < n_2 < n_3 < \dots$$

has a least upper bound. However, $\mathbb{N}$ extended with a top element ω where $n < \omega$ for all $n \in \mathbb{N}$ is a ccpo (the reason for using the symbol ω should be clear in Chapter 3).

On the other hand, $(\mathbb{Z}, \preceq)$ which is $\mathbb{N}$ augmented by negative integers, so that the positive numbers precede the negative ones: $0 \prec 1 \prec 2 \prec \cdots -3 \prec -2 \prec -1$ is not a ccpo (as $\mathbb{N}$ has upper bounds, but not the least one).

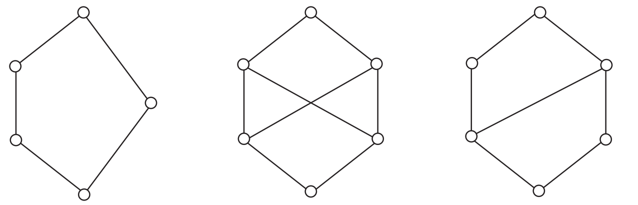
Example 14. *We show that the poset of partial functions $(A \twoheadrightarrow B, \subseteq)$ is a ccpo. Consider an ascending chain S in $A \twoheadrightarrow B$. We first show that its union $f = \bigcup S$ is a partial function. Let $(x, a), (y, b) \in f$ be two distinct tuples. We need to prove that $x \neq y$ or $a = b$. Let $f_1, f_2 \in S$ be two partial functions where $f_1(x) = a$ and $f_2(y) = b$. Since S is a chain, either $f_1 \subseteq f_2$, or $f_2 \subseteq f_1$, and we assume without loss of generality that $f_1 \subseteq f_2$. Then $x \in \mathrm{Domain}(f_2)$ and $f_2(x) = a$. Since f_2 is a partial function either $x \neq y$ or $a = b$. Note that $f = \bigcup S$ is also the lub of S in $(A \twoheadrightarrow B, \subseteq)$, since if $f \subseteq g$ for any $f \in S$ then $\bigcup S \subseteq g$.*

Informally, all the functions in a chain S agree on each $a \in A$, in the sense that they do not give distinct result for a (either $f_i(a) = (\bigcup S)(a)$ or f_i is not defined on a). In other words, the elements of the chain contain information consistent with one another. Moreover, the lub of the chain contains only the information provided by the chain elements ($\bigcup S$ is defined on a iff some $f_i \in S$ is defined on a).

Such posets of partial functions are important for instance in semantics of programming languages. In particular, an infinite (partial) function can be represented as the lub of a chain of finite functions. Similarly, the semantics of a loop statement can be represented as the lub of a chain of functions, each describing the loop performing up to n repetitions.

Exercises

2.1 Which of the following Hasse diagrams represent lattices?



2.2 Consider a set A and the set $E = \{ B \subseteq A \mid |B| \text{ is even} \}$ of its finite subsets of even size. Is $(E, \subseteq)$ a lattice?

2.3 Which of the following structures are complete lattices? Give a counter-example if not, otherwise give the lub and glb:

1. The set of all finite strings Σ^* under the (non-strict) lexicographical order, where e.g. $a \sqsubseteq ab \sqsubseteq abb \sqsubseteq ac$.
2. The set $\mathbb{N}$ under the partial order $m \preceq n$ iff there exists an $i \in \mathbb{N}$ such that $m \cdot i = n$.
3. The set of all equivalence relations on a set A, ordered by set inclusion.

2.4 Show that any finite lattice is a complete lattice.

2.5 Prove that the following definitions of a lattice are equivalent:

- A poset $(A, \leq)$ where all $x, y \in A$ have a lub and glb.
- A poset $(A, \leq)$ where every finite and non-empty subset has a lub and glb.

2.6 Consider a lattice $(A, \leq)$ with lub $\vee$ and glb $\wedge$. Show that the following conditions are equivalent:

1. $a \leq b$,
2. $a \wedge b = a$,
3. $a \vee b = b$.

2.7 If possible give an example of a 5-element lattice which is distributive, and one which is not.

2.8 Prove that if a lattice $(A, \leq)$ satisfies $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$, then it also satisfies $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

2.9 If possible give an example of a 5-element complemented lattice which has unique complements, and one which has not.

2.10 Use the algebraic laws $L_1 - L_4$ to prove that $a \wedge b = a$ iff $a \vee b = b$.

2.11 Prove that the componentwise product of two complete lattices is a complete lattice as well.

2.12 Let $(A, \leq)$ be a non-empty poset. Prove that the following are equivalent:

1. $(A, \leq)$ is a complete lattice,
2. $\bigwedge B$ exists for every $B \subseteq A$,
3. $\bigvee A$, and $\bigwedge B$ exist for every non-empty $B \subseteq A$.

Hint: To show that (2) implies (1), for a given $C \subseteq A$ consider the set $B = \{ x \in A \mid C \leq x \}$ and show that its glb, $\bigwedge B$, is the lub of C.

2.13 Prove that $(\mathcal{O}(A), \subseteq)$ (see Definition 1) is a complete lattice.

2.14 Prove that the two definitions of a lattice – the one based on an ordered set $(A, \leq)$ with least upper and greatest lower bounds, and the algebraic one $(A, \wedge, \vee)$ – are equivalent. That is

1. Given a lattice $(A, \leq)$, prove that the algebra $(A, \wedge, \vee)$ (where $\wedge$, $\vee$ are the lub and glb operations of the lattice) satisfies $L_1 - L_4$.
2. Given an algebra $(A, \wedge, \vee)$ satisfying $L_1 - L_4$, prove that $(A, \leq)$, where $x \leq y$ iff $x \vee y = y$, is a poset and that each $x, y \in A$ has a least upper and greatest lower bound.

2.15 Prove Theorem 1.

2.16 Give an example of a ccpo $(A, \leq)$ which has a top element, and for some its elements $a, b \in A$ their lub $a \vee b$ does not exist.

References

G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3rd edition, 1967.

G. Grätzer. *General Lattice Theory*. Academic Press, 1978.