Properties of an Approximability-related Parameter on Circular Complete Graphs

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Abstract

The instances of the *Weighted Maximum H-Colourable Subgraph* problem (MAX H-COL) are edge-weighted graphs G and the objective is to find a subgraph of G that has maximal total edge weight, under the condition that the subgraph has a homomorphism to H; note that for $H = K_k$ this problem is equivalent to MAX k-CUT. Färnqvist et al. have introduced a parameter on the space of graphs that allows close study of the approximability properties of MAX H-COL. Here, we investigate the properties of this parameter on circular complete graphs $K_{p/q}$, where $2 \leq p/q \leq 3$. The results are extended to K_4-minor-free graphs. We also consider connections with Šámal's work on fractional covering by cuts: we address, and decide, two conjectures concerning cubical chromatic numbers.

Keywords: graph H-colouring, circular colouring, combinatorial optimisation, graph theory

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1 Introduction

Denote by $\mathcal{G}$ the set of all simple, undirected and finite graphs. A *graph homomorphism* from $G \in \mathcal{G}$ to $H \in \mathcal{G}$, denoted by $G \to H$, is a vertex map which carries the edges in G to edges in H. Now, *Weighted Maximum H-Colourable Subgraph* (Max H-Col) is the maximisation problem with

Instance: An edge-weighted graph (G, w), where $G \in \mathcal{G}$ and $w : E(G) \to \mathbb{Q}^+$.

Solution: A subgraph G' of G such that $G' \to H$.

Measure: The sum of the weights of $E(G')$ with respect to w.

Given an edge-weighted graph (G, w), denote by $mc_H(G, w)$ the measure of an optimal solution to the problem Max H-Col. Denote by $mc_k(G, w)$ the (weighted) size of a largest k-cut in (G, w). This notation is well justified by the fact that $mc_k(G, w) = mc_{K_k}(G, w)$. In this sense, Max H-Col generalises Max k-cut which is a well-known and well-studied problem that is computationally hard when $k > 1$. Since Max H-Col is a hard problem to solve exactly, efforts have been made to find suitable approximation algorithms. Färnqvist et al. [3] introduce a method that can be used to extend previously known (in)approximability bounds on Max H-Col to new and larger classes of graphs. Assuming the *Unique Games Conjecture*, Raghavendra's semidefinite programming algorithms [7] have optimal performance for every *maximum constraint satisfaction problem* (Max CSP), a problem which generalises Max H-Col, but the exact approximation ratios are not yet known. In fact, even though an algorithm (doubly exponential in the domain size) for computing these ratios for specific Max Csp problems has emerged [8], this should be contrasted to the infinite classes of graphs the method of Färnqvist et al. gives new bounds for.

The fundament of this promising technique is the ability to compute (or closely approximate) a function $s : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ defined as follows:

$$s(M, N) = \inf_{\substack{G \in \mathcal{G} \\ w : E(G) \to \mathbb{Q}^+}} \frac{mc_M(G, w)}{mc_N(G, w)}. \tag{1}$$

It is shown [3] that s satisfies the following property: if $M \to N$ and mc_M can be approximated within α, then mc_N can be approximated within $\alpha \cdot s(M, N)$. Using this technique, Färnqvist et al. [3] present concrete approximation ratios for certain graphs (such as the odd cycles) and near-optimal asymptotic results for large graph classes. Let $\{A_i\}_{i=1}^r$ be the set of orbits of the edge automorphism group $\mathrm{Aut}^*(N)$, and for each $f : V(M) \to V(N)$, let f_i be the number of edges in A_i that f maps to some edge in N. Then, $s(M, N)$ can be obtained by the linear program which minimises the objective function, s, subject to $\sum_{i=1}^r f_i \cdot w_i \leq s$, for each vertex map

f, and $\sum_{i=1}^{r} |A_i| \cdot w_i = 1$. It is clear that $\mathrm{Aut}^*(N)$ may be prohibitively large for a direct application of this technique. However, bounds can be obtained by using the following lemma:

Lemma 1.1 ([3]) *Let $M \to H \to N$. Then, $s(M, H) \geq s(M, N)$ and $s(H, N) \geq s(M, N)$.*

In order to use this result effectively, we need a large selection of graphs M, N that are known to be close to each other with respect to s. For the moment, the set of such examples is quite meagre. In this paper, we initiate the study of s on the class of circular complete graphs. In particular, we take a careful look at 3-colourable circular complete graphs and, amongst other things, find that s is constant between a large number of these graphs. Moreover, we extend the results on s to K_4-minor-free graphs with some additional constraints.

Another way to bound the function s is to relate it to other known graph parameters. We do this by connecting our work with that of Šámal [9,10] on fractional covering by cuts to obtain a new family of 'chromatic numbers'. This reveals that $s(M, N)$ and the new chromatic numbers $\chi_M(N)$ are closely related quantities, which provides us with an alternative way of computing s. We also use our knowledge about the behaviour of s to disprove a conjecture by Šámal concerning the cubical chromatic number and, finally, we decide in the positive another conjecture by Šámal concerning the same parameter. All proofs are deferred to the technical report [2] available at `http://www.arxiv.org/abs/0904.4600`.

2 Calculating s for Circular Complete Graphs

A circular complete graph $K_{p/q}$ is a graph with vertex set $\{v_0, v_1, \ldots, v_{p-1}\}$ and edge set $E(K_{p/q}) = \{v_i v_j \mid q \leq |i - j| \leq p - q\}$. This can be seen as placing the vertices on a circle and connecting two vertices by an edge if they are at a distance at least q from each other. A fundamental property of these graphs is that $K_{p/q} \to K_{p'/q'}$ iff $p/q \leq p'/q'$. Due to this fact, when we write $K_{p/q}$, we will assume that p and q are relatively prime. We note that a homomorphism from a graph G to $K_{p/q}$ is called a (circular) (p/q)-colouring of G. The book by Hell and Nešetřil [4] and the survey by Zhu [11] give more information on this topic.

In the following, we investigate $s(K_r, K_t)$ for rational numbers $2 \leq r < t \leq 3$. First, we fix $r = 2$ and choose t so that $\mathrm{Aut}^*(K_t)$ has few orbits. We find some interesting properties of these numbers which lead us to look at the case $r = 2 + 1/k$. Our approach is based on relaxing the linear program for s that was mentioned in Section 1, combined with arguments that our chosen relaxations in fact find the optimum in the original program.

Proposition 2.1 *Let $k \geq 1$ be an integer and $2 \leq r < \frac{2k+1}{k} \leq t \leq \frac{4k}{2k-1}$. Then,*

(i) $s(K_2, K_{\frac{4k}{2k-1}}) = \frac{2k}{2k+1}$ *(Prop. 3.1 [2]),*

(ii) $s(K_r, K_t) = \frac{2k}{2k+1}$ *(Corr. 3.2 [2]),*

(iii) $s(K_2, K_{\frac{6k+5}{3k+1}}) = \frac{6k^2+8k+3}{6k^2+11k+5} = 1 - \frac{3k+2}{(k+1)(6k+5)}$ *(Prop. 3.3 [2]),*

(iv) $s(K_2, K_{\frac{8k+6}{4k+1}}) = \frac{8k^2+6k+2}{8k^2+10k+3} = 1 - \frac{4k+1}{(k+1/2)(8k+6)}$ *(Prop. 3.4 [2]).*

We see that there are intervals $I_k = \{t \in \mathbb{Q} \mid 2 + 1/k \leq t \leq 2 + 2/(2k-1)\}$ where $s(t) = s(K_r, K_t)$ is constant. In Figure 1 these intervals are shown for the first few values of k. The intervals I_k form an infinite sequence with endpoints tending to 2. Similar intervals appear throughout the space of circular complete graphs. More specifically, Färnqvist et al. [3] have shown that $s(K_n, K_{2m-1}) = s(K_n, K_{2m})$ for arbitrary integers $n, m \geq 2$. Furthermore, it can be proved that $s(K_2, K_n) = s(K_{8/3}, K_n)$ for $n \geq 3$. Two applications of Lemma 1.1 now shows that $s(K_r, K_t)$ is constant on each region $[2, 8/3] \times [2m-1, 2m]$, $m \in \mathbb{Z}^+$.

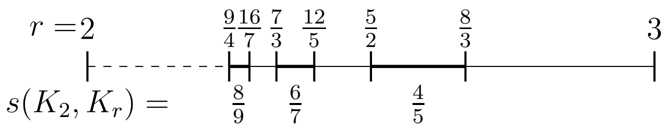

Fig. 1. The space between 2 and 3 with the intervals I_k marked for $k = 2, 3, 4$.

Since $s(K_r, K_t)$ is constant on the region $(r, t) \in [2, 2 + 1/k) \times I_k$, it is interesting to see what happens when t remains in I_k, but r is set to $2 + 1/k$. Smaller $t \in I_k$ can be expressed as $t = 2 + 1/(k - x)$, where $0 \leq x < 1/2$. We will write $x = m/n$ for positive integers m and n which implies the form $t = 2 + n/(kn - m)$, with $m < n/2$. For $m = 1$ and $m = 2$, we get the following results:

Proposition 2.2 (Prop. 3.5 [2]) *Let $k \geq 2$ and n be integers. Then,*

(i) $s(C_{2k+1}, K_{\frac{2(kn-1)+n}{kn-1}}) = \frac{(2(kn-1)+n)(4k-1)}{(2(kn-1)+n)(4k-1)+4k-2}$, *for $n \geq 2$.*

(ii) $s(C_{2k+1}, K_{\frac{2(kn-2)+n}{kn-2}}) \geq \frac{(2(kn-2)+n)(\xi_n(4k-1)+(2k-1))}{(2(kn-2)+n)(\xi_n(4k-1)+(2k-1))+(4k-2)(1-\xi_n)}$, *where $\xi_n = \left(\alpha_1^{(n-1)/2} + \alpha_2^{(n-1)/2}\right)/4$, and α_1, α_2 are the reciprocals of the roots of $\frac{2k-3}{4k-2}z^2 - 2z + 1$, for $n \geq 3$ and odd.*

3 Extensions and Connections

We start by applying known bounds on the circular chromatic number for certain classes of planar graphs. Much of the extensive study conducted in this direction was instigated by the restriction of a conjecture by Jaeger [5] to planar graphs, which is equivalent to the claim that every planar graph of girth at least $4k$ has a circular chromatic number at most $2 + 1/k$, for $k \geq 2$. We remark that Jaeger's conjecture implies a weaker statement in our setting. Namely, if G is a planar graph with girth greater than $4k$, then $G \to C_k$ implies $s(K_2, G) \geq s(K_2, C_k) = 2k/(2k+1)$. Deciding this to be true would certainly provide support for the original conjecture, and would be an interesting result in its own right. Currently, the best proven girth for when the circular chromatic number of a planar graph is guaranteed to be at most $2 + 1/k$ is $\frac{20k-2}{3}$ and due to Borodin et al. [1]. This result was used by Färnqvist et al. to achieve the bound $s(K_2, G) \leq \frac{4k}{4k+1}$ for planar graphs G of girth at least $\frac{40k-2}{3}$. Here, we improve on this result for the class of K_4-minor-free graphs by using a result by Pan and Zhu [6]:

Proposition 3.1 (Prop. 3.8 [2]) *Let G be a K_4-minor-free graph, and $k \geq 1$ an integer. If G has an odd girth of at least $6k-1$, then $s(K_2, G) \leq \frac{4k}{4k+1}$. If G has an odd girth of at least $6k+3$, then $s(K_2, G) \leq \frac{4k+2}{4k+3}$.*

Connections can also be made to the work of Šámal [9,10] on fractional covering by cuts. In this paper, we generalise Šámal's *cubical chromatic number* χ_q to a family of such 'chromatic numbers', $\chi_H(G)$, $H \in \mathcal{G}$, where $\chi_q(G)$ appears for $H = K_2$. Let H_k^n be the graph on vertex set $V(H)^n$ and an edge between $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ when $|\{i \mid (u_i, v_i) \in E(H)\}| \geq k$. $\chi_H(G) = \inf\{\frac{n}{k} \mid G \to H_k^n\}$. From a linear programming formulation of this parameter, we are able to show that $\chi_H(G) = 1/s(H, G)$ for all $G, H \in \mathcal{G}$. In particular $\chi_q(G) = 1/s(K_2, G)$.

In Section 2 we obtained lower bounds on s by relaxing a linear program. In most cases, the corresponding solution was proven feasible for the original linear program, and hence optimal. Another source of upper bounds on s can be derived as follows. Let $G, H \in \mathcal{G}$, with $G \to H$ and take an arbitrary S such that $G \to S \to H$. Then, applying Lemma 1.1 followed by Lemma 8 from [3] gives $s(G, H) \leq s(G, S) = \inf_{w: w(S)=1} mc_G(S, w) \leq mc_G(S, 1/|E(S)|)$. For $G = K_2$ it follows that $s(K_2, H) \leq \min_{S \subseteq G} b(S)$, where $b(S)$ denotes the bipartite density of S. Šámal [10] conjectured that this inequality, expressed on the form $\chi_q(S) \geq 1/(\min_{S \subseteq G} b(S))$, can be replaced by an equality. We answer this in the negative, using $K_{11/4}$ as our counterexample. Proposition 2.1(iii) with $k = 1$ gives $s(K_2, K_{11/4}) = 17/22$. If $s(K_2, K_{11/4}) = b(S)$ for some $S \subseteq K_{11/4}$ it means that S must have at least 22 edges. Since $K_{11/4}$ has exactly 22 edges, then $S = K_{11/4}$. However, a cut in a cycle must contain an even number of edges. Since the edges of $K_{11/4}$ can be partitioned into two cycles, we find that the maximum cut in $K_{11/4}$ must be of even size, hence $|E(K_{11/4})| \cdot b(K_{11/4}) \neq 17$. This is a contradiction.

As a part of his investigation of χ_q, Šámal [10] studied the value of $\chi_q(Q_{n/k})$

($Q_{n/k} = (K_2)^n_k$). He provided an upper bound and an approach to a lower bound using the largest eigenvalue of the Laplacian of a subgraph of $Q_{n/k}$. Computing this eigenvalue boils down to an inequality (Conjecture 5.4.6 [10]) involving some binomial coefficients. Using an inductive argument on a combinatorial interpretation of his Conjecture 5.4.2 [10], we complete the proof of the following proposition:

Proposition 3.2 (Prop. 4.2 [2]) *Let k, n be integers such that $k \leq n < 2k$. Then, $\chi_q(Q_{n/k}) = n/k$ if k is even and $(n+1)/(k+1)$ if k is odd.*

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