Refinements and Independence: A Simple Method for Identifying Tractable Disjunctive Constraints

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Abstract. The constraint satisfaction problem provides a natural framework for expressing many combinatorial problems. Since the general problem is NP-hard, an important question is how to restrict the problem to ensure tractability. The concept of independence has proven to be a useful method for constructing tractable constraint classes from existing classes. Since checking the independence property may be a difficult task, we provide a simple method for checking this property. Our method builds on a somewhat surprising connection between independence and refinements which is a recently established way of reducing one constraint satisfaction problem to another. Refinements have two interesting properties: (1) they preserve consistency; and (2) their correctness can be easily checked by a computer-assisted analysis. We show that all previous independence results of the point algebra for totally ordered and partially ordered time can be derived using this method. We also employ the method for deriving new tractable classes.

1 Introduction

The constraint satisfaction problem provides a framework for expressing combinatorial problems in computer science and elsewhere. The basic computational problem is NP-hard [12] so an important question is how to restrict the problem to ensure tractability. This research has mainly followed two different paths: restricting the scope of the constraints [8, 7], *i.e.*, which variables may be constrained with other variables, or restricting the constraints [6, 10, 17], *i.e.*, the allowed values for mutually constrained variables. In this paper, we will only consider problems where the constraints are restricted.

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As we already have noted, quite a large number of tractable subclasses of the CSP problem has been identified in the literature. Thus, it is of considerable interest to investigate how tractable constraint types may be combined in order to yield more general problems which are still tractable. Cohen *et al.* [5] have studied so-called "disjunctive constraints", *i.e.*, constraints which are disjunctions of constraints of specified types. They identified a certain property, *independence*, which allows for new tractable constraint classes to be constructed from existing classes. Several important classes of tractable constraints can be obtained by their method such as the Horn fragment of propositional logic, the ORD-Horn fragment [13] of Allen's Interval Algebra, and the class of max-closed constraints [10].

It is hardly surprising that deciding the independence property may be a highly non-trivial task in many cases. The main goal of this paper is to present a simple method for checking the independence property. Our method builds on a connection between the independence property and *refinements* [15]. Loosely speaking, a refinement is a way of reducing one CSP problem to another and it has the property that if the second problem can be decided by path-consistency, then path-consistency decides the first problem, too. Refinements were successful in proving tractability of large subsets of the Region Connection Calculus as well as Allen's Interval Algebra [15]. One important aspect of refinements is that their correctness can be easily checked by a computer-assisted analysis which implies that the independence property can be automatically checked in many cases.

Using our method, we show that all previous independence results on the time point algebra for partially ordered time [4, 3, 1] and the point algebra for linear time [4, 5] can be derived using refinements and that this is sufficient to identify *all* tractable sets of disjunctions of relations for the partially ordered time-point algebra as well as for the point algebra for linear time. We also use this method for deriving new tractable subclasses of the Region Connection Calculus [14].

The paper is organized as follows: In Section 2 we introduce the basic concepts that are needed in the rest of the paper. In Section 3 we relate refinements and independence and prove the main result that refinements imply independence. In Section 4 we apply this result to various tractable sets of relations and derive independent relations which form large tractable sets of disjunctions of relations. Finally, the last section contains some concluding remarks.

2 Preliminaries

2.1 CSPs, Disjunctions, and Independence

Let $\mathcal{A}$ be a finite set of jointly exhaustive and pairwise disjoint binary relations, also called *basic* relations. We denote the standard operations composition, intersection and converse by $\circ$, $\cap$ and $^{-1}$, respectively. Furthermore, we define the unary operation $\neg$ such that $\neg\mathcal{S} = \mathcal{A}\backslash\mathcal{S}$ for all $\mathcal{S} \subseteq \mathcal{A}$.

The consistency problem $\mathsf{CSPSAT}(\mathcal{S})$ for sets $\mathcal{S} \subseteq 2^{\mathcal{A}}$ over a domain $\mathcal{D}$ is defined as follows [16]:

Instance: A set V of variables over a domain D and a finite set Θ of binary constraints xRy, where $R \in \mathcal{S}$ and $x, y \in V$.
Question: Is there an instantiation of all variables in Θ such that all constraints are satisfied?

Naturally, a set of basic relations is to be interpreted as a disjunction of its member relations. Given an instance Θ of $\mathsf{CSPSAT}(R)$, let $\mathrm{Mods}(\Theta)$ denote the class of models of Θ (*i.e.* the satisfying instantiations) and $\mathrm{Vars}(\Theta)$ the variables appearing in Θ.

Next, we introduce operators for combining relations.

Definition 1 Let R_1, R_2 be relations of arity i, j and define the disjunction $R_1 \vee R_2$ of arity $i + j$ as follows:

$$R_1 \vee R_2 = \{(x_1, \dots, x_{i+j}) \in D^{i+j} | (x_1, \dots, x_i) \in R_1 \vee \\ (x_{i+1}, \dots, x_{i+j}) \in R_2\}$$

Thus, the disjunction of two relations with arity i, j is the relation with arity $i+j$ satisfying either of the two relations. Note that the CSPSAT problem trivially can be extended to handle disjunctive and non-binary constraints.

To give a concrete example, let $D = \{0, 1\}$ and let the relations $\mathsf{And} = \{\langle 1, 1\rangle\}$ and $\mathsf{Xor} = \{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ be given. The disjunction of And and Xor is given by:

$$\mathsf{And} \vee \mathsf{Xor} = \left\{ \begin{array}{l} \langle 0,0,0,1\rangle, \langle 0,1,0,1\rangle, \langle 1,0,0,1\rangle, \langle 1,1,0,1\rangle, \\ \langle 0,0,1,0\rangle, \langle 0,1,1,0\rangle, \langle 1,0,1,0\rangle, \langle 1,1,1,0\rangle, \\ \langle 1,1,0,0\rangle, \langle 1,1,0,1\rangle, \langle 1,1,1,0\rangle, \langle 1,1,1,1\rangle \end{array} \right\}$$

We see that the constraint x And $y \vee x$ Xor z is satisfiable when x, y and z has, for instance, been instantiated to $1, 0, 0$, respectively.

The definition of disjunction can easily be extended to sets of relations.

Definition 2 Let Γ_1, Γ_2 be sets of relations and define the disjunction $\Gamma_1 \overset{\times}{\vee} \Gamma_2$ as follows:

$$\Gamma_1 \overset{\times}{\vee} \Gamma_2 = \Gamma_1 \cup \Gamma_2 \cup \{R_1 \vee R_2 \mid R_1 \in \Gamma_1, R_2 \in \Gamma_2\}$$

The disjunction of two sets of relation $\Gamma_1 \overset{\times}{\vee} \Gamma_2$ is the set of disjunctions of each pair of relations in Γ_1, Γ_2 plus the sets Γ_1, Γ_2. It is sensible to include Γ_1 and Γ_2 since one wants to have the choice of using the disjunction or not. In many cases we shall be concerned with constraints that are specified by disjunctions of an arbitrary number of relations. Thus, we make the following definition: for any set of relations, Δ, define $\Delta^* = \bigcup_{i=0}^{\infty} \Delta^{\vee i}$ where $\Delta^{\vee 0} = \{\bot\}$ and $\Delta^{\vee i+1} = \Delta^{\vee i} \overset{\times}{\vee} \Delta$.

We continue by defining the independence property.

Definition 3 For any sets of relations Γ and Δ, we say that Δ is independent with respect to Γ if for any set of constraints C in $\mathsf{CSPSAT}(\Gamma \cup \Delta)$, C has a solution whenever every $C' \subseteq C$, which contains at most one constraint whose constraint relation belongs to Δ, has a solution.

Theorem 4 For any sets of relations Γ and Δ, if $\mathsf{CSPSAT}(\Gamma \cup \Delta)$ is tractable and Δ is independent with respect to Γ, then $\mathsf{CSPSAT}(\Gamma \overset{\times}{\vee} \Delta^*)$ is tractable.

The notion of independence can alternatively (but equivalently) be defined as follows: Let $C = \{c_1, \ldots, c_k\}$ and $D = \{d_1, \ldots, d_n\}$ be arbitrary finite sets of constraints over Γ and Δ, respectively. Then, Δ is independent of Γ iff for every possible choice of C and D, the following holds: if $C \cup \{d_i\}$, $1 \leq i \leq n$, is satisfiable, then $C \cup D$ is satisfiable.

2.2 Refinements

We review the basics of *refinements* in this subsection. For proofs and additional results, see Renz [15]. A *refinement* of a constraint xRy is a constraint $xR'y$ such that $R' \subseteq R$. A refinement of a set of constraints Θ is a set of constraints Θ' such that every constraint of Θ' is a refinement of a constraint of Θ. We assume that a set of constraints Θ contains n ordered variables $x_1, \ldots, x_n$. The following definition is central.

Definition 5 Let $\mathcal{S}, \mathcal{T} \subseteq 2^{\mathcal{A}}$. $\mathcal{S}$ can be *reduced by refinement* to $\mathcal{T}$, if for every relation $S \in \mathcal{S}$ there is a relation $T_S \in \mathcal{T}$ with $T_S \subseteq S$ and every path-consistent set Θ of constraints over $\mathcal{S}$ can be refined to a set Θ' by replacing $x_i S x_j \in \Theta$ with $x_i T_S x_j \in \Theta'$ for $i < j$, such that enforcing path-consistency to Θ' does not result in an inconsistency.

Lemma 6 If path-consistency decides $\mathsf{CSPSAT}(\mathcal{T})$ for a set $\mathcal{T} \subseteq 2^{\mathcal{A}}$, and $\mathcal{S}$ can be reduced by refinement to $\mathcal{T}$, then path-consistency decides $\mathsf{CSPSAT}(\mathcal{S})$.

In order to handle different refinements, we introduce a *refinement matrix* that contains for every relation $S \in \mathcal{S}$ all specified refinements.

Definition 7 A *refinement matrix* M of $\mathcal{S}$ has $|\mathcal{S}| \times 2^{|\mathcal{A}|}$ Boolean entries such that for $S \in \mathcal{S}$, $R \in 2^{\mathcal{A}}$, $M[S][R] = \mathit{true}$ only if $R \subseteq S$.

M is called the *basic refinement matrix* if $M[S][R] = \mathit{true}$ if and only if $S = R$.

The algorithm Check-Refinements (see Figure 1) takes as input a set of relations $\mathcal{S}$ and a refinement matrix M of $\mathcal{S}$ and either succeeds or fails. A similar algorithm, Get-Refinements, returns the revised refinement matrix if Check-Refinements returns `succeed` and the basic refinement matrix if Check-Refinements returns `fail`. Since $\mathcal{A}$ is a finite set of relations, M can be changed only a finite number of times, so both algorithms always terminate.

If Check-Refinements returns `succeed` and Get-Refinements returns M', we have pre-computed all possible refinements of every path-consistent triple of variables as given in the refinement matrix M'. Thus, applying these refinements to a path-consistent set of constraints can never result in an inconsistency when enforcing path-consistency.

Theorem 8 Let $\mathcal{S}, \mathcal{T} \subseteq 2^{\mathcal{A}}$, and let M be a refinement matrix of $\mathcal{S}$. Get-Refinements$(\mathcal{S}, M)$ returns the refinement matrix M'. If for every $S \in \mathcal{S}$ there is a $T_S \in \mathcal{T}$ with $M'[S][T_S] = \mathit{true}$, then $\mathcal{S}$ can be reduced by refinement to $\mathcal{T}$.

Algorithm: CHECK-REFINEMENTS
Input: A set $\mathcal{S}$ and a refinement matrix M of $\mathcal{S}$.
Output: `fail` if the refinements specified in M can make a path-consistent triple of constraints over $\mathcal{S}$ inconsistent; `succeed` otherwise.

1. changes $\leftarrow$ true
2. *while* changes *do*
3. $oldM \leftarrow M$
4.   *for every* path-consistent triple $T = (R_{12}, R_{23}, R_{13})$ of relations over $\mathcal{S}$ *do*
5.   *for every* refinement $T' = (R'_{12}, R'_{23}, R'_{13})$ of T with $oldM[R_{12}][R'_{12}] = oldM[R_{23}][R'_{23}] = oldM[R_{13}][R'_{13}]$ =true *do*
6. $T'' \leftarrow$ PATH-CONSISTENCY(T')
7.   *if* $T'' = (R''_{12}, R''_{23}, R''_{13})$ contains the empty relation *then return* `fail`
8.   *else do* $M[R_{12}][R''_{12}] \leftarrow$ true, $M[R_{23}][R''_{23}] \leftarrow$ true, $M[R_{13}][R''_{13}] \leftarrow$ true
9.   *if* $M = oldM$ *then* changes $\leftarrow$ false
10. *return* `succeed`

Fig. 1. Algorithm CHECK-REFINEMENTS

Now, the procedures CHECK-REFINEMENTS and GET-REFINEMENTS can be used to prove tractability for sets of relations.

Theorem 9 Let $\mathcal{S}, \mathcal{T} \subseteq 2^{\mathcal{A}}$ be two sets such that path-consistency decides $\mathsf{CSPSAT}(\mathcal{T})$, and let M be a refinement matrix of $\mathcal{S}$. GET-REFINEMENTS$(\mathcal{S}, M)$ returns M'. If for every $S \in \mathcal{S}$ there is a $T_S \in \mathcal{T}$ with $M'[S][T_S] = true$, then path-consistency decides $\mathsf{CSPSAT}(\mathcal{S})$.

Given this theorem, what is needed for proving a set $\mathcal{S}$ to be tractable is a set $\mathcal{T}$ for which path-consistency is known to decide consistency and a refinement matrix M. Although it might be difficult to find a suitable refinement matrix, the simple heuristic of eliminating all identity relations from disjunctive relations led to a suitable refinement matrix for many interesting sets of relations (cf. [15]). For the scope of this paper we are interested in a particular type of refinement matrices which we define as follows:

Definition 10 Let $R \in \mathcal{A}$. M^R is the *R-refinement matrix* of a set $S \subseteq 2^{\mathcal{A}}$ if for every $S \in \mathcal{S}$, $M^R[S][S'] = true$ iff $S' = S \cap R$ and $S' \neq \emptyset$ or $S' = S$.

Definition 11 Let $\mathcal{S} \subseteq 2^{\mathcal{A}}$ such that path-consistency decides $\mathsf{CSPSAT}(\mathcal{S})$ and $R \in 2^{\mathcal{A}}$. We say that R is a *refinement* of $\mathcal{S}$ if CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed`.

Since the refinement matrix we are interested in, namely M^R for a particular relation R is given, we do not face the difficulty of the refinement method of finding a suitable refinement matrix.

3 Relating Refinements and Independence

The independence property has been proven for many different relations [5, 4], but there is no general proof schema for proving this property, so it is usually a matter of luck or intuition if a proof of independence can be found. In contrast to this, it is possible to verify refinements automatically [15] by merely running the algorithm given in Figure 1. It would, hence, be a large improvement if the same could be done for proving independence. In this section, we study the relationship between the notion of refinements and independence. It turns out that the two notions are very similar and that the algorithm for verifying refinements can also be used for proving independence.

When looking at the definitions of refinements and independence one notes that refinements eliminate labels from given constraints without changing consistency while by the independence property it is possible to add additional constraints without changing consistency. Eliminating a label R from a given constraint xTy, however, is equivalent to adding the constraint $x\neg Ry$. The correspondence between the two notions is formulated in the following theorem.

Theorem 12 Given a set of relations $\mathcal{S} \subseteq 2^{\mathcal{A}}$ for which path-consistency decides consistency and a refinement matrix M^R. If CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed`, then R is independent of S.

Proof. Given a path-consistent set Θ of constraints over $\mathcal{S}$. Θ' is obtained from Θ by refining all constraints $x_iT_iy_i \in \Theta$ with $T_i \not\subseteq \neg R$ to $x_iT_i \cap Ry_i$. Since CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed`, all these refinements can be made without making Θ' inconsistent. Instead of refining a constraint $x_iT_iy_i$ to $x_iT_i \cap Ry_i$ it is equivalent to add the constraint $h_i \equiv x_iRy_i$ to Θ. Unless the constraint x_iSy_i with $S \subseteq \neg R$ is contained in Θ, $\Theta \cup \{h_i\}$ is consistent. Thus, $\Theta \cup H$ ($H = \{h_1, \ldots, h_n\}$) is consistent if and only if $\Theta \cup \{h_i\}$ is consistent for all i, and, therefore, R is independent of $\mathcal{S}$. $\square$

This theorem gives us the possibility to prove independence of a relation R with respect to a set $\mathcal{S}$ automatically by simply running CHECK-REFINEMENTS $(\mathcal{S}, M^R)$. If the algorithm returns `succeed`, we know that R is independent of $\mathcal{S}$. In order to make use of a negative answer of the algorithm, we also have to prove the opposite direction, *i.e.*, independence of a relation R with respect to a set $\mathcal{S}$ implies that CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed`. Proving this is equivalent to saying that $\Theta \cup H$ is consistent if and only if $\Theta \cup \{h_i\}$ is consistent for all i implies that $\Theta \cup \{h_i\}$ is always consistent for all i unless $\neg h_i \in \Theta$. Although this is a highly desirable property, we have not been able to prove this nor did we find a counterexample. There are, however, many examples for which this conjecture holds. As we will see in Section 4, this includes all independence results for the point algebra for partially ordered time given by Broxvall and Jonsson [4] as well as those given for the point algebra for linear time. We give a proof of a slightly limited version of this conjecture.

Definition 13 Let $\mathcal{S} \subseteq 2^{\mathcal{A}}$ and $R \in \mathcal{S}$. We say that *path-consistency makes R explicit* iff for every path-consistent instance Θ of $\mathsf{CSPSAT}(\mathcal{S})$, the following holds: if $M(x)RM(y)$ for every $M \in \text{Mods}(\Theta)$, then $xSy \in \Theta$ and $S \subseteq R$.

Theorem 14 Let $\mathcal{S} \subseteq 2^{\mathcal{A}}$ and assume that $R \in \mathcal{S}$ is independent of $\mathcal{S}$. Then, CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed` if and only if path-consistency makes $\neg R$ explicit.

Proof. only-if: Assume to the contrary that there exists a path-consistent instance Θ of $\mathsf{CSPSAT}(\mathcal{S})$ and there exists $x, y \in \text{Vars}(\Theta)$ such that for all $M \in \text{Mods}(\Theta)$, $M(x)\neg RM(y)$ but $xSy \in \Theta$ and $S \cap R \neq \emptyset$. Since CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed`, the instance

$$\Theta' = \Theta \cup \{uRv \mid uTv \in \Theta \text{ and } T \cap R \neq \emptyset\}$$

is consistent. However, $S \cap R \neq \emptyset$ so $xRy \in \Theta'$. We know that all models M of Θ have the property $M(x)\neg RM(y)$ so every model M' of Θ' must also have this property. This contradicts the fact that Θ' has a model and, consequently, $S \cap R = \emptyset$ and $S \subseteq \neg R$. We have thus shown that path-consistency makes $\neg R$ explicit.

if: Let Θ be a path-consistent instance of $\mathsf{CSPSAT}(\mathcal{S})$ and arbitrarily choose a constraint $xSy \in \Theta$ such that $S \cap R \neq \emptyset$. The fact that path-consistency makes $\neg R$ explicit gives that $\Theta \cup \{xRy\}$ is consistent and, by independence, $\Theta' = \Theta \cup \{uRv \mid uTv \in \Theta \text{ and } T \cap R \neq \emptyset\}$ is consistent. However, Θ' is equivalent to Θ refined by the matrix M^R so CHECK-REFINEMENTS$(\mathcal{S}, M^R)$ returns `succeed` by Theorem 8 □

Corollary 15 Given a set of relations $\mathcal{S} \subseteq 2^{\mathcal{A}}$ for which path-consistency computes minimal labels and a refinement matrix M^R. Then, CHECK-REFINEMENTS-$(\mathcal{S}, M^R)$ returns `succeed` if and only if R is independent of S.

Proof. Simply note that if path-consistency computes minimal labels, then it makes $\neg R$ explicit. □

Examples of when path-consistency computes minimal labels can, for instance, be found in Bessière *et al.* [2].

4 Applications

We will now demonstrate that many known independence results can be obtained using refinements. We will also employ the method on constraint satisfaction problems where no independence results has yet been derived. In the following, we will not discuss the empty relation and the top relation (*i.e.* the relation containing all basic relations) since they are always independent of any set of relations.

4.1 The Region Connection Calculus

A well-known framework for qualitative spatial reasoning is the so-called Region Connection Calculus (RCC) [14] which models topological relations between spatial regions using first-order logic. Of particular interest is the RCC-8 calculus which is based on eight basic relations definable in the RCC theory. The eight basic relations are denoted as DC, EC, PO, EQ, TPP, NTPP, TPP^{-1}, and NTPP^{-1}, with the meaning of *DisConnected, Externally Connected, Partial Overlap, EQual, Tangential Proper Part, Non-Tangential Proper Part*, and their converses. RCC-5 is a subclass of RCC-8 where the boundary of spatial regions is not taken into account. Hence, it is not distinguished between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 relations DR for *DiscRete* and PP for *Proper Part*, respectively. Thus, RCC-5 contains the five basic relations DR, PO, PP, PP^{-1} and EQ. The consistency problem of both RCC-8 and RCC-5 is NP-complete [16], but large maximal tractable subsets have been identified [16, 11, 15]. In the following we demonstrate the usefulness of our method by identifying tractable disjunctive constraint classes of RCC-8 and RCC-5.

We begin with RCC-5 which contains four maximal tractable subsets, R_{28} (the only maximal tractable subset which contains all basic relations [16]),R_{20}, R_{17} (which consists of all relations containing the equality relation) and R_{14} [11]. We have applied the algorithm Check-Refinements on these sets using all different R-refinement matrices and found the following refinements (where Δ_{28}, Δ_{20}, Δ_{17} and Δ_{14} contain all refinements of R_{28}, R_{20}, R_{17} and R_{14}, respectively).

The sets Δ_{28}, Δ_{20} and Δ_{14} are defined by the following graphs where a relation R is present in Δ_X iff there exists a path from the initial node $\emptyset$ in the given graph to some other node such that exactly those relations present in R are visited, or if such a path exists for R's converse. Δ_{17} is given by $\Delta_{17} = R^{17}$.

$$\Delta_{28}\colon\quad \emptyset \rightarrow \{\mathsf{DR}\} \rightarrow \{\mathsf{PO}\} \rightarrow \{\mathsf{PP}\} \rightarrow \{\mathsf{PP}^{-1}\}$$

$$\Delta_{20}\colon\quad \begin{tikzcd}
 & & & \{\mathsf{EQ}\} & \{\mathsf{EQ}\} \\
\emptyset \arrow[r] \arrow[dr] & \{\mathsf{DR}\} \arrow[r] \arrow[dr] & \{\mathsf{PO}\} \arrow[r] \arrow[ur] & \{\mathsf{PP}\} \arrow[r] \arrow[ur] & \{\mathsf{PP}^{-1}\} \\
 & \{\mathsf{PO}\} & \{\mathsf{PP}\} \arrow[r] & \{\mathsf{PPI}\} &
\end{tikzcd}$$

$$\Delta_{14}\colon\quad \begin{tikzcd}
 & & \{\mathsf{PO}\} & \\
\emptyset \arrow[r] & \{\mathsf{PP},\mathsf{PP}^{-1}\} \arrow[r] \arrow[ur] & \{\mathsf{DR}\} \arrow[r] & \{\mathsf{PO}\}
\end{tikzcd}$$

In order to apply Theorem 12 and use these refinement results as independence results we must show that R_{28}, R_{20}, R_{17} and R_{14} are decidable by path-consistency. As shown in [16], R_{28} is decidable by path-consistency. From the refinements given above, it can be shown that R_{20} can be reduced by refinement to R_{28} and hence R_{20} is also decidable by path-consistency by Theorem 9. Sets of constraints over R_{17} are trivially consistent, thus R_{17} also is decidable by path-consistency. In order to show that R_{14} is decided by path-consistency we define R_{Tot} as the following:

$$R_{\text{Tot}} = \{\, \{\mathsf{PP}\},\ \{\mathsf{PP}\}^{-1},\ \{\mathsf{PP},\ \mathsf{PP}^{-1}\},\ \{\mathsf{EQ}\},\ \{\mathsf{PP},\ \mathsf{EQ}\},\ \{\mathsf{PP}^{-1},\ \mathsf{EQ}\}\,\}$$

From the previous refinements it follows that R_{14} can be reduced by refinement to R_{Tot}. It is easy to show that R_{Tot} is equivalent to the point algebra for linear time [18] by making a straightforward translation of the basic relations PP, PP^{-1}, EQ into $<, >, =$, respectively. Since the point algebra for linear time is decidable by path consistency, Theorem 9 gives that R_{14} is also decidable by path-consistency.

Having proven that R_{28}, R_{20}, R_{17} and R_{14} are decidable by path-consistency, Theorem 12 gives that $\Delta_{28}, \Delta_{20}, \Delta_{17}$, and Δ_{14} are independent of R_{28}, R_{20}, R_{17}, and R_{14}, respectively. Thus, $\mathsf{CSPSAT}(R_i \overset{\times}{\vee} \Delta_i)$ is tractable for $i \in \{28, 20, 17, 14\}$.

RCC-8 contains three maximal tractable subsets $\widehat{\mathcal{H}}_8, \mathcal{C}_8$ and $\mathcal{Q}_8$ which all contain the basic relations and which are all decidable by path-consistency [16, 15]. By using our method we can easily identify all relations which are refinements of the three sets. We let $\Delta_{\widehat{\mathcal{H}}_8}$, $\Delta_{\mathcal{C}_8}$, and $\Delta_{\mathcal{Q}_8}$ contain all refinements of the maximal tractable subsets which implies that $\mathsf{CSPSAT}(\Gamma \overset{\times}{\vee} \Delta_\Gamma)$ is tractable for $\Gamma \in \{\widehat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8\}$.

The sets $\Delta_{\widehat{\mathcal{H}}_8}, \Delta_{\mathcal{Q}_8}$ and $\Delta_{\mathcal{C}_8}$ are defined by the following graphs which are to be interpreted in the same way as the previous graphs for RCC-5.

$$\begin{tikzcd}
 & & & & & \{\mathsf{NTPP}\} & \{\mathsf{EQ}\} & \{\mathsf{EQ}\} \\
\Delta_{\widehat{\mathcal{H}}_8}\!: \emptyset \arrow[r] & \{\mathsf{DC}\} \arrow[r] & \{\mathsf{EC}\} \arrow[r] & \{\mathsf{PO}\} \arrow[r] & \{\mathsf{TPP}\} \arrow[r] \arrow[ur] & \{\mathsf{TPP}^{-1}\} \arrow[r] \arrow[ur] & \{\mathsf{NTPP}\} \arrow[r] \arrow[ur] & \{\mathsf{NTPP}^{-1}\}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & \{\mathsf{EQ}\} & \{\mathsf{EQ}\} & \{\mathsf{EQ}\} & \{\mathsf{EQ}\} \\
\Delta_{\mathcal{Q}_8}\!: \emptyset \arrow[r] & \{\mathsf{DC}\} \arrow[r] & \{\mathsf{EC}\} \arrow[r] & \{\mathsf{PO}\} \arrow[r] \arrow[ur] & \{\mathsf{TPP}\} \arrow[r] \arrow[ur] \arrow[dr] & \{\mathsf{TPP}^{-1}\} \arrow[r] \arrow[ur] & \{\mathsf{NTPP}\} \arrow[r] \arrow[ur] & \{\mathsf{NTPP}^{-1}\} \\
 & & & & & \{\mathsf{NTPP}\} \arrow[r] & \{\mathsf{EQ}\} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & \{\mathsf{NTPP}^{-1}\} & & & \{\mathsf{EQ}\} \\
\Delta_{\mathcal{C}_8}\!: \emptyset \arrow[r] & \{\mathsf{DC}\} \arrow[r] & \{\mathsf{PO}\} \arrow[r] \arrow[dr] & \{\mathsf{NTPP}\} \arrow[r] \arrow[ur] & \{\mathsf{TPP}\} \arrow[r] & \{\mathsf{NTPP}^{-1}\} \arrow[r] & \{\mathsf{TPP}^{-1}\} \arrow[r] \arrow[ur] & \{\mathsf{EC}\} \\
 & & & \{\mathsf{EC}\} \arrow[r] \arrow[dr] & \{\mathsf{NTPP}\} \arrow[r] & \{\mathsf{TPP}\} \arrow[r] & \{\mathsf{NTPP}^{-1}\} & \\
 & & & & \{\mathsf{NTPP}^{-1}\} & & &
\end{tikzcd}$$

	Γ'_A	Δ'_A	Γ_A	Δ_A	Γ'_B	Δ'_B	Γ_B	Δ_B	Γ'_C	Δ'_C	Γ_C	Δ_C	Δ_D
$\{<\}$			•				•		•		•		
$\{<,=\}$	•		•		•		•						•
$\{<,>\}$							•	•					
$\{<,>,=\}$					•	•	•	•					•
$\{\|\}$	•	•	•	•							•	•	
$\{\|,=\}$	•		•								•	•	•
$\{=\}$			•				•		•		•		•
$\{<,>,\|\}$	•	•	•	•	•	•	•	•	•	•	•	•	
$\{<,\|\}$			•	•							•	•	
$\{<,\|,=\}$			•						•	•	•	•	•

Table 1. Tractable classes of the point algebra for partially ordered time.

4.2 The point algebra for partially ordered time

After having demonstrated that new independence results can be derived using refinements we will now show that many previously presented independence results can also be derived using refinements. We begin by showing that all independence results for the point algebra for partially ordered time can be derived using refinements and, moreover, that *every* maximal tractable set of disjunctions of relations for partially ordered time can be derived using refinements. This, of course, requires a definition of a maximal tractable set of disjunctions of relations.

Let Γ be a set of disjunctive relations constructed from a set $\mathcal{B}$ of binary relations by applying the $\overset{\times}{\vee}$ operator. We say that Γ is a *maximal tractable subclass* iff Γ is tractable and for every set $X \not\subseteq \Gamma$ of relations which can be constructed by the relations in $\mathcal{B}$ and $\overset{\times}{\vee}$, $\Gamma \cup X$ is intractable.

The point algebra for partially ordered time is based on the notion of *relations* between pairs of variables interpreted over a partially-ordered set. We consider four basic relations which we denote by $<, >, =$ and $\|$. If x, y are points in a partial order $\langle T, \leq \rangle$ then we define these relations in terms of the partial ordering $\leq$ as follows:

1. $x\{<\}y$ iff $x \leq y$ and not $y \leq x$
2. $x\{>\}y$ iff $y \leq x$ and not $x \leq y$
3. $x\{=\}y$ iff $x \leq y$ and $y \leq x$
4. $x\{\|\}y$ iff neither $x \leq y$ nor $y \leq x$

The point algebra for partially ordered time has been throughly investigated earlier and a total classification with respect to tractability has been given in Broxvall and Jonsson [3]. In Broxvall and Jonsson [4] the sets of relations in Table 1 are defined and it is proven that $\Gamma_A \overset{\times}{\vee} \Delta^*_A$,$\Gamma_B \overset{\times}{\vee} \Delta^*_B$,$\Gamma_C \overset{\times}{\vee} \Delta^*_C$ and Δ^*_D are the unique maximal tractable disjunctive classes of relations for partially ordered time. The proofs of tractability for those sets rely on several handmade independence proofs. We will now derive these independence results using refinements.

To do so, we need to show that the classes $\Gamma_A, \Gamma_B, \Gamma_C$ and Δ_D are decidable by path-consistency. We begin by proving a useful connection between RCC-5 and the point algebra for partially ordered time which in turn will be needed to prove that path-consistency decides Γ_A.

Lemma 16 Let Γ be a set of relations in the point algebra for partially ordered time and define the function σ such that

1. $\sigma(<) = \{\mathsf{PP}\}$;
2. $\sigma(>) = \{\mathsf{PP}^{-1}\}$;
3. $\sigma(=) = \{\mathsf{EQ}\}$; and
4. $\sigma(||) = \{\mathsf{DR}, \mathsf{PO}\}$.

Then, Γ can be decided by path-consistency if the set

$$\Gamma' = \{\bigcup_{r \in R} \sigma(r) \mid R \in \Gamma\}$$

of RCC-5 relations can be decided by path-consistency.

Proof. Let Π be an arbitrary CSP instance over the relations in Γ. Define the set Σ of RCC-5 formulae as follows: for each $x_i \; R \; x_j \in \Pi$, add the formula $x_i \bigcup_{r \in R} \sigma(r) x_j$. Note that Σ is a CSP instance over Γ' that can be decided by path-consistency by our initial assumptions.

We begin by comparing the composition tables for partially-ordered time and the RCC-5 relations $\{\mathsf{PP}\}, \{\mathsf{PP}^{-1}\}, \{\mathsf{EQ}\}, \{\mathsf{DR}, \mathsf{PO}\}$:

	$\{<\}$	$\{>\}$	$\{=\}$	$\{\parallel\}$
$\{<\}$	$\{<\}$	$\top$	$\{<\}$	$\{<, \parallel\}$
$\{>\}$	$\top$	$\{>\}$	$\{>\}$	$\{>, \parallel\}$
$\{=\}$	$\{<\}$	$\{>\}$	$\{=\}$	$\{\parallel\}$
$\{\parallel\}$	$\{<, \parallel\}$	$\{>, \parallel\}$	$\{\parallel\}$	$\top$

	$\{\mathsf{PP}\}$	$\{\mathsf{PP}^{-1}\}$	$\{\mathsf{EQ}\}$	$\{\mathsf{DR}, \mathsf{PO}\}$
$\{\mathsf{PP}\}$	$\{\mathsf{PP}\}$	$\top$	$\{\mathsf{PP}\}$	$\{\mathsf{PP}, \mathsf{DR}, \mathsf{PO}\}$
$\{\mathsf{PP}^{-1}\}$	$\top$	$\{\mathsf{PP}^{-1}\}$	$\{\mathsf{PP}^{-1}\}$	$\{\mathsf{PP}^{-1}, \mathsf{DR}, \mathsf{PO}\}$
$\{\mathsf{EQ}\}$	$\{\mathsf{PP}\}$	$\{\mathsf{PP}^{-1}\}$	$\{\mathsf{EQ}\}$	$\{\mathsf{DR}, \mathsf{PO}\}$
$\{\mathsf{DR}, \mathsf{PO}\}$	$\{\mathsf{PP}, \mathsf{DR}, \mathsf{PO}\}$	$\{\mathsf{PP}^{-1}, \mathsf{DR}, \mathsf{PO}\}$	$\{\mathsf{DR}, \mathsf{PO}\}$	$\top$

After having made this comparison, it should be fairly obvious that the empty relation can be derived from Π by enforcing path-consistency if and only if it can be derived from Σ. Thus, we only have to show that whenever Σ has a model, Π also has a model.

Let M be a model that assigns sets to the variables $x_1, \ldots, x_n$ that appear in Σ. We define an interpretation N from the variables in Π to the partial order

$\langle\{M(x_i) \mid 1 \leq i \leq n\}, \subseteq\rangle$ as follows: $N(x_i) = M(x_i)$ for $1 \leq i \leq n$. To conclude the proof, we pick an arbitrary constraint $x_i R x_j$ in Σ and show that it is satisfied by the interpretation N. Assume now, for instance, that $M(x_i)\ \{\mathsf{PP}\}\ M(x_j)$. By the definition of σ, we know that $\{<\} \subseteq R$ and it follows immediately that $N(x_i) < N(x_j)$ and the constraint $x_i R x_j$ is satisfied. The remaining cases can easily be proved analogously.

Theorem 17 Path-consistency decides consistency for Γ_A, Γ_B, Γ_C and Δ_D.

Proof. Let $\Gamma' = \{\bigcup_{r \in R} \sigma(r) \mid R \in \Gamma_A\}$ (where σ is defined as in Lemma 16) and note that $\Gamma' \subseteq R_{28}$. Since R_{28} can be decided by path-consistency [16], Lemma 16 implies that path-consistency decides Γ_A.

By using CHECK-REFINEMENTS, it can be verified that Γ_B can be reduced by refinements to Γ_A which by Theorem 9 gives that Γ_B is decided by path-consistency. For Γ_C the result follows from the fact that it is a subset of Γ_A. Finally, Δ_D is trivially decided by path-consistency. □

Using the algorithm CHECK-REFINEMENTS, we can automatically verify that $\Delta_A, \Delta_B, \Delta_C$ and Δ_D are valid refinements of $\Gamma_A, \Gamma_B, \Gamma_C$ and Δ_D, respectively. Theorems 17 now gives that $\Delta_A, \Delta_B, \Delta_C$ and Δ_D are independent of $\Gamma_A, \Gamma_B, \Gamma_C$ and Δ_D, respectively and we have proven tractability of *all* maximal tractable sets of disjunctions of relations for the point algebra for partially ordered time.

4.3 The point algebra for linear time

In Broxvall and Jonsson [4] the time-point algebra for linear time is also investigated and the following two classes are defined:

$$\mathcal{X}_1 = \{\{<\}, \{<,=\}, \{<,>\}, \{=\}\} \overset{\times}{\vee} \{\{<,>\}\}$$

$$\mathcal{X}_2 = \{\{<,=\}, \{=\}\}^*$$

Furthermore, proof is also given that these two classes are the only two maximal tractable sets of disjunctions of relations. Both independence results needed for that classification can be derived using refinements. In Renz [15] it is noted that disequality is a refinement in the point algebra for linear time. The other independence result consisting of all relations containing equality can easily be verified using the refinement algorithm, and decidability by path consistency is trivial.

It should thus be noted that both in the case of the point algebra for partially ordered time and for the point algebra for linear time it is sufficient only to look at the refinements in order to derive *all* tractable sets of disjunctions. However, we have no guarantee that this holds in the general case.

5 Discussion and Conclusions

Independence of relations with respect to tractable sets of relations is a very useful tool for generating expressive tractable disjunctive constraint classes. However, proving independence is often a highly non-trivial task. In this paper we proposed a method for proving independence which we obtained by relating the notion of refinement to the notion of independence. We found that if a relation R is a refinement of a set of relations $\mathcal{S}$, then R is also independent of $\mathcal{S}$. Since refinements can be checked by running a simple algorithm, this allows us to automatically generate independence results. The only requirement for applying this method is the sufficiency of path-consistency for deciding consistency in $\mathcal{S}$. In many cases this can, however, also be shown by using refinement techniques.

In order to demonstrate the usefulness of our method, we applied it to the Region Connection Calculi RCC-5 and RCC-8 and derived many previously unknown independence results. Furthermore, using our method we were able to obtain all previously known independence results of the point algebra for partially ordered time as well as for linear time. This is particularly interesting since in this case refinements are sufficient for identifying all maximal tractable sets of disjunctions, *i.e.*, in this case independence seems to imply refinement. We have not been able so far to prove this implication in the general case and instead specified a certain condition of when independence implies refinements. It would be very interesting to know whether the correspondence between refinements and independence holds in the general case or alternatively which restrictions must be made in order to have this correspondence. Then, refinements can be used to derive all independence results in a simple way.

So far we have only used a restricted form of refinement matrices, namely, an R-refinement matrix M^R for some relation R. We did this because previously tractable disjunctive constraint classes formed by two sets Γ and Δ required that all relations of Δ are independent of all relations of Γ. Using the refinement method it is possible to verify more complex refinement matrices. These refinement matrices can be used for proving an advanced notion of independence such as "Δ is *subset independent* of Γ iff there are (non-disjoint) subsets $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$ and $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ such that Δ_i is independent of Γ_i". It might well be possible that this advanced notion of independence allows to generate new types of tractable disjunctive constraint classes whose tractability can, again, be proven using refinements techniques.

Another piece of further work which seems to be worthwhile is to analyze the relationship between the refinement method and a method for proving tractability which was developed by Jeavons *et al.* [9].

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