A Galois Connection for Universally Quantified Conjunctive Formulas

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1 Background

Let $D \subseteq \mathbb{N}$ be a finite set of values. A *clone* over D is a set of functions C over D such that (1) C is closed under composition of functions and (2) C contains all *projection functions*, i.e., k-ary functions of the form $e_i^k(x_1, \ldots, x_k) = x_i$ for some $i \in \{1, \ldots, k\}$. Equivalently, a clone over D can be defined as the set of term operations over an algebra (D, F), where F is a set of functions over D. Hence, from a given set F of functions we may form new functions by combining functions from F in various ways, and if no new functions can be obtained in this way, the resulting set is called a clone.

It is well-known that there exists a so-called *Galois connection* between clones and sets of relations satisfying certain closure properties. More specifically, a set of relations Γ is said to be a *relational clone*, or a *co-clone*, if Γ is closed under first-order formulas consisting of existential quantification over conjunctions of positive atoms from Γ, *primitive positive formulas*. Hence, a relational clone may be specified by a set of relations from which one may form a larger set by combining the "basic relations" in the ways prescribed by primitive positive formulas. The Galois connection between clones and relational clones then says that each clone (in a unique way) can equivalently well be described by a relational clone, and vice versa. This essentially offers two very different interpretations of the same mathematical object, and this duality is a fundamental part of modern algebraic approaches to study computational complexity.

Given these notions, it is natural to ask whether similar Galois connections are possible if one considers different classes of logical formulas. This question has been studied extensively in the literature and a survey of known results can be found in Pöschel [4] and Börner [3]. For example, it is known that sets of relations closed under pp-definitions without equality corresponds to *clones of multifunctions*, *hyperclones*, and that sets of relations closed under pp-definitions without existential quantification corresponds to *strong partial clones*.

2 Goals

Let us consider yet another variation of pp-formulas, namely first-order formulas where existential quantification is replaced by universal quantification. Surprisingly, while this results in a rather natural closure operator on relations, nothing is known about dual classes of functions which would arise in the presence of a Galois connection. Hence, does there exist a Galois connection between sets of relations closed under such formulas and certain sets of functions?

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3   Reading Material

The surveys by Pöschel [4] and Börner [3] provide a good starting point. For a gentler introduction to clones and co-clones, the more practically inclined survey by Böhler et al. [1, 2] is also a good reference.

References

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