Some Observations on Durations, Scheduling and Allen's Algebra

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Abstract. We continue the study of complexity issues in Allen's algebra extended for handling metric durations. The eighteen known maximal tractable subclasses are studied and we show that three of them can be extended with metric constraints on the durations without sacrificing tractability, while the remaining fifteen subclasses become NP-complete even if we restrict ourselves to the case of fixed lengths on the durations. We also consider the scheduling problem and the main result is that ORD-Horn is the only known maximal subclass for which the scheduling problem is tractable. This extends previous tractability results on scheduling in Allen's algebra since we allow far more temporal relations and we allow intervals with uncertain durations.

1 Introduction

Representing and reasoning about time has for a long time been acknowledged as one of the core areas of artificial intelligence and a large number of formalisms for temporal constraint reasoning (TCR) have been proposed in the literature. Important examples are the time point algebra [VKvB89], Allen's algebra [All83], simple temporal constraints [DMP91] and the qualitative algebra [Mei96]. These formalisms are almost exclusively dealing with the relative positions of time points (qualitative information) and/or the absolute position of time points on the time line (quantitative or metric information).

Recently, the TCR community has shown an increasing interest in reasoning about the *distance* between time points on the time line. As an example, suppose one wants to put a bound c on the distance between two time points x and y whose order is possibly unknown. This can be accomplished by imposing the constraint $abs(x-y) \leq c$ (where abs denotes the absolute value function). Recent examples of formalisms that can handle such constraints can be found in Navarrete & Marin [NM97], Wetprasit & Sattar [WS98] and Pujari & Sattar [PS99]. The full formalisms are NP-hard in all cases but the authors have also identified nontrivial tractable subclasses.

An interesting variation on reasoning about distances is reasoning about *durations*, *i.e.*, reasoning about the length of intervals. Such problems occurs naturally in, for instance, scheduling applications. In an (abstract) scheduling problem, we are given a collection of tasks of known durations and constraints on their relative order (for instance in the form of limited resources and/or precedence relations) and the question is whether the tasks can be completed within a given time frame. As has been observed by Anger & Rodriguez [AR94][1], many problems of this type can conveniently be expressed within an extended version of Allen's interval algebra. Anger & Rodriguez also presents two algorithms for solving this scheduling problem—the first solves the problem exactly and in polynomial time when the only allowed relations are the basic relations of Allen's algebra. The second algorithm is a heuristic algorithm for the general case. Clearly, the second algorithm cannot run in polynomial time since even the satisfiability problem is NP-complete when arbitrary relations are allowed.

In this paper, we continue the study of complexity issues in Allen's algebra extended for handling metric durations. We consider the eighteen known maximal tractable subclasses [NB95,DJ97a,DJ97c] of Allen's algebra and show that three of them (ORD-Horn [NB95], $\mathcal{S}(\succ)$ and $\mathcal{E}(\prec)$ [DJ97a]) can be extended with metric constraints on durations without sacrificing tractability of the satisfiability problem. The other fifteen subclasses become NP-complete even if the only allowed duration information is the assignment of fixed lengths to the intervals. We also consider the scheduling problem and our main result shows that ORD-Horn is the only known maximal subclass such that the scheduling problem is tractable. This result extends Anger and Rodriguez's [AR94] previously mentioned tractability result in at least two ways: (1) we allow far more temporal relations; and (2) we allow intervals with uncertain durations.

The structure of the paper is as follows: Section 2 defines Allen's algebra and the notions of qualitative and metric time as well as the known maximal tractable subclasses of Allen's algebra. In section 3 we add metric duration information to three of these subclasses, show that ORD-Horn remains tractable in this case, and give a polynomial time algorithm for deciding satisfiability of the remaining two cases. Section 4 shows intractability of the 15 remaining subclasses and in section 5 another problem, that of *scheduling* a set of given tasks within a given time, is defined and we show that the only known maximal tractable subclass of Allen's algebra for which this problem is solvable in polynomial time is the ORD-Horn algebra. Finally, in section 6, we conclude with a discussion of what we can expect from the currently unknown subclasses of Allen's algebra.

2 Preliminaries

We start with defining the basic notions needed.

2.1 Allen's algebra

The notion of *relations between pairs of intervals* forms the basis of Allen's interval algebra [All83]. An interval I is represented as a tuple $\langle I^-, I^+\rangle$ where $I^-, I^+ \in \mathbb{R}$ with $I^- < I^+$ (to ensure that the length of the interval is strictly

[1] Other authors, such as Belhadji & Isli [BI98], have made the same observation.

positive) denotes the left and right endpoints of the interval. Relations between intervals are composed as disjunctions of *basic interval relations* which are defined in table 1. These disjunctions are represented as sets of basic relations using the following notation; The disjunction of the basic relations b_1 and b_2 is written $(b_1\ b_2)$, and, for example, given the disjunctions $(b_1\ b_2\ b_3)$ and $(b_1\ b_2)$ it holds that $(b_1\ b_2) \subseteq (b_1\ b_2\ b_3)$. The disjuction of all basic relations is written $\top$ and the empty relation is written $\perp$.

The algebra is provided by the operations *converse, intersection* and *composition* on intervals. The converse operation takes an interval relation r to its converse r^{-1}, obtained by inverting each basic relation in r, that is, exchanging x and y in the endpoint relations in table 1. The *negation* of a set of relations is defined as follows: If R is a set of basic relations, by $\neg R$ we mean the disjunction of every basic relation *except* those in R. Since there are 13 basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. The set of all interval relations we denote $\mathcal{A}$. Subclasses of the full algebra are given by considering subsets of $\mathcal{A}$. Classes closed under intersection, converse and composition are said to be *algebras.*

The most common computational problem associated with Allen's interval algebra is *satisfiability*, *i.e.*, deciding whether or not there exists an assignment of intervals on the real line for the variables, such that all of the relations between them are satisfied. This is defined as follows.

Definition 1 (ISAT($\mathcal{I}$)). *Let $\mathcal{I} \subseteq \mathcal{A}$ be a set of interval relations. An instance of* ISAT($\mathcal{I}$) *is a tuple $Q = \langle V, E\rangle$, where V is a set of interval variables and E a subset of $V \times \mathcal{I} \times V$. An element $\langle I, r, J\rangle \in E$ means that I and J are related by r.*

A function M taking an interval variable I to its interval representation $M(I) = \langle I^-, I^+\rangle$ with $I^-, I^+ \in \mathbb{R}$ and $I^- < I^+$ is said to be an interpretation *of Q.*

An instance $Q = \langle V, E\rangle$ is said to be satisfiable *iff there exists an interpretation M such that for each $\langle I, r, J\rangle \in E$, $M(I) r M(J)$ holds,* i.e., *the endpoint relations required by r (see table 1) are satisfied by the assignments of I and J. We say that M is a* model *of Q.*

By the size *of an instance Q we mean $|V| + |E|$.*

Vilain, Kautz & van Beek [VKvB89], have shown the ISAT($\mathcal{A}$) problem to be NP-complete.

2.2 Qualitative and Metric Time

We start with a brief summary of the ORD-Horn formalism.

Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables, and let α, β linear polynomials (polynomials of degree one) over X with integer coefficients. A *linear relation* over X is a mathematical expression of the form $\alpha R \beta$, where $R \in \{<, \leq, =, \neq, \geq, >\}$.

Basic relation		Example	Endpoints
x before y	$\prec$	xxx	$x^- < y^-$
y after x	$\succ$	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met-by x	m^{-1}	yyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overl.-by x	o^{-1}	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	d^{-1}	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$
y started by x	s^{-1}	yyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^- > y^-$
y finished by x	f^{-1}	yyyyyy	$x^+ = y^+$
x equals y	$\equiv$	xxxx	$x^- = y^-$
		yyyy	$x^+ = y^+$

Table 1. The thirteen basic relations.

A *disjunctive linear relation* (DLR) over X is a disjunction of a nonempty finite set of linear relations. A DLR is said to be *Horn* iff at most one of its disjuncts is *not* of the form $\alpha \neq \beta$.

The problem of *satisfiability* for finite sets of DLRs is denoted DLRSAT, which is checking whether there exists an assignment M of variables in X to real numbers such that all DLRs are satisfied in M. Such an M is said to be a *model* of D. The satisfiability problem for finite sets of Horn DLRs is denoted HORNDLRSAT.

Proposition 1. *There is an algorithm for solving* HORNDLRSAT *which runs in polynomial time wrt the size of the instance.*

Proof. See Koubarakis [Kou96] or Jonsson & Bäckström [JB98].

The ALG-HORNDLRSAT algorithm is, while still polynomial, rather expensive and relies on a linear-programming algorithm, so when there is no need for specifying metric information, the following subclass of Horn-DLRs gives us an algorithm of lower complexity.

Definition 2 (Point algebra). *The* point algebra *[Vil82] is the subclass of Horn DLRs consisting of the set of expressions xRy, where x and y are variables and $R \in \{<, \leq, =, \neq, \geq, >\}$.*

Proposition 2. PASAT *is solvable in linear time in the size of the instance.*

Proof. See Gerevini, *et al* [GSS93].

The following problem will be studied in section 3, where we extend the subclasses of Allen's algebra to include metric duration information.

Definition 3 (D-ISAT). *For an interval $I \in V$, we let $d(I)$ denote the duration of I,* i.e., *$I^+ - I^-$. An instance of the problem of* interval satisfiability with metric duration information *for a set $\mathcal{I}$ of interval relations, denoted* D-ISAT($\mathcal{I}$), *is a tuple $Q = \langle V, E, X \rangle$ where X is a set of duration relations, expressed as Horn DLRs on $d(I)$.*

2.3 Maximal tractable subclasses of Allen's algebra

The following defines the subclasses of Allen's algebra we are interested in.

Definition 4 (Maximal tractable subclass). *A* maximal tractable subclass *of Allen's algebra is a set $M \subseteq \mathcal{A}$, such that* ISAT(M) *is polynomial but for any $N \subseteq \mathcal{A}$ with $M \subset N$,* ISAT(N) *is not polynomial (assuming $P \neq NP$).*

The following five definitions define the 18 known maximal tractable subclasses of Allen's algebra.

Definition 5 (ORD-Horn algebra). *An* ORD clause *is a disjunction of relations of the form xRy where $R \in \{\geq, =, \neq\}$. The* ORD-Horn *subclass $\mathcal{H}$ [NB95] is the set of relations in $\mathcal{A}$ which can be written as ORD clauses containing only disjunctions with at most one relation of the form $x = y$ or $x \geq y$ and an arbitrary number of relations of the form $x \neq y$.*

ORD-Horn is the unique subclass containing all of the basic relations in table 1.

Definition 6 (The subclasses $\mathcal{S}(b)$ and $\mathcal{E}(b)$). *Set $r_s = (\succ\ \mathsf{d}\ \mathsf{o}^{-1}\ \mathsf{m}^{-1}\ \mathsf{f})$ and $r_e = (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})$.*

For $b \in \{\succ, \mathsf{d}, \mathsf{o}^{-1}\}$, define $\mathcal{S}(b)$ to be the set of relations r, such that at least one of the following holds:

$$\begin{aligned} (b\ b^{-1}) &\subseteq r \\ (b) &\subseteq r \subseteq r_s \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) \\ (b^{-1}) &\subseteq r \subseteq r_s^{-1} \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) \\ r &\subseteq (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) \end{aligned}$$

By switching starting and ending points of intervals, for $b \in \{\prec, \mathsf{d}, \mathsf{o}\}$, $\mathcal{E}(b)$ is defined to be the set of relations r, such that at least one of the following holds:

$$\begin{aligned} (b\ b^{-1}) &\subseteq r \\ (b) &\subseteq r \subseteq r_e \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}) \\ (b^{-1}) &\subseteq r \subseteq r_e^{-1} \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}) \\ r &\subseteq (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}) \end{aligned}$$

The two classes $\mathcal{S}(\succ)$ and $\mathcal{E}(\prec)$ turn out to be of special interest, as we shall see in the next section, since the D-ISAT problem is tractable for these.

Definition 7 (The subclasses $\mathcal{S}^*$ and $\mathcal{E}^*$). *Let r_s and r_e be as in definition 6, and define $\mathcal{S}^*$ to be the set of relations r such that at least one of the following holds:*

$$
\begin{aligned}
(\equiv \ \mathsf{f}\ \mathsf{f}^{-1}) &\subseteq r \\
(\mathsf{f}\ \mathsf{f}^{-1}) &\subseteq r \subseteq r_s \cup r_s^{-1} \\
(\equiv \ \mathsf{f}) &\subseteq r \subseteq r_s \cup (\equiv \ \mathsf{s}\ \mathsf{s}^{-1}) \\
(\equiv \ \mathsf{f}^{-1}) &\subseteq r \subseteq r_s^{-1} \cup (\equiv \ \mathsf{s}\ \mathsf{s}^{-1}) \\
(\mathsf{f}) &\subseteq r \subseteq r_s \\
(\mathsf{f}^{-1}) &\subseteq r \subseteq r_s^{-1} \\
(\equiv) &\subseteq r \subseteq (\equiv \ \mathsf{s}\ \mathsf{s}^{-1}) \\
r &\subseteq \perp
\end{aligned}
$$

Symmetrically, by replacing f *by* s, f^{-1} *by* s^{-1}, $(\equiv \ \mathsf{s}\ \mathsf{s}^{-1})$ *by* $(\equiv \ \mathsf{f}\ \mathsf{f}^{-1})$ *and* r_s *by* r_e, *we get the subclass* $\mathcal{E}^*$.

Definition 8 (The subclass $A_{\equiv}$). *Define the subclass $A_{\equiv}$ as $\{\perp\} \cup \{r \mid \{\equiv\} \subseteq r\}$.*

Definition 9 (The subclasses $A(r,b)$). *Let $b \in \{\mathsf{s}, \mathsf{s}^{-1}, \mathsf{f}, \mathsf{f}^{-1}\}$ and r one of the relations*

$$
\begin{aligned}
&(\prec \ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{-1}) \\
&(\prec \ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{-1}\ \mathsf{f}^{-1}) \\
&(\prec \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\
&(\prec \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{-1})
\end{aligned}
$$

containing b. First define the subclasses $A_1(b), A_2(r,b)$ and $A_3(r,b)$ by

$$
\begin{aligned}
A_1(b) &= \{r' \cup (b\ b^{-1}) \mid r' \in \mathcal{A}\} \\
A_2(r,b) &= \{r' \cup (b) \mid r' \subseteq r\} \\
A_3(r,b) &= \{r' \cup (\equiv) \mid r' \in A_2(r,b)\} \cup \{(\equiv)\}
\end{aligned}
$$

Then set

$$B = A_1(b) \cup A_2(r,b) \cup A_3(r,b)$$

and finally

$$A(r,b) = B \cup \{x^{-1} \mid x \in B\} \cup \{\perp\}$$

We denote the set of all $A(r,b)$ subclasses by $\mathbf{A}_{r,b}$.

Theorem 1. *The subclasses defined in definitions 5, 6, 7, 8 and 9 are the 18 known maximal tractable subclasses of $\mathcal{A}$.*

Proof. See Nebel & Bürckert [NB95] for the algebra $\mathcal{H}$, Drakengren & Jonsson [DJ97c] for the subclasses in $\mathbf{A}_{r,b}$ and $A_{\equiv}$, and Drakengren & Jonsson [DJ97a] for $\mathcal{S}(b), \mathcal{E}(b), \mathcal{S}^*$ and $\mathcal{E}^*$.

Algorithm: $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$
input: Instance $Q = \langle V, E \rangle$ of $\textsc{Isat}(\mathcal{A})$
$Q' = \langle V, E, H' \rangle \leftarrow \mathit{expl}^-(Q)$
if not $\textsc{Alg-PAsat}(H')$ **then**
  reject
$K \leftarrow \emptyset$
for each $\langle I, r, J \rangle \in E$ **do**
  if not $\textsc{Alg-PAsat}(H' \cup \{I^- \neq J^-\})$ **then**
    $K \leftarrow K \cup \{I^- = J^-\}$
  else
    $K \leftarrow K \cup \{I^- \neq J^-\}$
  end if
end for
$P \leftarrow \{I^+ \mathit{eprel}^-(r) J^+ \mid \langle I, r, J \rangle \in E \wedge I^- = J^- \in H' \cup K\}$
if not $\textsc{Alg-PAsat}(P)$ **then**
  reject
accept

Fig. 1. The $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$ algorithm.

3 Tractability results

Here we present the known tractable subclasses of $\mathcal{A}$ which remain tractable when duration information is added.

The ORD-Horn subclass can easily be extended to encompass durations. To see this, we start by noting that ORD-Horn is a subset of Horn DLRs. It is possible to add an arbitrary number of constraints of the form $I^+ - I^- = d_I$, where I^+ and I^- are end points of an interval $I \in V$ and d_I is the required duration of this interval, and still only deal with Horn DLRs.

The algorithm $\textsc{Alg-D-Isat}_{\mathcal{S}(\succ)}$ uses the $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$ algorithm [DJ97a], which is presented in figure 1. In order to understand it, we need the following definitions.

Definition 10 ($\mathit{sprel}(r)$, $\mathit{eprel}(r)$, $\mathit{sprel}^+(r)$, $\mathit{eprel}^-(r)$). *Take the relation $r \in \mathcal{A}$, let I and J be interval variables, and consider the instance S of* $\textsc{Isat}(\{r\})$ *which relates I and J with the relation r only. Define the relation $\mathit{sprel}(r)$ on real numbers to be the implied relation between the starting points of I and J. That is, for basic relations define (the quotation marks are only to avoid notational confusion; the actual relations are intended)*

$$\begin{aligned}
\mathit{sprel}(\equiv) &= \text{“}{=}\text{”} \\
\mathit{sprel}(\prec) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{d}) &= \text{“}{>}\text{”} \\
\mathit{sprel}(\mathsf{o}) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{m}) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{s}) &= \text{“}{=}\text{”} \\
\mathit{sprel}(\mathsf{f}) &= \text{“}{>}\text{”} \\
\mathit{sprel}(r^{\smile}) &= (\mathit{sprel}(r))^{-1},
\end{aligned}$$

Algorithm: $\textsc{Alg-D-Isat}_{\mathcal{S}(\succ)}$
input: Instance $Q = \langle V, E, X \rangle$ of $\textsc{D-Isat}(\mathcal{S}(\succ))$

1 **if not** $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}(\langle V, E \rangle)$ **then**
2   **reject**
3 $K := \emptyset$
4 **for** each $\langle I, r, J \rangle \in E$ **do**
5   **if not** $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}(\langle V, E \cup \{I \neg \{\mathsf{s}, \mathsf{s}^{-1}\} J\} \rangle)$ **then**
6     $K := K \cup \{d(I) \neq d(J)\}$
7   **if not** $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}(\langle V, E \cup \{I \neg \{\equiv, \mathsf{s}\} J\} \rangle)$ **then**
8     $K := K \cup \{d(I) \leq d(J)\}$
9   **if not** $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}(\langle V, E \cup \{I \neg \{\equiv, \mathsf{s}^{-1}\} J\} \rangle)$ **then**
10     $K := K \cup \{d(I) \geq d(J)\}$
11 **end for**
12 $X' := X \cup \{d(I) > 0 \mid I \in V\}$
13 **if not** $\textsc{Alg-hornDLRsat}(X' \cup K)$ **then**
14   **reject**
15 **accept**

Fig. 2. The $\textsc{Alg-D-Isat}_{\mathcal{S}(\succ)}$ algorithm.

and for disjunctions $sprel(r)$ is the relation corresponding to $\bigvee_{b \in r} sprel(b)$. Symmetrically, we define $eprel(r)$ to be the implied relation between ending points given r. Note that $sprel(r)$ and $eprel(r)$ have to be one of $<$, $\leq$, $=$, $\geq$, $>$, $\neq$, $\top$ or $\perp$.

Furthermore, we define specialisations of these, by $sprel^{+}(r) = sprel(r \cap (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}))$ and $eprel^{-}(r) = eprel(r \cap (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}))$, i.e., *the implied relations on starting (ending) points by r, given that the ending (starting) points are known to be equal.*

Definition 11 (Explicit starting (ending) point relations). *Let $\mathcal{I} \subseteq \mathcal{A}$ and define the function $expl^{-}$ on instances $Q = \langle V, E \rangle$ of* $\textsc{Isat}(\mathcal{I})$ *by setting*

$$expl^{-}(Q) = \{I^{-}\, sprel(r)\, J^{-} \mid \langle I, r, J \rangle \in E\}.$$

$expl^{-}(Q)$ is said to be obtained from Q by making starting point relations explicit.

Symmetrically, using eprel and ending points instead of sprel and starting points, $expl^{+}(Q)$ is said to be obtained from Q by making ending points explicit.

The algorithm whose existence was shown in proposition 2 is used by $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$. We call it $\textsc{Alg-PAsat}$, but do not state it verbatim. Similary, the algorithm for solving the $\textsc{hornDLRsat}$ problem is denoted $\textsc{Alg-hornDLRsat}$.

Lemma 1. *Let $Q = \langle V, E \rangle$ be a satisfiable instance of* $\textsc{Isat}(\mathcal{S}(\succ))$. *There exists a model M of Q such that for all $I, J \in V$, $M(I^{-}) \neq M(J^{-})$ unless $M(I^{-}) = M(J^{-})$ has to hold in every model.*

Proof. Follows from the correctness of the $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$ algorithm [DJ97a] (pay special attention to the activities on lines 7 and 9).

Before proving the correctness of ALG-D-ISAT$_{\mathcal{S}(\succ)}$ (see figure 2), we note that the corresponding algorithm for solving satisfiability for D-ISAT($\mathcal{E}(\prec)$) is gained by exchanging s and s^{-1} for f and f^{-1} in the algorithm in figure 2. The calls to the ALG-ISAT$_{\mathcal{S}(\succ)}$ algorithm are replaced with calls to ALG-ISAT$_{\mathcal{E}(\prec)}$.

Theorem 2. *The* ALG-D-ISAT$_{\mathcal{S}(\succ)}$ *and* ALG-D-ISAT$_{\mathcal{E}(\prec)}$ *algorithms correctly solves satisfiability for* $\mathcal{S}(\succ)$ *and* $\mathcal{E}(\prec)$*, respectively.*

Proof. First assume the algorithm accepts. Since the tests on lines 1 and 13 do not reject, there exist models M_1, M_2 of $\langle V, E\rangle$ and X, respectively. The starting points of M_1, ordered by $\leq$, give rise to a function $f : V \to \mathbb{N}$ mapping an interval $I \in V$ to the natural number representing its order. Now let $c = \max_{I\in V}\{d(I)\} + 1$, *i.e.*, the length of the longest interval in $V + 1$, and let $d_{M_2}(I)$ be the function giving the duration of interval I in M_2 and define the model M as;

$$M(I^-) = f(I) \cdot c$$
$$M(I^+) = M(I^-) + d_{M_2}(I)$$

By construction, M satisfies the duration constraints, and thus it remains to be shown that M is indeed a model of Q.

In the definition of $\mathcal{S}(b)$ and $\mathcal{E}(b)$, note that r_s contains all basic relations b such that whenever IrJ for interval variables I, J, $I^- > J^-$ has to hold in any model and, symmetrically, r_e is equivalent to $I^+ < J^+$ holding in any model.

Given a tuple $\langle I, r, J\rangle \in E$, it is the case that $M_1(I^-) < M_1(J^-)$, $M_1(I^-) = M_1(J^-)$ or $M_1(I^-) > M_1(J^-)$, and we get the following three cases to analyse:

- If $M_1(I^-) < M_1(J^-)$ then r is a relation from line 1 or 2[2]. $M(I^-) < M(I^+) < M(J^-) < M(J^+)$ and $I(\prec)J$ is satisfied.
- If $M_1(I^-) > M_1(J^-)$ then r is a relation from line 2 or 3. $M(J^-) < M(J^+) < M(I^-) < M(I^+)$ and $I(\succ)J$ is satisfied.

The case when $M_1(I^-) = M_1(J^-)$ is trickier since then $M(I^-) = M(J^-)$ and the relation depends on the duration of the intervals. By lemma 1 we can restrict our analysis to $r \cap (\equiv\ \mathsf{s}\ \mathsf{s}^{-1})$ and view r as a relation from line 4. The set K in the algorithm is constructed as follows:

- If $\langle V, E \cup \{I\neg\{\mathsf{s}, \mathsf{s}^{-1}\}J\}\rangle$ is unsatisfiable then $I(\mathsf{s}\ \mathsf{s}^{-1})J$ holds in any model of Q and $d(I) \neq d(J)$ is added to K.
- If $\langle V, E \cup \{I\neg\{\equiv, \mathsf{s}\}J\}\rangle$ is unsatisfiable then $I(\equiv\ \mathsf{s})J$ holds in any model of Q and $d(I) \leq d(J)$ is added to K.
- If $\langle V, E \cup \{I\neg\{\equiv, \mathsf{s}^{-1}\}J\}\rangle$ is unsatisfiable then $I(\equiv\ \mathsf{s}^{-1})J$ holds in any model of Q and $d(I) \geq d(J)$ is added to K.

The idea behind this is as follows. Assume $r \cap (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) = (\mathsf{s})$. The first and second tests add the constraints $d(I) \neq d(J)$ and $d(I) \leq d(J)$ to K and these

[2] The line numbering refers to definition 6.

two combine to the constraint $d(I) < d(J)$ and thus $I(\mathsf{s})J$ is satisfied. So given any two intervals I, J with $I^- = J^-$ we know from K which the relation between $d(I)$ and $d(J)$ is. Thus M satisfies Q and we have a model for $\langle V, E, X\rangle$.

Next assume the algorithm rejects an instance $Q = \langle V, E, X\rangle$, *i.e.*, either or both of the tests on lines 1 and 13 rejects. If the first test rejects then $\langle V, E\rangle$ is unsatisfiable and obviously $\langle V, E, X\rangle$ cannot have a model. If the test on line 13 fails, then $X' \cup K$ is unsatisfiable. Since K and $\{d(I) \mid I \in V\}$ has to hold in every model of Q, it must be the case that X combined with these causes the rejection and thus is unsatisfiable, so Q is unsatisfiable.

Symmetrically, by exchanging starting and ending points of the algorithm, we get the the proof for $\textsc{Alg-D-Isat}_{\mathcal{E}(\prec)}$.

Theorem 3. *The* $\textsc{Alg-D-Isat}_{\mathcal{S}(\succ)}$ *and* $\textsc{Alg-D-Isat}_{\mathcal{E}(\prec)}$ *algorithms run in polynomial time.*

Proof. By corollary 5.13, Drakengren & Jonsson [DJ97a], the $\textsc{Alg-Isat}_{\mathcal{S}(\succ)}$ algorithm runs in $\mathcal{O}(|E|^2)$ time, where E is the set of constraints. The tests on lines 1, 5, 7 and 8 thus each take $\mathcal{O}(|E|^2)$ time. The loop is iterated $|E|$ times and thus the total running time for the loop is $\mathcal{O}(|E|^3)$. By proposition 1 the $\textsc{Alg-hornDLRsat}$ algorithm runs in polynomial time, but its complexity is very much depending on the linear programming approach used, so we simply observe that its upper bound is given by a polynomial P in the size of the instance, and the total running time for the algorithms is then $\mathcal{O}(|E|^3 + P)$, which is polynomial.

4 Intractability results

In this section we consider the remaining 15 maximal tractable subclasses and show that the $\textsc{D-Isat}$ problem is intractable for these. Several of the proofs make use of the $\textsc{3-Colourability}$ problem, a well-known NP-complete problem, which is defined as follows:

Definition 12 (The 3-Colourability problem).

- $\textsc{Instance}$: *A graph* $G = \langle V, E\rangle$
- $\textsc{Question}$: *Is* G *3-colourable,* i.e., *does there exist a function* $f : V \to \{1, 2, 3\}$ *such that* $f(u) \neq f(v)$ *whenever* $\{u, v\} \in E$*?*

Preparing for the next two lemmata, let $\hat{r}_1 = (\prec\ \succ\ \ \mathsf{d}\ \mathsf{d}^{-1}\ \mathsf{s}\ \mathsf{s}^{-1}\ \mathsf{f}\ \mathsf{f}^{-1})$, $\hat{r}_2 = \hat{r}_1 - \{\mathsf{s}\ \mathsf{s}^{-1}\}$, $\hat{r}_3 = \hat{r}_1 - \{\mathsf{f}\ \mathsf{f}^{-1}\}$ and $\cap = \mathcal{A} - (\prec\succ)$.

Lemma 2. $\textsc{D-Isat}(\{\hat{r}_i, \cap\})$, $1 \leq i \leq 3$, *is NP-complete.*

Proof. Pe'er & Shamir [PS97] have shown that $\textsc{D-Isat}(\{(\prec\ \succ), \cap\})$ is NP-complete even if all intervals are required to have length one. Observe that under this restriction, there are no intervals I, J such that $I(\mathsf{d}\ \mathsf{d}^{-1}\ \mathsf{s}\ \mathsf{s}^{-1}\ \mathsf{f}\ \mathsf{f}^{-1})J$. Consequently, the problem $\textsc{D-Isat}(\{\hat{r}_i, \cap\})$ is NP-complete.

Lemma 3. D-ISAT($\mathcal{X}$) *is NP-complete for every* $\mathcal{X} \in \mathbf{A}_{r,b} \cup \{\mathcal{S}(\mathsf{d}), \mathcal{E}(\mathsf{d}), \mathcal{S}^*, \mathcal{E}^*\}$.

Proof. Arbitrarily choose $\mathcal{X} \in \mathbf{A}_{r,b} \cup \{\mathcal{S}(\mathsf{d}), \mathcal{E}(\mathsf{d}), \mathcal{S}^*, \mathcal{E}^*\}$, note that $\{\hat{r}_i, \cap\} \subseteq \mathcal{X}$ for some $1 \leq i \leq 3$, and apply the previous lemma.

Lemma 4. D-ISAT($A_{\equiv}$) *is NP-complete.*

Proof. Polynomial-time reduction from 3-COLOURABILITY. Let $G = \langle V, E \rangle$ be an arbitrary undirected graph. Construct an instance Γ of D-ISAT($A_{\equiv}$) as follows:

1. introduce a variable x and the duration constraint $d(x) = 2$;
2. for each $v \in V$, let v denote a fresh variable and add the constraints $v(\equiv \mathsf{m}\ \mathsf{m}^{-1})x$ and $d(v) = 2$.
3. for each edge $(v, w) \in E$, introduce a fresh variable S (which we refer to as the *separating interval*) and the constraints $d(S) = 3$, $S(\equiv\ \succ\ \mathsf{o}\ \mathsf{o}^{-1})v$ and $S(\equiv\ \mathsf{m}\ \mathsf{m}^{-1}\ \mathsf{s}^{-1})w$.

We show that Γ has a model iff G is 3-colourable. First of all, it is obvious that each node variable is assigned a value from three possible choices. We have to prove that for each edge $(v, w) \in E$, the corresponding interval variables are assigned distinct values *and* that the separating interval can be assigned a value consistent with the choice of v and w. It is clear that v and w never can be equal since $S(\equiv\ \succ\ \mathsf{o}\ \mathsf{o}^{-1})v$, $S(\equiv\ \mathsf{m}\ \mathsf{m}^{-1}\ \mathsf{s}^{-1})w$, $d(S) \neq d(v)$ and $d(S) \neq d(w)$. Figure 3 shows how the interval S can be chosen in order to satisfy the constraints; the two cases where $v = (x^-, x^+), w = (x^+, x^+ + 1)$ and $w = (x^-, x^+), v = (x^+, x^+ + 1)$ are omitted since they are analogous to the two cases in the top part of the figure.

Before proving the NP-completeness of D-ISAT($\mathcal{S}(\mathsf{o}^{-1})$), we need a construction that places seven intervals "in a row", *i.e.*, $I_1(\mathsf{m})I_2(\mathsf{m})...(\mathsf{m})I_7$. Define $Q = \langle V, E, X \rangle$ to be an instance of D-ISAT.

Let $R \in \mathcal{S}(\mathsf{o}^{-1})$ be the relation $(\mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{m}\ \mathsf{m}^{-1})$. Let V contain the constraints

$$I_1(\mathsf{s})I,\ I_2RI_1,\ I_2RI_3,\ I_3RI_2,\ I_3RI_4,\ I_4RI_3$$
$$I_4RI_5,\ I_5RI_4,\ I_5RI_6,\ I_6RI_5,\ I_6RI_7,\ I_7(\mathsf{o}^{-1}\ \mathsf{f})I$$

and set $X = \{d(I_i) = 1 \mid 1 \leq i \leq 7\} \cup \{d(I) = 7\}$.

We can deduce $I_1^- = I^-$ and $I_7^+ \geq I^+$. Having thus constrained the positions of the first and last intervals, in order for the seven intervals to 'reach' all the way from I^- to I^+, it has to be the case that $I_1(\mathsf{m})I_2(\mathsf{m})I_3, \ldots, I_6(\mathsf{m})I_7$ and $I_7(\mathsf{f})I$. The resulting order is shown in figure 4.

Lemma 5. D-ISAT($\mathcal{S}(\mathsf{o}^{-1})$) *and* D-ISAT($\mathcal{E}(\mathsf{o})$) *are NP-complete.*

Proof. Polynomial-time reduction from 3-COLOURABILITY. Let $G = \langle V, E \rangle$ be an arbitrarily chosen undirected graph. We begin by using the construction with seven intervals $I_1, \ldots, I_7$ presented above. For each $v \in V$, introduce a fresh variable v, add the following four constraints:

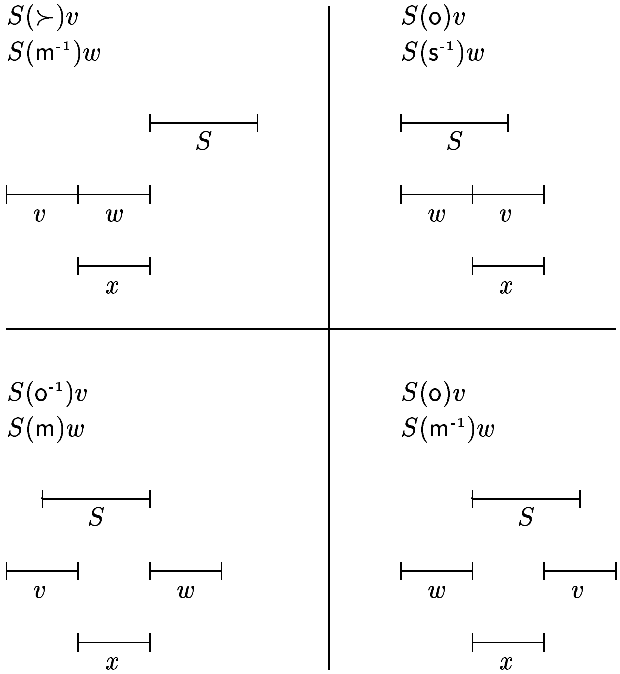

Fig. 3. The separating interval in the proof of lemma 4.

1. $v(\succ\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{s}^{-1})I_1$;
2. $v(\succ\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{s}^{-1}\ \mathsf{f}^{-1})I_3$;
3. $v(\prec\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{s}^{-1}\ \mathsf{f}^{-1})I_5$;
4. $v(\prec\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{f}^{-1})I_7$.

and the duration constraint $d(v) = 3$. By noting that none of the constraints 1.-4. above contains the relation d^{-1} and remembering that we have assigned the duration 3 to v, v cannot overlap or be overlapped by any of the intervals I_k, $k \in \{1, 3, 5, 7\}$. It is now quite easy to realize that v can only be assigned three different intervals (depicted in figure 4). Finally, for each $(v, w) \in E$, we add the constraint $v \neq w$[3] thereby forcing adjacent nodes to be coloured differently.

To make the proof complete, we observe that it is trivial to construct a model of the given constraints whenever we have a valid 3-colouring of the graph G.

By exchanging $I_1(\mathsf{s})I$ for $I_1(\mathsf{o}\ \mathsf{s})I$ and $I_7(\mathsf{o}^{-1}\ \mathsf{f})I$ for $I_7(\mathsf{f})I$ in the construction of the seven intervals, we get the proof for the NP-completeness of D-ISAT($\mathcal{E}(\mathsf{o})$).

To summarize this section, we state the following theorem.

Theorem 4. *$\mathcal{H}$, $\mathcal{S}(\succ)$ and $\mathcal{E}(\prec)$ are the only known maximal tractable subclasses of $\mathcal{A}$ for which the* D-ISAT *problem is tractable.*

[3] I.e., any relation can hold between v, w except $\equiv$.

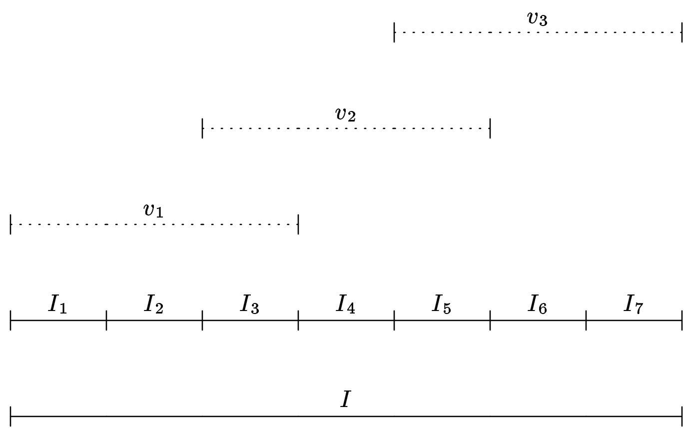

Fig. 4. The construction of the seven intervals and the three possible positionings of v in lemma 5.

Proof. By theorem 3, lemmata 3, 4 and 5 and the discussion at the beginning of the previous section.

5 Scheduling

With the results from the previous sections in mind, we now turn our attention to a related problem, that of *scheduling* a set of tasks within a given time frame.

We define the computational problem $\textsc{Sched}(\mathcal{S})$ as follows:
$\textsc{Instance}$: An instance Θ of $\mathrm{ISAT}(\mathcal{S})$ with variable set V together with

1. a duration $d(I) \in \mathbb{Z}^+$ for each $I \in V$; and
2. a *schedule length* $K \in \mathbb{Z}^+$.

$\textsc{Question}$: Is there a model M of Θ such that

1. $M(I^+) - M(I^-) = d(I)$ for all $I \in V$; and
2. $K \geq (\max_{I \in V} M(I^+)) - (\min_{I \in V} M(I^-))$.

Lemma 6. $\textsc{Sched}(\mathcal{H})$ *is tractable.*

Proof. Let Θ be an instance of $\textsc{Sched}(\mathcal{H})$ with schedule length K, variable set V and constraint set E. Let $\nu = \{I^+ - I^- = d(I) \mid I \in V\}$ where $d(I)$ is the duration of interval I. Consider the relation $r = (\equiv\ \mathsf{s}\ \mathsf{f}\ \mathsf{d}) \in \mathcal{H}$. This relation has the property that given $J_0 r J_1$, $J_0^- \geq J_1^-$ and $J_0^+ \leq J_1^+$, *i.e.*, r expresses the fact that J_0 is inside J_1. Now add a fresh interval variable J together with the constraint $J^+ - J^- = K$ to ν, and for each $I \in V$ add the constraint IrJ. $\nu \cup E$ give us an instance Θ' of $\textsc{D-Isat}(\mathcal{H})$, which is tractable by the previous section.

Lemma 7. $\textsc{Sched}(\mathcal{S}(\succ))$ *and* $\textsc{Sched}(\mathcal{E}(\prec))$ *are NP-complete.*

Proof. Polynomial-time reduction from 3-COLOURABILITY. Let $G = \langle V, E \rangle$ be an arbitrary undirected graph. Construct an instance Θ of SCHED($\mathcal{S}(\succ)$) as follows:

1. introduce three variables A, B and C with durations $d(A) = d(B) = d(C) = 1$ together with the constraints $A(\prec \succ \ \mathsf{m})B$ and $B(\prec \succ \ \mathsf{m})C$;
2. for each $v \in V$, introduce a fresh variable v with duration $d(v) = 1$ and the constraint $v(\equiv \prec \succ \ \mathsf{m}\ \mathsf{m}^{-1})B$;
3. for each $(v, w) \in E$, add the constraint $v(\prec \succ \ \mathsf{m}\ \mathsf{m}^{-1})w$;
4. let the schedule length K have the value 3.

We claim that G is 3-colourable iff Θ has a solution. To see this, we begin by noting that in every model of Θ, $A(\mathsf{m})B$ and $B(\mathsf{m})C$ and that every variable corresponding to a node in G equals exactly one of A, B, C. By interpreting A, B, C as the three colours, it is obvious that the constraints imposed in step 3 above forces adjacent nodes to have different colours. For the other direction, we merely observe that it is trivial to construct a valid model of Θ given a 3-colouring of the graph G.

The proof for SCHED($\mathcal{E}(\prec)$) is symmetric.

This section can be summarized in the following theorem.

Theorem 5. *Out of the 18 known maximal tractable subclasses of Allen's algebra, $\mathcal{H}$ is the only subclass for which the scheduling problem is tractable.*

Proof. We note that if the D-ISAT problem is intractable for a given subclass, then this must also be the case for the scheduling problem for this subclass. This excludes all but $\mathcal{H}$, $\mathcal{S}(\succ)$ and $\mathcal{E}(\prec)$. By lemma 7 we know that neither of the problems SCHED($\mathcal{S}(\succ)$) and SCHED($\mathcal{E}(\prec)$) are tractable. The only remaining subclass, $\mathcal{H}$, is tractable by lemma 6 and the result follows.

6 Discussion and conclusions

In this paper we have studied complexity issues in Allen's algebra extended for handling metric durations. Fifteen of the eighteen known maximal tractable subclasses have been shown to become intractable when metric durations are added, leaving only three for which the satisfiability problem is tractable.

We have extended previous results regarding the scheduling problem and shown that the only known maximal tractable subclass of Allen's algebra for which this problem is tractable is the ORD-Horn subalgebra.

While a complete classification of the subclasses of Allen's algebra is yet to be made, we already know much about the 'undiscovered' classes. It was shown in [DJ97b] that $\mathcal{S}(\succ)$ and $\mathcal{E}(\prec)$ are the only possible maximal tractable subclasses containing the relation $(\prec \succ)$ and thus can express *sequentiality*, which is considered an important notion in many AI contexts, for example planning. In the same paper it was shown that a new tractable subclass can contain at most the basic relations ($\equiv$), (b) and (b^{-1}), where $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This quells most hope of finding a subclass more expressive than any known today, while still retaining tractability.

It is important to note that while we have shown intractability of the scheduling problem for seventeen of the known maximal tractable subclasses of Allen's algebra, this does not rule out the existence of interesting subclasses which are *not* maximal.

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