Assignment 4

for Discrete Structures 2

4.1 Explain what is wrong in the presented erroneous inductive proof on page 2.

4.2 Let $(A, \leq), (B \sqsubseteq)$ be complete lattices, and $f\colon A \to B$ be a monotonic function[1]. Prove that

$$f(\bigvee C) \sqsupseteq \bigsqcup \{ f(x) \mid x \in C \}$$

for any subset $C \subseteq A$.

$\bigvee$ and $\bigsqcup$ stand for the lub's, respectively, in $(A, \leq)$ and in $(B \sqsubseteq)$.

[This problem is not related to induction, but rather to lattices and ccpo's. But a special case of this property is used in the lecture notes, as a lemma in the context of transfinite induction.]

4.3 Assume that $f\colon A \to A$ is a monotonic function on a complete lattice $(A, \leq)$. Prove that $f^\beta \leq f^\alpha$ for any ordinals $\beta < \alpha$.

Make it clear which induction principle you apply. You may use the fact (proved in the lecture note) that $f^\alpha \leq f^{\alpha+1}$. Note that it may be not necessary to refer to the inductive assumption in one of the cases within your proof.

[1] I.e., if $x \leq y$ then $f(x) \sqsubseteq f(y)$.

An erroneous inductive proof

Let M be an infinite set, f a function on M, and $L \subseteq M$. Property $P(L)$ says that f is constant on L. $\boxed{P(L)\text{: If } x, y \in L \text{ then } f(x) = f(y)}$

We "prove" $P(L)$ for any finite non empty L.

Base case, $|L| = 1$: $\quad x, y \in L \;\Rightarrow\; x = y \;\Rightarrow\; f(x) = f(y)$.

Inductive step: Assume $P(L')$ holds for any $|L'| = n$.

Let $|L| = n + 1$, $x, y \in L$.

1. $x = y \;\Rightarrow\; f(x) = f(y)$.
2. $x \neq y$

$$L\colon\ \boxed{\rlap{$\overbrace{}^{L\setminus\{y\}}$}x\ \underbrace{\cdots z\cdots\ y}_{L\setminus\{x\}}}$$

$\big|L \setminus \{x\}\big| = n = \big|L \setminus \{y\}\big|$,
$f(x) = f(z)$ for each $z \in L \setminus \{y\}$,
$f(y) = f(z)$ for each $z \in L \setminus \{x\}$.
Thus $f(x) = f(z) = f(y)$.

Hence $f(x) = f(y)$ for any $x, y \in L$.

What is wrong?