Extending the Point Algebra into the Qualitative Algebra

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Abstract

We study the computational complexity of the qualitative algebra *which is a temporal formalism that combines the point algebra, the point-interval algebra and Allen's interval algebra. We identify all tractable fragments containing the point algebra and show that, for all other fragments containing the point algebra, the problem is NP-complete.*

1 Introduction

Reasoning about temporal knowledge is a common task in many branches of computer science and elsewhere, *cf.* Golumbic and Shamir [6] for a list of examples from a wide range of applications. Knowledge of temporal constraints is typically expressed in terms of collections of relations between time points and/or time intervals. Reasoning tasks include determining the satisfiability of such collections and deducing new relations from those that are known.

Basic temporal formalisms can only be used for reasoning about objects of a single type—for instance, the point algebra [16] is only useful for time points and Allen's interval algebra [1] is only useful for time intervals. Such restricted languages may not be sufficient for modelling real-world problems so several formalisms for multisorted temporal reasoning have been proposed [2, 7, 9, 13, 15]. However, the basic temporal formalisms are much easier to analyse from a complexity-theoretic standpoint; all tractable subclasses of Allen's interval algebra are known, for instance [10].

The goal of this paper is to study the computational complexity of Meiri's [13] *Qualitative Algebra* which is a temporal formalism able to represent both time points and time intervals. Thus, we can relate points with points, points with intervals and intervals with intervals using an expressive set of qualitative relations. In contrast with Allen's algebra and the point-interval algebra, the point algebra is known to be tractable. In this paper, we show how to maximally extend the point algebra with point-interval and interval-interval relations so that the obtained extension remains tractable. Moreover, we prove that all other extensions are NP-complete. In fact, our classification result holds under the following weaker assumption: the point-point relation 'less than' is a member of the set under consideration. By using combinatorial techniques, we can prove this result without using computer-assisted enumeration methods. A related result has been proved by Broxvall and Jonsson [3]—they identify all tractable fragments of the point algebra extended with disjunctions.

The paper is organised as follows: in Section 2 we give the basic definitions and present the maximal tractable subclasses. In Section 3 we formally state the classification result and prove it; Subsection 3.1 contains some tractability results and Section 3.2. contains the classification proof together with a few proof techniques. Some concluding remarks are collected in Section 4.

2 Preliminaries

In the Qualitative Algebra (QA) [13], a qualitative constraint between two objects O_i and O_j (each may be a point or an interval), is a disjunction of the form

$$(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$$

where each one of the $r_i' s$ is a *basic qualitative relation* that may exist between two objects. There are three types of basic relations.

1. *Point-point* (PP) relations that can hold between a pair of points.
2. *Point-interval* (PI) and *interval-point* (IP) relations that can hold between a point and an interval and vice-versa.
3. *Interval-interval* (II) relations that can hold between a pair of intervals.

The basic relations are shown in Table 1. Note that we use different fonts to distinguish between PI- and II-relations. The endpoint relation $I^- < I^+$ that is required for all intervals has been omitted. For the sake of brevity, we will write expressions of the form $(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$ as $O_i(r_1 \ldots r_k)O_j$. Let $\emptyset$ denote the empty relation. Let $\mathcal{PP}$, $\mathcal{PI}$ and $\mathcal{II}$ denote the sets of all PP-relations, PI-relations and II-relations, respectively, and let $\mathcal{QA} = \mathcal{PP} \cup \mathcal{PI} \cup \mathcal{II}$.

The problem of *satisfiability* (QA-SAT) of a set of point and interval variables with relations between them is that of deciding whether there exists an assignment of points and intervals on the real line for the variables, such that all of the relations are satisfied. This is defined as follows.

Definition 1 *Let $X \subseteq \mathcal{QA}$. An instance Π of* QA-SAT(X) *consists of a set V_p of point variables, a set V_I of interval variables and a set of constraints of the form xry where $x, y \in V_p \cup V_I$ and $r \in X$. We require that $V_p \cap V_I = \emptyset$.*

The question is whether Π is satisfiable *or not, i.e. whether there exists a function M, called a* model, *satisfying the following:*

1. *for each $v \in V_p$, $M(v) \in \mathcal{R}$;*
2. *for each $v \in V_I$, $M(v) = (I^-, I^+) \in \mathcal{R} \times \mathcal{R}$ and $I^- < I^+$.*
3. *for each constraint $xry \in C$, $M(x)rM(y)$ holds.*

It is easy to show that QA-SAT is in NP. Let $X \subseteq \mathcal{QA}$ and assume that $\Pi = (V_p, V_I, C)$ is an instance of QA-SAT. We define $\mathrm{Var}(\Pi)$ as the set of variables in Π and $X_{\mathcal{PP}}$, $X_{\mathcal{PI}}$, $X_{\mathcal{II}}$ as $X \cap \mathcal{PP}$, $X \cap \mathcal{PI}$, $X \cap \mathcal{II}$, respectively. We extend the notation to sets of constraints and problem instances, i.e. $\Pi_{\mathcal{II}}$ denotes the subinstance only containing II-constraints: $(\emptyset, V_I, \{IrJ \in C \mid I, J \in V_I\})$.

If there exists a polynomial-time algorithm solving all instances of QA-SAT(X) then we say that X is tractable. On the other hand, if QA-SAT(X) is NP-complete then we say that X is NP-complete. Since $\mathcal{QA}$ is finite, the problem of describing tractability in $\mathcal{QA}$ can be reduced to the problem of describing the *maximal* tractable subclasses in $\mathcal{QA}$, i.e., subclasses that cannot be extended without losing tractability.

The complexity of QA-SAT$(\mathcal{S})$ has been completely determined earlier when $\mathcal{S}$ is a subset of $\mathcal{PP}$, $\mathcal{PI}$ or $\mathcal{II}$. In order to simplify the presentation of tractable subclasses, we use the symbol $\pm$, which should be interpreted as follows. A condition involving $\pm$ means the conjunction of two conditions: one corresponding to $+$ and one corresponding to $-$. For example, condition $(\mathsf{o})^{\pm 1} \subseteq r \Leftrightarrow (\mathsf{d})^{\pm 1} \subseteq r$ means that both $(\mathsf{o}) \subseteq r \Leftrightarrow (\mathsf{d}) \subseteq r$ and $(\mathsf{o}^{-1}) \subseteq r \Leftrightarrow (\mathsf{d}^{-1}) \subseteq r$ hold. Note that implication takes precedence over conjunction in the descriptions.

The following notation is used in Figure 3 to describe the tractable subalgebras of $\mathcal{A}$. Let $q_0 = (\mathsf{pmo})$. Further, let $q_1 = q_0 \cup (\mathsf{d}^{-1}\mathsf{f}^{-1})$, $q_2 = q_0 \cup (\mathsf{df})$, $q_3 = q_0 \cup (\mathsf{df}^{-1})$, $q_4 = q_0 \cup (\mathsf{ds})$, $q_5 = q_0 \cup (\mathsf{d}^{-1}\mathsf{s}^{-1})$, $q_6 = q_0 \cup (\mathsf{d}^{-1}\mathsf{s})$.

Theorem 2 (Vilain *et al.* [16]) *$\mathcal{PP}$ is tractable.*

Theorem 3 (Jonsson *et al* [8]) *$\mathcal{PI}$ contains 5 maximal tractable subclasses $\mathcal{V_H}, \mathcal{V_S}, \mathcal{V_E}, \mathcal{V_S}$ and $\mathcal{V_F}$ (see Table 2).*

Theorem 4 (Krokhin *et al* [10]) *$\mathcal{II}$ contains 18 maximal tractable subclasses (see Table 3).*

Let $\mathcal{II}_{\mathrm{tr}}$ denote the set of maximal tractable subclasses of II-relations. In some previous papers, the subclasses in Tables 2 and 3 were defined in other ways. However, in all cases except for $\mathcal{H}$, it is very straightforward to verify that our definitions are equivalent to the original ones. The subclass $\mathcal{H}$ was originally defined as the 'ORD-Horn algebra' [14], but has also been characterized as the set of 'pre-convex' relations (see, e.g., [12]). Using the latter description it is not hard to show that our definition of $\mathcal{H}$ is equivalent.

3 Main Result

Our main result is the identification of all tractable subclasses $\mathcal{S}$ of $\mathcal{QA}$ satisfying the side condition $(<) \in \mathcal{S}_{\mathcal{PP}}$. Let $\mathcal{W} \subseteq \mathcal{II}$ and $\mathcal{V} \subseteq \mathcal{PI}$. Let $\mathcal{WV}$ denote the set $\mathcal{W} \cup \mathcal{V} \cup \mathcal{PP}$. We remind the reader that exact definitions of the subalgebras can be found in Tables 2 and 3.

Theorem 5 *Let $X \subseteq \mathcal{QA}$ satisfy $(<) \in X$. Then* QA-SAT(X) *is tractable iff X is a included in one of the subclasses defined below. Otherwise,* QA-SAT(X) *is NP-complete.*

- $\mathcal{WV}_b$ *and* $\mathcal{WV}_a$ *if* $\mathcal{W} \in \mathcal{II}_{\mathrm{tr}} - \{\mathcal{S}_p, \mathcal{E}_p\}$
- $\mathcal{WV}_d$ *if* $\mathcal{W} \in \mathcal{II}_{\mathrm{tr}} - \{\mathcal{H}, \mathcal{S}_p, \mathcal{E}_p\}$
- $\mathcal{HV}_{\mathcal{H}}$, $\mathcal{S}_p\mathcal{V}_{\mathcal{S}}$, $\mathcal{E}_p\mathcal{V}_{\mathcal{E}}$
- $\mathcal{WV}_{\mathcal{SH}}$ *if* $\mathcal{W} \in \{\mathcal{S}_d, \mathcal{S}_o, \mathcal{S}^*\}$
- $\mathcal{WV}_{\mathcal{EH}}$ *if* $\mathcal{W} \in \{\mathcal{E}_d, \mathcal{E}_o, \mathcal{E}^*\}$

The rest of this section is structured as follows. In Subsection 3.1, we prove the tractability of a number of subclasses and we give the proof of Theorem 5 in Subsection 3.2.

Basic relation		Example	Endpoints
p before q	$<$	`p` `  q`	$p < q$
p equals q	$=$	`  p` `  q`	$p = q$
p after q	$>$	`    p` `  q`	$p > q$

Basic relation		Example	Endpoints
p before I	`b`	`p` `   III`	$p < I^-$
p starts I	`s`	`   p` `   III`	$p = I^-$
p during I	`d`	`    p` `   III`	$I^- < p < I^+$
p finishes I	`f`	`     p` `   III`	$p = I^+$
p after I	`a`	`       p` `   III`	$p > I^+$

Basic relation		Example	Endpoints
I precedes J	p	`III`	$I^+ < J^-$
J preceded by I	p^{-1}	`     JJJ`	
I meets J	m	`IIII`	$I^+ = J^-$
J met by I	m^{-1}	`    JJJJ`	
I overlaps J	o	`IIII`	$I^- < J^- < I^+$,
J overl. by I	o^{-1}	`  JJJJ`	$I^+ < J^+$
I during J	d	`  III`	$I^- > J^-$,
J includes I	d^{-1}	`JJJJJJJ`	$I^+ < J^+$
I starts J	s	`III`	$I^- = J^-$,
J started by I	s^{-1}	`JJJJJJJ`	$I^+ < J^+$
I finishes J	f	`    III`	$I^+ = J^+$,
J finished by I	f^{-1}	`JJJJJJJ`	$I^- > J^-$
I equals J	$\equiv$	`IIII` `JJJJ`	$I^- = J^-$, $I^+ = J^+$

Table 1. Basic PP-, PI- and II-relations.

$\mathcal{V}_{\mathcal{H}} = \{r \mid r \cap (\mathtt{bs}) \neq \emptyset \;\&\; r \cap (\mathtt{fa}) \neq \emptyset \Rightarrow (\mathtt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{SH}} = \{r \mid \; r \cap (\mathtt{fa}) \neq \emptyset \Rightarrow (\mathtt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{EH}} = \{r \mid r \cap (\mathtt{bs}) \neq \emptyset \Rightarrow (\mathtt{d}) \subseteq r\}$
$\mathcal{V}_{\mathcal{S}} = \{r \mid r \cap (\mathtt{df}) \neq \emptyset \Rightarrow (\mathtt{a}) \subseteq r\}$
$\mathcal{V}_{\mathcal{E}} = \{r \mid r \cap (\mathtt{sd}) \neq \emptyset \Rightarrow (\mathtt{b}) \subseteq r\}$
$\mathcal{V}_{\mathsf{r}} = \{r \mid r \neq \emptyset \Rightarrow (\mathsf{r}) \subseteq r\}$ where $\mathsf{r} \in \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}, \mathtt{a}\}$

Table 2. Subsets of PI-relations.

3.1 Tractability results

We shall now show that all subclasses in Theorem 5 are tractable.

Lemma 6 *$\mathcal{WV}_{\mathsf{b}}$ and $\mathcal{WV}_{\mathsf{a}}$ are tractable if $\mathcal{W} \in \mathcal{II}_{\rm tr} - \{\mathcal{S}_{\mathsf{p}}, \mathcal{E}_{\mathsf{p}}\}$.*

Proof. We give a proof for the case $X = \mathcal{WV}_{\mathtt{b}}$; the other case is analogous. Let Π be an arbitrary instance of QA-SAT(X) and assume without loss of generality that no constraint is trivially unsatisfiable, i.e. of the form $x \emptyset y$. We claim that Π is satisfiable iff $\Pi_{\mathcal{PP}}$ and $\Pi_{\mathcal{II}}$ are satisfiable—obviously, this can be checked in polynomial time by the choice of $\mathcal{W}$.

If $\Pi_{\mathcal{PP}}$ or $\Pi_{\mathcal{II}}$ are not satisfiable, then Π is not satisfiable. Otherwise, there exists two models $M_{\mathcal{PP}}$ and $M_{\mathcal{II}}$ of $\Pi_{\mathcal{PP}}$ and $\Pi_{\mathcal{II}}$, respectively. We can, without loss of generality, assume that $M_{\mathcal{PP}}$ has the following additional property: $M_{\mathcal{PP}}(p) < M_{\mathcal{II}}(I^-)$ for all $p \in \mathrm{Var}(\Pi_{\mathcal{PP}})$ and $I \in \mathrm{Var}(\Pi_{\mathcal{II}})$. We construct a model M of Π as follows:

$$M(x) = \begin{cases} M_{\mathcal{PP}}(x) & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{PP}}) \\ M_{\mathcal{II}}(x) & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{II}}) \end{cases}$$

It follows that M is a model of Π since every constraint in $\Pi_{\mathcal{PI}}$ contains the relation $\mathtt{b}$. $\square$

Lemma 7 *$\mathcal{WV}_{\mathsf{d}}$ is tractable if $\mathcal{W} \in \mathcal{II}_{\rm tr} - \{\mathcal{H}, \mathcal{S}_{\mathsf{p}}, \mathcal{E}_{\mathsf{p}}\}$.*

Proof. Assume Π is a satisfiable instance of QA-SAT(X) where $X \in \mathcal{II}_{\rm tr} - \{\mathcal{H}, \mathcal{S}_{\mathsf{p}}, \mathcal{E}_{\mathsf{p}}\}$ By studying the correctness proofs of the algorithms for these subclasses [4, 5], one notices that Π always has a model M such that the intersection of all intervals is itself a non-empty interval, say J.

Thus, we can use a similar trick as in the proof of Lemma 6: instead of moving the points to a position before or after the intervals, we scale the points and move them to a position within the interval J. $\square$

For proving tractability of the remaining subclasses, we define the function $S : \mathcal{QA} \to \mathcal{II}$ such that

$$\begin{array}{ll} S(<) = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) & S(=) = (\equiv \mathsf{ss}^{-1}) \\ S(>) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{df}) & S(\mathtt{b}) = (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \\ S(\mathtt{s}) = (\equiv \mathsf{ss}^{-1}) & S(\mathtt{d}) = (\mathsf{o}^{-1}\mathsf{df}) \\ S(\mathtt{f}) = (\mathsf{m}^{-1}) & S(\mathtt{a}) = (\mathsf{p}^{-1}) \end{array}$$

and $S(r) = r$ if r is a basic II-relation. We extend S such that $S(r) = S(r_1) \cup \ldots \cup S(r_n)$ if $r = (r_1, \ldots, r_n)$, and given a set $X \subseteq \mathcal{QA}$, we define $S(X) = \{S(r) \mid r \in X\}$.

The idea is to transform instances of QA-SAT(X) into instances of QA-SAT$(X \cap \mathcal{II})$—this will avoid the need for constructing completely new algorithms.

Lemma 8 *Let* $\Pi = (V_p, V_I, C)$ *be an instance of* QA-SAT(X). *Let* $V_I' = V_I$ *and* $V_p' = \{I_p' \mid p \in V_p\}$ *(where we assume that* $V_I' \cap V_p' = \emptyset$*). Define an instance*

$$\Pi' = (\emptyset, V_I' \cup \{I_p' \mid p \in V_p\}, C')$$

of QA-SAT$(\mathcal{II})$ *where* $C' = \{I_p' S(r) I_q' \mid prq \in C_{\mathcal{PP}}\} \cup \{I_p' S(r) I' \mid prI \in C_{\mathcal{PI}}\} \cup \{I' S(r) J' \mid IrJ \in C_{\mathcal{II}}\}$.

Then, Π *is satisfiable iff* Π' *is satisfiable.*

Proof. *only-if:* Let M be a model of Π. Construct an interpretation M' of Π' as follows:

1. for each interval $I' \in V_I'$, let $M'(I') = M(I)$; and
2. for each interval $I_p' \in V_p'$, let $M'(I_p') = [M(p), M(p)+1]$.

It is straightforward to verify that M' is a model of Π'. As an example, assume that $p(\mathtt{bs})I \in C$, $M(p) = 1$ and $M(I) = [2,4]$. Then, $I_p'(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})I' \in C'$, $M'(I_p') = [1,2]$ and $M'(I') = [2,4]$; consequently, the relation between I_p' and I' is satisfied.

if: Let M' be a model of Π'. Construct an interpretation M of Π as follows:

1. for each point $p \in V_p$, let $M(p) = M'(I_p^-)$; and
2. for each interval $I \in V_I$, let $M(I) = M'(I')$.

Once again, it is straightforward to verify that M is a model of Π. We take the same example as before: Assume $I_p'(\equiv \mathsf{pmod}^{-1}\mathsf{ss}^{-1}\mathsf{f}^{-1})I' \in C'$, $M'(I_p') = [1,2]$ and $M'(I') = [2,4]$. Then, we know that $p(\mathtt{bs})I \in C$, $M(p) = 1$ and $M(I) = [2,4]$. □

As is evident in the proof, function S identifies the points with the left endpoint of intervals while the relations between the right endpoints are arbitrary; thus, we can symmetrically define a function E that identifies points with the right endpoint of intervals.

$E(<) = (\mathsf{pmods})$ $\quad E(=) = (\equiv \mathsf{ff}^{-1})$
$E(>) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1})$ $\quad E(\mathtt{b}) = (\mathsf{p})$
$E(\mathtt{s}) = (\mathsf{m})$ $\quad E(\mathtt{d}) = (\mathsf{ods})$
$E(\mathtt{f}) = (\equiv \mathsf{ff}^{-1})$
$E(\mathtt{a}) = (\mathsf{p}^{-1}\mathsf{m}^{-1}\mathsf{o}^{-1}\mathsf{d}^{-1}\mathsf{s}^{-1})$

Lemma 9 *Let* X *be one of the subclasses in Theorem 5 that is not covered by Lemmata 6 or 7. Then,* X *is tractable.*

Proof. Assume X' is a tractable subset of $\mathcal{II}$. If $S(X) \subseteq X'$ or $E(X) \subseteq X'$, then X is tractable by Lemma 8. It can be verified that either $S(X)$ or $E(X)$ is a subset of $X \cap \mathcal{II}$ and the lemma follows since $X \cap \mathcal{II}$ is tractable. □

3.2 Proof of Theorem 5

One of our main tools for proving the result is the notion of *derivations*. Suppose $X \subseteq \mathcal{QA}$ and Π is an instance of QA-SAT(X). Let the two variables x, y appear in Π. Furthermore, let $r \in \mathcal{QA}$ be the relation defined as follows: a basic relation r' is included in r if and only if the instance obtained from Π by adding the constraint $xr'y$ is satisfiable. In this case, we say that r is *derived* from X.

It should be noted that if the instance $\Pi_1 = \Pi \cup \{xr'y\}$ is satisfiable, then, for any two points or intervals i_1, j_1 such that $i_1 r' j_1$, there is a model M of Π such that $M(x) = i_1$ and $M(y) = j_1$. This can be established as follows: since Π_1 is satisfiable, it has a model M'. Denote $M'(x)$ by i_2 and $M'(y)$ by j_2; then $i_2 r' j_2$. There exists a continuous monotone injective mapping ϕ of the real line into itself such that ϕ takes i_2 to i_1 and j_2 to j_1. Obviously, ϕ maps intervals to intervals, and it does not change the relative order between points and intervals. Therefore, by combining ϕ and M' we obtain the required model M.

It can easily be checked that adding a derived relation r to X does not change the complexity of QA-SAT(X) because, in any instance, any constraint involving r can be replaced by the set of constraints in Π (introducing fresh variables when needed), and this can be done in polynomial time.

We will sometimes use a principle of *duality* for simplifying proofs. We make use of a function $\mathsf{reverse}$ which is defined on the basic relations of $\mathcal{QA}$ by the following table:

r	$<$	$=$	$>$				
$\mathsf{reverse}(r)$	$>$	$=$	$<$				
r	$\mathtt{b}$	$\mathtt{s}$	$\mathtt{d}$	$\mathtt{f}$	$\mathtt{a}$		
$\mathsf{reverse}(r)$	$\mathtt{a}$	$\mathtt{f}$	$\mathtt{d}$	$\mathtt{s}$	$\mathtt{b}$		
r	$\equiv$	p	p^{-1}	m	m^{-1}	o	o^{-1}
$\mathsf{reverse}(r)$	$\equiv$	p^{-1}	p	m^{-1}	m	o^{-1}	o
r	d	d^{-1}	s	s^{-1}	f	f^{-1}	
$\mathsf{reverse}(r)$	d	d^{-1}	f	f^{-1}	s	s^{-1}	

and is defined for all other elements in $\mathcal{QA}$ by setting $\mathsf{reverse}(R) = \bigcup_{r \in R} \mathsf{reverse}(r)$.

Let Π be any instance of QA-SAT, and let Π' be obtained from Π by replacing every relation r with $\mathsf{reverse}(r)$. It is easy to check that Π has a model M if and only if Π' has a model M' given by

$$M'(x) = \begin{cases} -M(x) & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{PP}}) \\ [-M(x)^+, -M(x)^-] & \text{if } x \in \mathrm{Var}(\Pi_{\mathcal{II}}) \end{cases}$$

In other words, M' is obtained from M by redirecting the real line and leaving all intervals (as geometric objects) in their places. This observation leads to the following lemma.

$$\begin{array}{ll}
\mathcal{S}_{\mathsf{p}} = \{r \mid r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\} & \mathcal{E}_{\mathsf{p}} = \{r \mid r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{p})^{\pm 1} \subseteq r\} \\
\mathcal{S}_{\mathsf{d}} = \{r \mid r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d}^{-1})^{\pm 1} \subseteq r\} & \mathcal{E}_{\mathsf{d}} = \{r \mid r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r\} \\
\mathcal{S}_{\mathsf{o}} = \{r \mid r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\} & \mathcal{E}_{\mathsf{o}} = \{r \mid r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r\} \\
\mathcal{A}_1 = \{r \mid r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s}^{-1})^{\pm 1} \subseteq r\} & \mathcal{B}_1 = \{r \mid r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\} \\
\mathcal{A}_2 = \{r \mid r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\} & \mathcal{B}_2 = \{r \mid r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f})^{\pm 1} \subseteq r\} \\
\mathcal{A}_3 = \{r \mid r \cap q_2^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\} & \mathcal{B}_3 = \{r \mid r \cap q_5^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\} \\
\mathcal{A}_4 = \{r \mid r \cap q_3^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r\} & \mathcal{B}_4 = \{r \mid r \cap q_6^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r\}
\end{array}$$

$$\mathcal{S}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap q_1^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{f}^{-1})^{\pm 1} \subseteq r, \\ 2)\ r \cap (\mathsf{ss}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{E}^* = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap q_4^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{s})^{\pm 1} \subseteq r, \\ 2)\ r \cap (\mathsf{ff}^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r \end{array} \right\}$$

$$\mathcal{A}_{\equiv} = \{r \mid r \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$$

$$\mathcal{H} = \left\{ r \;\middle|\; \begin{array}{l} 1)\ r \cap (\mathsf{os})^{\pm 1} \neq \emptyset\ \&\ r \cap (\mathsf{o}^{-1}\mathsf{f})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{d})^{\pm 1} \subseteq r, \text{ and} \\ 2)\ r \cap (\mathsf{ds})^{\pm 1} \neq \emptyset\ \&\ r \cap (\mathsf{d}^{-1}\mathsf{f}^{-1})^{\pm 1} \neq \emptyset \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r, \text{ and} \\ 3)\ r \cap (\mathsf{pm})^{\pm 1} \neq \emptyset\ \&\ r \not\subseteq (\mathsf{pm})^{\pm 1} \Rightarrow (\mathsf{o})^{\pm 1} \subseteq r \end{array} \right\}$$

Table 3. The tractable subalgebras of Allen's algebra.

Lemma 10 *Let $X = \{r_1, \ldots, r_n\} \subseteq \mathcal{QA}$ and $X' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{QA}$ be such that, for all $1 \leq k \leq n$, $r'_k = \mathsf{reverse}(r_k)$. Then X is tractable (NP-complete) if and only if X' is tractable (NP-complete).*

Lemma 10 implies that a proof of NP-completeness for, say, $\{(<), (\mathtt{bf}), (\mathsf{ods}^{-1})\}$, immediately yields a proof of NP-completeness for $\{(>), (\mathtt{sa}), (\mathsf{o}^{-1}\mathsf{df}^{-1})\}$.

The classification proof has four step. In each step, it is proved that if a subclass $\mathcal{S}$ satisfies a certain condition, then either $\mathcal{S}$ is NP-complete, contained in one of the tractable subclasses or $\mathcal{S}$ satisfies the conditions of some earlier step. Throughout the proof, we assume that $\mathcal{S}$ is closed under derivations and $(<) \in \mathcal{S}$. We say that a relation is *non-trivial* if it is not equal to the empty relation.

Step 1. We begin by proving that $\mathcal{S}$ is NP-complete unless $\mathcal{S}_{\mathcal{PI}}$ is a subset of $\mathcal{V}_{\mathcal{H}}, \mathcal{V}_{\mathcal{S}}$ or $\mathcal{V}_{\mathcal{E}}$.
Step 2. Assume now that $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \subseteq (\mathtt{fa})$ and $r_2 \subseteq (\mathtt{bs})$. This implies that $\mathcal{S}$ is NP-complete or $\mathcal{S}$ is included in one of $\mathcal{H}\mathcal{V}_{\mathcal{H}}$, $\mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}$ or $\mathcal{E}_{\mathsf{p}}\mathcal{V}_{\mathcal{E}}$.
Step 3. We note that if $(\mathtt{b}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$ or $(\mathtt{a}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$, then $\mathcal{S}$ is NP-complete or contained in one of the tractable subclasses. Thus, we assume the existence of $r_1, r_2 \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r_1$ and $(\mathtt{a}) \not\subseteq r_2$ and show that $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{SH}}$ or $\mathcal{V}_{\mathcal{EH}}$, or else the previous step applies.
Step 4. Finally, we show that if $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$ or $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{EH}}$, then either $\mathcal{S}$ is NP-complete or is contained in one of the tractable subclasses listed in Theorem 5.

Before the proof, we present a number of derivations that will be frequently used.

Lemma 11 *Assume $r \in \mathcal{S}$ is a non-trivial relation. Then,*

1. *if $(\mathtt{b}) \not\subseteq r$ and $r \cap (\mathtt{sd}) \neq \emptyset$, then $(\mathtt{dfa}) \in \mathcal{S}$;*
2. *if $(\mathtt{b}) \not\subseteq r$ and $r \cap (\mathtt{sd}) = \emptyset$, then $(\mathtt{a}) \in \mathcal{S}$;*
3. *if $(\mathtt{a}) \not\subseteq r$ and $r \cap (\mathtt{df}) \neq \emptyset$, then $(\mathtt{bsd}) \in \mathcal{S}$;*
4. *if $(\mathtt{a}) \not\subseteq r$ and $r \cap (\mathtt{df}) = \emptyset$, then $(\mathtt{b}) \in \mathcal{S}$;*

Proof. The cases are similar so we only consider the first one: the relation $p(\mathtt{dfa})I$ is derived from $\{qrI, p > q\}$. □

Lemma 12 *$\mathcal{S}$ is NP-complete or $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$.*

Proof. Suppose that $\mathcal{S}_{\mathcal{PI}}$ is not NP-complete. By Theorem 3, it is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$, $\mathcal{V}_{\mathtt{s}}$, $\mathcal{V}_{\mathtt{f}}$. Assume that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{s}}$. If $(\mathtt{b}) \subseteq r$ for every non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$. Suppose there is a non-trivial $r \in \mathcal{S}_{\mathcal{PI}}$ such that $(\mathtt{b}) \not\subseteq r$. Then $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{a}), (\mathtt{dfa})\} \neq \emptyset$ by Lemma 11, a contradiction. The argument is dual when $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathtt{f}}$. □

From now on we will assume that $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$, $\mathcal{V}_{\mathcal{E}}$. We define the relation as $r_{\mathtt{d}} = \bigcap\{r \in \mathcal{S}_{\mathcal{PI}} \mid (\mathtt{d}) \subseteq r\}$ and note that $r_{\mathtt{d}} \in \mathcal{S}_{\mathcal{PI}}$ since it is derived from the relations in $\mathcal{S}_{\mathcal{PI}}$.

Lemma 13 *Suppose that $\mathcal{S}_{\mathcal{PI}}$ contains two non-trivial relations r_1, r_2 such that $r_1 \subseteq (af)$ and $r_2 \subseteq (bs)$. Then either $\mathcal{S}$ is NP-complete or is contained in one of $\mathcal{H}\mathcal{V}_{\mathcal{H}}$, $\mathcal{S}\mathsf{p}\mathcal{V}_{\mathcal{S}}$ or $\mathcal{E}\mathsf{p}\mathcal{V}_{\mathcal{E}}$.*

Proof. First note that $\{(\mathtt{a}), (\mathtt{b})\} \subseteq \mathcal{S}_{\mathcal{PI}}$ by Lemma 11. Now, $I(\mathsf{p})J$ is derived from $\{p(\mathtt{a})I, p(\mathtt{b})J\}$. It follows from Theorem 4 that either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or it is contained in one of $\mathcal{H}$, $\mathcal{S}\mathsf{p}$, $\mathcal{E}\mathsf{p}$.

Suppose first that we have $(\mathtt{d}) \subseteq r_{\mathtt{d}} \subseteq (\mathtt{dsf})$. By using Lemma 12, we conclude that either $\mathcal{S}_{\mathcal{PI}}$ is NP-complete or $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$. Furthermore, $I(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1})J$ is derived from $\{pr_{\mathtt{d}}I, pr_{\mathtt{d}}J\}$. Therefore we have $(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ which now implies that either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{H}$. We conclude that either $\mathcal{S}$ is NP-complete or $\mathcal{S} \subseteq \mathcal{H}\mathcal{V}_{\mathcal{H}}$.

We can now assume that $r_{\mathtt{d}}$ contains $(\mathtt{a})$ or $(\mathtt{b})$ (or both). Suppose we have $(\mathtt{a}) \subseteq r_{\mathtt{d}}$; the second case is dual. It follows that, for every $r \in \mathcal{S}_{\mathcal{PI}}$, $(\mathtt{d}) \subseteq r$ implies $(\mathtt{a}) \subseteq r$. If there exists $r' \in \mathcal{S}_{\mathcal{PI}}$ such that $r' \cap (\mathtt{fa}) = (\mathtt{f})$ then $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{bsd})\} \neq \emptyset$ by Lemma 11 which contradicts the assumption just made. It can now be checked that $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$ and we complete the proof by considering two cases.

<u>Case 1.</u> $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{E}}$.
If $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}\mathsf{p}$ or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{E}\mathsf{p}$ then we get the required result. Otherwise there exist $r_3, r_4 \in \mathcal{S}_{\mathcal{II}}$ such that $r_3 \notin \mathcal{S}\mathsf{p}$ and $r_4 \notin \mathcal{E}\mathsf{p}$, that is, $r_3 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_3$, and $r_4 \cap (\mathsf{pmods}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_3$. Now one can check that the constraint $p(\mathtt{d})y$ is derived from $\{Ir_4J, Jr_3K, p(\mathtt{a})I, p(\mathtt{b})K\}$. Indeed, suppose these constraints are satisfied. Then $p(\mathtt{a})I$, $p(\mathtt{b})K$ imply $I^+ < p < K^-$. Since $(\mathsf{p}) \not\subseteq r_4$ and $(\mathsf{p}) \not\subseteq r_3$, we have $J^- \leq I^+$ and $K^- \leq J^+$. It follows that $J^- < p < J^+$, that is $p(\mathtt{d})J$. On the other hand, if $p(\mathtt{d})J$ then, for any choice of $r_3 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1})$ and $r_4 \cap (\mathsf{pmods})$, it is easy to find intervals I and K such that the constraints $\{Ir_4J, Jr_3K, p(\mathtt{a})I, p(\mathtt{b})K\}$ are satisfied. This contradicts the fact that $r_{\mathtt{d}}$ contains $\mathtt{a}$ and/or $\mathtt{b}$.

<u>Case 2.</u> $\mathcal{S}_{\mathcal{PI}} \not\subseteq \mathcal{V}_{\mathcal{E}}$.
It is easy to check that $\mathcal{S}_{\mathcal{PI}}$ contains $r_5 \in \{(\mathtt{sa}), (\mathtt{da}), (\mathtt{sda}), (\mathtt{sfa}), (\mathtt{dfa}), (\mathtt{sdfa})\}$. Then, $p(\mathtt{dfa})I \in \mathcal{S}$ by Lemma 11, and we have $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ because $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{dfa})I, p(\mathtt{b})J\}$. In particular, we obtain that $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{H}$ or $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}\mathsf{p}$. If $\mathcal{S}_{\mathcal{II}} \subseteq \mathcal{S}\mathsf{p}$ then $\mathcal{S} \subseteq \mathcal{S}\mathsf{p}\mathcal{V}_{\mathcal{S}}$. Otherwise there is a relation $r_6 \in \mathcal{S}_{\mathcal{II}}$ such that $r_6 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \neq \emptyset$ but $(\mathsf{p}) \not\subseteq r_6$. If $r_6 \cap (\mathsf{mo}) \neq \emptyset$, then $p(\mathtt{d})J$ is derived from $\{Ir_6J, Jr_6K, p(\mathtt{a})I, p(\mathtt{b})K\}$ and we have a contradiction. Otherwise we get $r_7 = r_6 \cap (\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \subseteq (\mathsf{d}^{-1}\mathsf{f}^{-1})$. Note that $r_7 \in \mathcal{S}_{\mathcal{II}}$. Now one can check that the constraint $p(\mathtt{d})I$ is derived from $\{Ir_7J, p(\mathtt{dfa})I, p(\mathtt{b})J\}$ which leads to a contradiction. □

Assume that $(\mathtt{b}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$ or $(\mathtt{a}) \subseteq r$ for all $r \in \mathcal{S}_{\mathcal{PI}}$. By using Lemma 6, we see that either $\mathcal{S}$ is NP-complete (if $\mathcal{S}_{\mathcal{II}}$ is NP-complete) or contained in one of the tractable subclasses $\mathcal{W}\mathcal{V}_{\mathsf{a}}$ or $\mathcal{W}\mathcal{V}_{\mathsf{b}}$ where $\mathcal{W} \in \mathcal{II}_{\mathrm{tr}}$.

Lemma 14 *Suppose there exist $r_1, r_2 \in \mathcal{S}_{\mathcal{PI}}$ such that $(b) \not\subseteq r_1$ and $(a) \not\subseteq r_2$. Then, $\mathcal{S}$ is NP-complete, $\mathcal{S}_{\mathcal{PI}}$ is contained in one of $\mathcal{V}_{\mathcal{SH}}, \mathcal{V}_{\mathcal{EH}}$, or Lemma 13 applies.*

Proof. $\mathcal{S}$ is NP-complete if $\mathcal{S}_{\mathcal{PI}}$ is not a subset of $\mathcal{V}_{\mathcal{H}}$, $\mathcal{V}_{\mathcal{S}}$ or $\mathcal{V}_{\mathcal{E}}$ by Lemma 12. Thus, we consider three cases depending on which of these sets $\mathcal{S}_{\mathcal{PI}}$ is included in. The claim obviously holds if $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{H}}$ by the definitions of $\mathcal{V}_{\mathcal{SH}}$ and $\mathcal{V}_{\mathcal{EH}}$. Suppose $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$; then $r_2 \subseteq (\mathtt{bs})$. If r_1 can be chosen so that $r_1 \subseteq (\mathtt{sfa})$ and $r_1 \neq (\mathtt{s})$, then we can apply Lemma 13 with r_1 if $(\mathtt{s}) \not\subseteq r_1$ and with $r_1 \cap (\mathtt{dfa})$ otherwise (since $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11). If there is no such r_1 then $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{EH}}$. For $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{E}}$ the argument is dual. □

By duality, it is sufficient to consider $\mathcal{S}_{\mathcal{PI}}$ with $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$.

Lemma 15 *If $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{SH}}$ then either $\mathcal{S}$ is NP-complete or is contained in one of the tractable subclasses listed in Theorem 5.*

Proof. We consider three different cases depending on the value of $r_{\mathtt{d}} \cap (\mathtt{ba})$.

<u>Case 1.</u> $r_{\mathtt{d}} \cap (\mathtt{ba}) \in \{(\mathtt{b}), (\mathtt{ba})\}$ (i.e. $(\mathtt{b}) \subseteq r_{\mathtt{d}}$).
In this case we have $(\mathtt{s}) \notin \mathcal{S}_{\mathcal{PI}}$, since otherwise $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11 and $r_{\mathtt{d}} \subseteq (\mathtt{dfa})$. Thus $(\mathtt{b})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result by Lemma 6.

<u>Case 2.</u> $r_{\mathtt{d}} \cap (\mathtt{ba}) = (\mathtt{a})$.
Note that in this case we also have $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}}$ so $\mathcal{S}_{\mathcal{PI}} \subseteq \mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{SH}}$. We have $(\mathtt{dfa}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11 since $(\mathtt{d}) \subseteq r_{\mathtt{d}} \in \mathcal{S}_{\mathcal{PI}}$. If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} = \emptyset$ then $(\mathtt{a})$ is contained in every non-trivial relation from $\mathcal{S}_{\mathcal{PI}}$, and we get the required result by Lemma 6. Otherwise we have $(\mathtt{b}) \in \mathcal{S}_{\mathcal{PI}}$ (repeating the argument from the beginning of Lemma 13).

Then $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{dfa})I, p(\mathtt{b})J\}$. If $(\mathsf{pmod}^{-1}\mathsf{f}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ then, as follows from Theorem 4, either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or it is contained in one of $\mathcal{H}$, $\mathcal{S}_{\mathsf{p}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}^*$. Thus, if $\mathcal{S}_{\mathcal{II}}$ is not NP-complete then $\mathcal{S}$ is contained in one of the known tractable subclasses $\mathcal{HV}_{\mathcal{H}}$ (since $\mathcal{V}_{\mathcal{SH}} \subseteq \mathcal{V}_{\mathcal{H}}$), $\mathcal{S}_{\mathsf{p}}\mathcal{V}_{\mathcal{S}}, \mathcal{S}_{\mathsf{o}}\mathcal{V}_{\mathcal{SH}}, \mathcal{S}_{\mathsf{d}}\mathcal{V}_{\mathcal{SH}}, \mathcal{S}^*\mathcal{V}_{\mathcal{SH}}$.

Case 3. $r_{\mathsf{d}} \cap (\mathtt{ba}) = \emptyset$.
Since $p(\mathtt{d})I$ is derived from $\{q_1 r_{\mathsf{d}} I, q_2 r_{\mathsf{d}} I, q_1 < p < q_2\}$, it follows that $r_{\mathsf{d}} = (\mathtt{d})$. We have $(\equiv \mathsf{oo}^{-1}\mathsf{dd}^{-1}\mathsf{ss}^{-1}\mathsf{ff}^{-1}) \in \mathcal{S}_{\mathcal{II}}$ because this relation is derived from $\{p(\mathtt{d})I, p(\mathtt{d})J\}$. In particular, either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or is contained in some maximal tractable subclass of $\mathcal{A}$ other than $\mathcal{S}_{\mathsf{p}}$ and $\mathcal{E}_{\mathsf{p}}$.

If $\mathcal{S}_{\mathcal{PI}} \cap \{(\mathtt{b}), (\mathtt{s}), (\mathtt{bs})\} \neq \emptyset$ then $(\mathtt{b}) \in \mathcal{S}_{\mathcal{PI}}$ by Lemma 11, and $I(\mathsf{pmod}^{-1}\mathsf{f}^{-1})J$ is derived from $\{p(\mathtt{d})I, p(\mathtt{b})J\}$. Therefore either $\mathcal{S}_{\mathcal{II}}$ is NP-complete or contained in one of $\mathcal{H}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}^*$. Thus, if $\mathcal{S}_{\mathcal{II}}$ is not NP-complete then $\mathcal{S}$ is contained in one of the tractable subclasses $\mathcal{HV}_{\mathcal{H}}, \mathcal{S}_{\mathsf{o}}\mathcal{V}_{\mathcal{SH}}, \mathcal{S}_{\mathsf{d}}\mathcal{V}_{\mathcal{SH}}, \mathcal{S}^*\mathcal{V}_{\mathcal{SH}}$.

Otherwise, every non-trivial relation in $\mathcal{S}_{\mathcal{PI}}$ contains $(\mathtt{d})$. If $\mathcal{S}_{\mathcal{II}}$ is included in some tractable subclass except $\mathcal{H}$, the result follows immediately from Lemma 7. If that is not the case, then $\mathcal{S} \subseteq \mathcal{HV}_{\mathcal{H}}$. $\square$

4 Conclusion

We have studied the computational complexity of a temporal formalism that combines the point algebra, the point-interval algebra and Allen's interval algebra. By assuming that $(<)$ is always included in the set of relations under consideration, we have identified all maximal tractable fragments of this formalisms.

There is an obvious two-step continuation of this work. In the first step, the additional requirement on point-point relations must be removed so a complete classification of the formalism is obtained. In the second step, metric constraints should be added to the formalism—thus yielding a classification of the extended QA suggested by Meiri [13]. The first step can probably be carried out using methods similar to those found in [10]. The second step is more of a challenge even though there has been some progress in the study of metric temporal formalisms, *cf.* [11].

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