Exploiting Bipartiteness to Identify Yet Another Tractable Subclass of CSP

Marcus Bjäreland*
Peter Jonsson**

Dept. of Comp. and Info. Sci.
Linköpings universitet
S-581 83 Linköping, Sweden
{marbj, petej}@ida.liu.se

Abstract. The class of constraint satisfaction problems (CSPs) over finite domains has been shown to be NP-complete, but many tractable subclasses have been identified in the literature. In this paper we are interested in restrictions on the types of constraint relations in CSP instances. By a result of Jeavons *et al.* we know that a key to the complexity of classes arising from such restrictions is the closure properties of the sets of relations. It has been shown that sets of relations that are closed under constant, majority, affine, or associative, commutative, and idempotent (ACI) functions yield tractable subclasses of CSP. However, it has been unknown whether other closure properties may generate tractable subclasses.
In this paper we introduce a class of tractable (in fact, SL-complete) CSPs based on bipartite graphs. We show that there are members of this class that are not closed under constant, majority, affine, or ACI functions, and that it, therefore, is incomparable with previously identified classes.

1 Introduction

In general, the class of *constraint satisfaction problems* (CSPs) over finite domains is NP-complete. However, much more is known about the complexity of CSP and its variants. For example, Schaefer [14] provided a complete complexity classification of subproblems of CSP with domains of size 2. Other examples of complexity analyses of CSPs can be found in [12, 4, 5], and in more recent work, such as [1, 10, 3].

By looking at subclasses of CSP restricted by the types of constraint relations allowed, Jeavons *et al.* [8] showed that the complexity of such subclasses can be characterized by the functions under which the respective sets of relations are closed. In particular, sets of relations closed under constant, majority, affine, or associative, commutative, and idempotent (ACI) functions are shown to be tractable. It has been unknown whether any tractable CSP subclass exists that is not closed under any of those four types of functions. In this paper we introduce such a class, which is based on a strong result in graph theory by Hell and Nešetřil [6] stating that the H-coloring problem (that is, the problem of finding homomorphisms from graphs to a fixed graph H) is tractable if H is bipartite, and NP-complete otherwise (discussed in Sect. 2). This result is relevant since a solution to a CSP can be seen as a homomorphism from the structure of variables in a CSP instance to the constraint relations [9]. Thus, for CSPs with only one binary relation we have tractability if the relation defines a bipartite graph, and NP-completeness otherwise. In fact, this result does by itself yield a new class of tractable CSPs. We show that the bipartite graph C_6, that is, the cycle of length 6, is a counterexample to each of the four closure functions (Sect. 4.1). However, for CSPs with more than one constraint relation bipartiteness alone does not provide

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tractability. We identify (in Sect. 4) three global properties on the sets of relations (seen as graphs) in a CSP, namely that every relation is in itself bipartite (**l**ocal bipartiteness), that all relations have the same partitions (**p**artition equivalence), and that all relations have at least one edge in common (**n**on-disjoint). The class of all sets of relations with these three properties is denoted **lpn**. We show (in Theorem 4) that the problem CSP(Γ) for $\Gamma \in$ **lpn** is tractable. In Corollary 1 we strengthen this by showing that this problem in fact is complete for the complexity class SL (Symmetric Logspace).

In Sect. 5 we show that any attempt to remove any of the three restrictions results in NP-completeness, and here we again rely on Hell and Nešetřil's theorem.

As a result of the work presented in this paper we have identified a number of open questions, which we state in Sect. 6.

2 Preliminaries

In this section we define the concepts that will be used in this paper.

Definition 1. *Let $\Gamma = \{R_1, \ldots, R_n\}$ be a set of relations over a domain D, such that R_i has arity k_i. An* instance *of the constraint satisfaction problem over Γ, $\mathcal{P}$, is a tuple,*

$$\mathcal{P} = \langle V, D, R_1(S_1), \ldots, R_n(S_n) \rangle$$

where

- *V is a finite set of* variables*;*
- *D is the finite domain;*
- *Each pair $R_i(S_i)$ is a* constraint*, where $R_i \in \Gamma$, and S_i is an ordered list of k_i variables.*

Definition 2 (Solution). *A* solution *to $\mathcal{P} = \langle V, D, R_1(S_1), \ldots, R_n(S_n) \rangle$ is a function $h : V \to D$ such that $h(S_i) \in R_i$ for all i, where $h(S_i)$ denotes the coordinate-wise application of h to the variables in S_i (i.e. if $S_i = \langle v_1, \ldots, v_{k_i} \rangle$, then $h(S_i) = \langle h(v_1), \ldots, h(v_{k_i}) \rangle$). The set of solutions to an instance $\mathcal{P}$ is denoted $Sol(\mathcal{P})$.*

Given a finite set of relations Γ, we define the computational problem CSP(Γ) as follows: given a CSP instance over Γ, does it have a solution?

Following the work on constraints and universal algebra in [9] we can equivalently define a solution to an instance as a homomorphism between two algebraic structures.

Definition 3 (CSP Homomorphism). *Let $\mathcal{P} = \langle V, D, R_1(S_1), \ldots, R_n(S_n) \rangle$ be a CSP where each relation R_i has arity k_i. Then construct $\Sigma = \langle V, \{S_1\}, \ldots, \{S_n\} \rangle$ and $\Sigma' = \langle D, R_1, \ldots, R_n \rangle$. A CSP* homomorphism *is a function $g : V \to D$ such that for all $i = 1, \ldots, n$,*

$$\langle v_1, \ldots, v_{k_i} \rangle \in \{S_i\} \Rightarrow \langle g(v_1), \ldots, g(v_{k_i}) \rangle \in R_i.$$

The set of all homomorphisms for $\mathcal{P}$ is denoted $Hom(\mathcal{P})$.

We establish the relation between the solutions to a CSP instance and CSP homomorphism.

Proposition 1 ([9]).

$$Sol(\mathcal{P}) = Hom(\mathcal{P}),$$

for an instance $\mathcal{P}$.

As we will rely heavily on viewing binary relations as graphs, we define the necessary concepts here.

Definition 4. *A* graph, *G, is a tuple, $G = \langle V, E\rangle$ where V is a (non-empty) set of* vertices, *and E a set of* edges, *$\langle v_i, v_j\rangle$, such that $v_i, v_j \in V$. Two vertices $v_i, v_j \in V$ are* adjacent *if $\langle v_i, v_j\rangle \in E$ or $\langle v_j, v_i\rangle \in E$. A* cycle *in a graph is a sequence of vertices, $v_{i_1}, v_{i_2}, \ldots, v_{i_m}$ such that $\langle v_{i_j}, v_{i_{j+1}}\rangle \in E$ for $j = 1, \ldots, n-1$, and $\langle v_{i_n}, v_{i_1}\rangle \in E$. The natural number m is the* length *of the cycle. A graph is* undirected *if, whenever $\langle v_i, v_j\rangle \in E$, then $\langle v_j, v_i\rangle \in E$. A graph is* bipartite *if it is possible to partition V into two disjoint sets X and Y, such that every edge with the left vertex in one of the partitions has the right vertex in the other partition, and no edges contain vertices from the same partition. Equivalently, a bipartite graph is a graph without cycles of odd length.*

Unless otherwise stated, we will assume that graphs are irreflexive, that is, that for any $v \in V$, $\langle v, v\rangle \notin E$. Moreover, all isolated vertices (that is, vertices that does not belong to any edge) are assumed to belong to one partition.

Definition 5 (Graph Homomorphism). *Let $G = \langle V, E\rangle$ and $G' = \langle V', E'\rangle$ be two graphs. A graph* homomorphism *from G to G' is a function $f : V \to V'$ such that, if v_i and v_j are adjacent vertices in G, then $f(v_i)$ and $f(v_j)$ are adjacent vertices in G'.*

For graphs, the k-coloring problem, for natural numbers k, is defined as the problem of finding a function $f : V \to \{0, 1, \ldots, k-1\}$ such that adjacent vertices in the graph are not mapped to the same number. We will use the fact that 2-coloring is a tractable problem.

A more general problem, for a fixed graph H, is the problem of deciding whether there exists a graph homomorphism from a graph G to H. This problem is called is called $H-coloring$. For H-coloring we have the following strong complexity result, proven in [6]. Note that for a non-irreflexive graph H, H-coloring is trivial.

Theorem 1 ([**6**])**.** *Let H be a fixed undirected graph. If H is bipartite then the H-coloring problem is in P. If H is not bipartite then the H-coloring problem is NP-complete.*

We can now prove

Proposition 2. *For a symmetric binary relation, R, CSP($\{R\}$) is tractable if the graph $\langle D, R\rangle$, where D is the domain of R, is bipartite, and NP-complete otherwise.*

Proof. Follows immediately from Proposition 1 and Theorem 1.

For instances with more than one constraint, it does not suffice that every relation is bipartite for tractability. As we will show in Sect. 5, it is necessary to impose global restrictions on the set of relations.

Since we only consider binary constraint relations in this paper, we will use the words "graph" and "relation" interchangeably.

3 Closure and complexity

We will be interested in closure properties of sets of binary relations, which motivates the following definition.

Definition 6. *Given a binary relation R, and a function $\phi : D^n \to D$, we say that R is closed under ϕ, if for all sets of tuples*

$$\begin{gathered}\langle d_1^1, d_2^1\rangle \in R \\ \vdots \\ \langle d_1^n, d_2^n\rangle \in R\end{gathered}$$

the tuple

$$\langle \phi(d_1^1, \ldots, d_1^n), \phi(d_2^1, \ldots, d_2^n)\rangle$$

also belongs to R.

Below we will assume that Γ is a set of relations over a finite set D with at least two elements. The set of all functions $\phi : D^n \to D$, any n, under which every member of Γ is closed, will be denoted $Fun(\Gamma)$.

If we define $\phi(R)$ to be the binary relation

$$\{\phi(d_1, d_2) \mid \langle d_1, d_2\rangle \in R\},$$

we can equivalently define R to be closed under ϕ iff $\phi(R) \subseteq R$. From this definition it is easy to prove

Theorem 2 ([8]). *For any set of finite relations Γ, and any $\phi \in Fun(\Gamma)$, there is a polynomial reduction from CSP(Γ) to CSP($\phi(\Gamma)$), with $\phi(\Gamma) = \{\phi(R) \mid R \in \Gamma\}$. That is, under the polynomial reduction CSP(Γ) is satisfiable iff CSP($\phi(\Gamma)$) is satisfiable.*

From Theorem 2 it follows that if $Fun(\Gamma)$ contains a non-injective unary function, then CSP(Γ) can be reduced to a problem with smaller domain. We say that Γ is *reduced* if $Fun(\Gamma)$ does not contain any non-injective unary functions.

Theorem 3 ([8]). *For any reduced set of relations Γ over a finite set D the set $Fun(\Gamma)$ must contain at least one of the following six types of functions:*

1. *A constant function;*
2. *A binary* idempotent *function, that is, a function ϕ such that $\phi(d, d) = d$ for all $d \in D$;*
3. *A ternary* majority *function, that is, a function ϕ such that $\phi(d, d, d') = \phi(d, d', d) = \phi(d', d, d) = d$ for all $d, d' \in D$;*
4. *A ternary* affine *function, that is, a function ϕ such that $\phi(d_1, d_2, d_3) = d_1 - d_2 + d_3$ for all $d_1, d_2, d_3 \in D$, where $\langle D, +\rangle$ is an abelian group;*
5. *A* semiprojection, *ϕ, that is, for $n \geq 3$, there exists $i \in \{1, \dots, n\}$ such that for all $d_1, \dots, d_n \in D$ with $|\{d_1, \dots, d_n\}| < n$, we have $\phi(d_1, \dots, d_n) = d_i$;*
6. *An* essentially unary *function, that is, a function ϕ of arity n such that $\phi(d_1, \dots, d_n) = f(d_i)$ for some i and some non-constant unary function f, for all $d_1, \dots, d_n \in D$.*

The complexity and closure function results in [8] can be summarized as follows:

- If $Fun(\Gamma)$ contains a constant function, then CSP(Γ) is tractable.
- If $Fun(\Gamma)$ contains a binary function that is associative, commutative, and idempotent (ACI), then CSP(Γ) is tractable.
- If $Fun(\Gamma)$ contains a majority function, then CSP(Γ) is tractable.
- If $Fun(\Gamma)$ contains an affine function, then CSP(Γ) is tractable.
- If $Fun(\Gamma)$ contains only semiprojections, then CSP(Γ) is NP-complete.
- If $Fun(\Gamma)$ contains only essentially unary functions, then CSP(Γ) is NP-complete.

Classes of problems that are tractable due to the closure properties have been extensively studied in the literature. For instance, the class of max-closed constraints [10] are closed under an ACI function [8], and the class of CRC constraints [2] are closed under a majority function [7].
In [8] Jeavons *et al.* state that

> It is currently unknown whether there are tractable sets of relations closed under some combination of semiprojections, unary operations, and binary operations which are not included in any of the tractable classes above.

Below, we will introduce a class that is not closed under constant, ACI, majority, or affine functions.

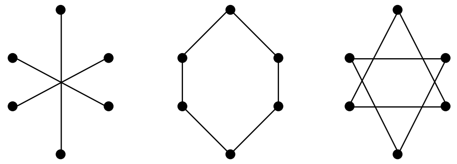

Fig. 1. Composition of the two bipartite relations R_1 and R_2 yielding a non-bipartite relation $R_1 \circ R_2$.

4 A new tractable subclass

We saw in Proposition 2 that for a single binary and bipartite constraint relation CSP was tractable. However, if we introduce more relations it is easy to see that bipartiteness alone on the union of the relations does not yield tractability.

Definition 7. Intersection *and* union *of relation is defined as set theoretic intersection and union.* Composition *of binary relations R_1 and R_2 is defined as* $R_1 \circ R_2 = \{\langle x, z\rangle \mid \exists y.\langle x, y\rangle \in R_2 \wedge \langle y, z\rangle \in R_1\}$. *We often write R^2 instead of $R \circ R$.*

Note that if CSP($\{R_1 \circ R_2\}$) is NP-complete, then CSP($\{R_1, R_2\}$) is too, since we can go back and forth from $\{R_1 \circ R_2\}$ to $\{R_1, R_2\}$ in polynomial time. This also holds for intersection.

Example 1. In Fig. 1 two bipartite relations, R_1 and R_2, are depicted. By composing them we get a relation $R_1 \circ R_2$ that contains cycles of odd length, which means that the composition is not bipartite. Thus, by Theorem 2, CSP($\{R_1, R_2\}$) is NP-complete.

Definition 8 (lpn). *Let $\Gamma = \{R_1, \dots, R_n\}$ be a set of binary symmetric relations over a finite domain D, and construct the graphs $G_1 = \langle D, R_1\rangle, \dots, G_n = \langle D, R_n\rangle$. Γ is said to be* locally bipartite *if G_i is bipartite, for all i. If Γ is locally bipartite, with partitions X_i, Y_i for G_i we say that Γ is* partition equivalent *if $X_1 = X_2 = \dots = X_n$ and $Y_1 = Y_2 = \dots = Y_n$. Furthermore, Γ is said to be* non-disjoint *if $R_1 \cap \dots \cap R_n \neq \emptyset$.*
We will refer to the class of locally bipartite, partition equivalent, non-disjoint sets of relations as **lpn**.

Henceforth, when we write CSP(Γ), we assume that $\Gamma \in$ **lpn**.

Theorem 4. CSP(Γ) *is tractable.*

Proof. Given $\mathcal{P} = \langle V, D, R_1(x_1, x_1'), \dots, R_n(x_n, x_n')\rangle \in$ CSP(Γ) we show that there exists a solution to $\mathcal{P}$ iff the graph $G_V = \langle V, \{\langle x_1, x_1'\rangle, \dots, \langle x_n, x_n'\rangle\}\rangle$ is 2-colorable.
$\Leftarrow$). Assume that G_V is 2-colorable, and choose $\langle d, d'\rangle \in \bigcap_{i=1}^{n} R_i$ (which exists since the set of relations is non-disjoint). Color G_V with d and d'. Clearly, this coloring is a solution to $\mathcal{P}$.
$\Rightarrow$). Let $h : V \to D$ be a solution to $\mathcal{P}$. Since the set of relations is bipartite and partition equivalent we name the partitions X and Y and construct a function $f : V \to \{0, 1\}$, as follows:

$$f(v) = \begin{cases} 0 & \text{if } h(v) \in X, \\ 1 & \text{if } h(v) \in Y. \end{cases}$$

Clearly, f is a 2-coloring of G_V.

It is known that 2-coloring is $co - SL$-complete [11], that is, it is complete for the complement of the class of symmetric logspace problems. Moreover, Nisan and Ta-Schma have shown that $SL = co - SL$ [13], which gives us the following:

Corollary 1. CSP(Γ) *is SL-complete.*

Proof. In the proof of Theorem 4, CSP(Γ) is trivially reduced to 2-colorability (and *vice versa*). Thus we can immediately apply Reif's result [11] followed by Nisan and Ta-Schma's result [13].

4.1 Non-closure properties of lpn

In this section we will show that some sets of relations in **lpn** are not closed under constant, majority, ACI, or affine functions. Consider the graph C_6 in Fig. 2 representing a bipartite relation

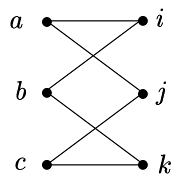

Fig. 2. The graph C_6, with partitions $\{a, b, c\}$ and $\{i, j, k\}$.

R. Since the graph is irreflexive, we can immediately see that $\{R\}$ is not closed under a constant function.
For the existence of a majority function d we can note that

$$\begin{aligned} d(a, a, b) &= a \\ d(b, a, b) &= b \\ d(b, c, c) &= c \\ d(i, j, k) &= x, \end{aligned}$$

with $x \in \{a, b, c, i, j, k\}$. We will show that x cannot be chosen such that C_6 is closed under a majority function. Consider the following three edges of C_6: $\langle a, i\rangle, \langle a, j\rangle, \langle b, k\rangle$ we can see that if we apply d to the edges component-wise, we get that $x \in \{i, j\}$ for C_6 to be closed under any majority function. For the following three edges: $\langle b, i\rangle, \langle a, j\rangle, \langle b, k\rangle$, we get $x \in \{i, k\}$. Finally, for $\langle b, i\rangle, \langle c, j\rangle, \langle c, k\rangle$ we get $x \in \{j, k\}$. Thus, there is no choice of x that satisfies the three triplets simultaneously, and therefore there cannot exist a majority function under which C_6 is closed.
Next, we turn our attention to ACI functions. We can easily see that undirected and irreflexive graphs are not closed under any *commutative* function, that is, if d is commutative then $d(x, y) = d(y, x)$, for all $x, y \in D$. If the graph is undirected there exists a pair of edges, $\langle x, x'\rangle$ and $\langle x', x\rangle$, and if we apply d component-wise to the edges we get $\langle d(x, x'), d(x', x)\rangle = \langle y, y\rangle$, for some $y \in D$. Since the graph was irreflexive, it cannot be closed under an ACI function.
Finally, we prove that C_6 is not closed under affine functions. Consider the three edges of C_6: $\langle a, i\rangle, \langle a, j\rangle, \langle c, j\rangle$. Choose $+$ so that $\langle D, +\rangle$ is an abelian group, and let $-x$ denote the inverse of the element $x \in D$. Next, we consider $\langle a + (-a) + c, i + (-j) + j\rangle$ which is the component-wise application of any affine function on the three edges. Since $\langle D, +\rangle$ is associative and that there exists a neutral group element, we have

$$\langle a + (-a) + c, i + (-j) + j\rangle = \langle (a + (-a)) + c, i + ((-j) + j)\rangle = \langle c, i\rangle,$$

which is not an edge in C_6. Thus, we have showed the following

Proposition 3. *Members in* **lpn** *are not in general closed under constant, majority, ACI, or affine functions.*

We have shown that members of **lpn** do not have closure properties that are known to yield tractable CSP instances. However, in similar spirit as the proof of Theorem 4, we can construct a unary non-injective function under which members of **lpn** are closed.

Let $\Gamma = \{R_1, \ldots, R_n\} \in$ **lpn**, X and Y be the two partitions, and $\langle d_X, d_Y\rangle \in \bigcap_{i=1}^{n} R_i$, such that $d_X \in X$ and $d_Y \in Y$. Then, construct the unary function $f : D \to D$

$$f(d) = \begin{cases} d_X & \text{if } d \in X, \\ d_Y & \text{if } d \in Y. \end{cases}$$

Since every edge in any relation has one vertex in X, and the other in Y, Γ is closed under f. Whenever the domain of Γ is larger than 2, f is non-injective. If we then look at $f(\Gamma)$ we can see that the remaining domain only contains 2 elements, namely d_X and d_Y, and that there are no non-injective unary functions under which this new relation is closed. The relation $\{\langle d_X, d_Y\rangle, \langle d_Y, d_X\rangle\}$ is thus reduced, and it is a trivial exercise to find a majority function under which it is closed.

5 The restrictions on lpn cannot be removed

The tractability proof of CSP(Γ) (Theorem 4) relies on the locally bipartite, partition equivalent, and non-disjoint properties of the sets of relations of the instances. We will now show that we cannot remove any of the three properties and still maintain tractability. The arguments will be similar to that in Example 1, that is, we start with some graphs, put them together somehow and end up with a non-bipartite result on which we can apply Hell and Nešetřil's theorem (Theorem 1). Clearly, if we remove the local bipartiteness property, we have an NP-complete problem, so we direct the attention to the other two cases.

5.1 Disjoint relations

In Example 1 we saw an example of two bipartite and partition equivalent relations, without any common edge (that is, disjoint relations). By composition we constructed a non-bipartite relation, which by Proposition 2 gives us NP-completeness.

5.2 Non-partition equivalent relations

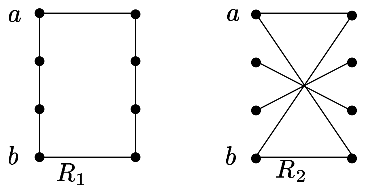

Fig. 3. Two relations R_1 and R_2 that are bipartite and non-disjoint, but does not have the same partitions.

In Fig. 3 we can see two relations that are locally bipartite and non-disjoint, but that does not have the same partitions (the two vertices a and b cannot belong to the same partition of both relations). In Fig. 4 we see that by composing R_1 with itself, we get a reflexive relation (remember

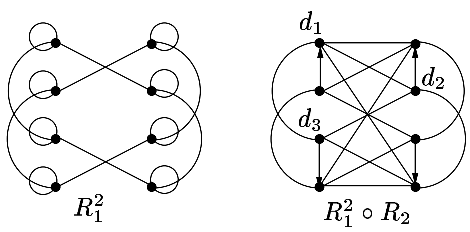

Fig. 4. R_1 composed with itself, and the result of $R_1^2 \circ R_2$, which is not bipartite due to the odd cycle between d_1, d_2, and d_3.

that the corresponding CSP then is trivially tractable), but when R_1^2 is composed with R_2 we get an odd cycle between the vertices d_1, d_2, and d_3. Note that the second graph in Fig. 4 is directed. We can, however, easily restore the undirectedness by intersecting the graph with its complement. NP-completeness then follows from Proposition 2.

6 Conclusion and open problems

In this paper we have exploited the notion of bipartite graphs to identify a new class of tractable CSPs. For a deeper understanding of the results presented in this paper a number of open problems need to solved. Here, we state some of the more interesting of them.

How do we extend $\mathrm{CSP}(\Gamma)$ to a *maximal* tractable class?
It is easy to see that $\mathrm{CSP}(\Gamma)$ is not a maximal tractable class, since we, for example, can add non-irreflexive relations to the sets of relations and still have tractability. However, Hell and Nešetřil's theorem hints that we are quite "close" to a maximal tractable class, but it remains to be investigated how close we really are.

Can the results be generalized to relations with higher arity?
In our tractability proof we rely on 2-coloring. However, for hypergraphs (that is, relations with arity > 2) it is known that 2-coloring is NP-complete. Thus, a generalization would have to rely on some other concept.

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