# Computational Complexity of Temporal Constraint Problems

Thomas Drakengren
Carlstedt Research & Technology
Stora Badhusgatan 18-20, SE-411 21 Göteborg, Sweden
email: draken@crt.se

Peter Jonsson
Department of Computer and Information Science
Linköping University, SE-581 83 Linköping, Sweden
email: petej@ida.liu.se

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**Abstract**

This chapter surveys results on the computational complexity of temporal constraint reasoning. The focus is on the satisfiability problem, but also the problem of entailed relations is treated. More precisely, results for formalisms based upon relating time points and/or intervals with qualitative and/or metric constraints are reviewed. The main purpose of the chapter is to distinguish between tractable and NP-complete cases.

# Contents

# 1 Introduction

The purpose of this chapter is to survey results on the computational complexity of temporal constraint reasoning. To keep the presentation reasonably short, we make a few assumptions:

1. We assume that time is linear, dense and unbounded. This implies that, for instance, we do not consider branching, discrete or finite time structures.

2. We focus on the satisfiability problem, that is, the problem of deciding whether a set of temporal formulae has a model or not. However, we also treat the problem of entailed relations, in the context of Allen's algebra.

3. Initially, we follow standard mathematical praxis and allow temporal variables to be unrelated, *i.e.*, we allow problems where variables may not be explicitly tied by any constraint. In the final section, we study some cases where this assumption is dropped.

Our main purpose is to distinguish between problems that are solvable in polynomial time and problems that are not[1]. As a consequence, we will not necessarily present the most efficient algorithms for the problems under consideration. We will instead emphasize simplicity and generality, which means that we will use standard mathematical tools whenever possible.

This chapter begins, in Section 2, with an in-depth treatment of *disjunctive linear relations* (DLR), here serving two purposes:

1. DLRs will be used as a unifying formalism for temporal constraint reasoning, since it subsumes most approaches that have been proposed in the literature.

2. DLRs will be used extensively for dealing with metric time.

We continue in Section 3 by introducing Allen's interval algebra, and presenting all tractable subclasses of that algebra. We also provide some results on the complexity of computing entailed relations.

[1]Assuming $P \neq NP$, of course.

Section 4 is concerned with *point-interval* relations, in which time points are related to intervals. A complete enumeration of all maximal tractable subclasses are given, together with algorithms for solving the corresponding problems.

In Section 5, the problem of handling metric time is studied. Extensions to Horn DLRs are considered, as well as methods based on arc and path consistency.

Finally, Section 6 contains some "non-standard" techniques in temporal constraint reasoning. We consider, for instance, temporal reasoning involving durations, and the implications of not allowing unrelated variables.

# 2 Disjunctive Linear Relations

## 2.1 Definitions

**Definition 2.1** Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables, and $\alpha, \beta$ linear polynomials (polynomials of degree one) over $X$, with rational coefficients. A *linear relation* over $X$ is a mathematical expression of the form $\alpha R \beta$, where $R \in \{<, \leq, =, \neq, \geq, >\}$.

A *disjunctive linear relation* (DLR) over $X$ is a disjunction of a nonempty finite set of linear relations. A DLR is said to be *Horn* if at most one of its disjuncts is not of the form $\alpha \neq \beta$.

The problem of *satisfiability* for finite sets $D$ of DLRs is denoted $\mathrm{DLR}\textsc{sat}(D)$, which is checking whether there exists an assignment $M$ of variables in $X$ to real numbers, such that all DLRs in $D$ are satisfied in $M$. Such an $M$ is said to be a *model* of $D$. The satisfiability problem for finite sets $H$ of Horn DLRs is denoted $\textsc{horn}\mathrm{DLR}\textsc{sat}(H)$. $\Box$

**Example 2.2**

$$x + 2y \leq 3z + 42.3$$

is a linear relation,

$$(x + 2y \leq 3z + 42.3) \vee (x > \frac{3}{12})$$

is a disjunctive linear relation, and

$$(x + 2y \leq 3z + 42.3) \vee (x \neq \frac{3}{12})$$

is a Horn disjunctive linear relation. □

In principle, the framework of DLRs makes it unnecessary to distinguish between qualitative and metric information. Nevertheless, when it comes to identifying tractable subclasses, the distinction is still convenient.

## 2.2 Algorithms and Complexity

In this section, we present the two main results for computing with DLRs. We also provide a polynomial-time algorithm for checking the satisfiability of Horn DLRs.

**Proposition 2.3** The problem DLRSAT is NP-complete.
**Proof:** The satisfiability problem for propositional logic, which is known to be NP-complete, can easily be coded as DLRs. For the details, see Jonsson and Bäckström [1998]. □

**Proposition 2.4** HORNDLRSAT is solvable in polynomial time.
**Proof:** See Jonsson and Bäckström [1998] or Koubarakis [1996]. □

We will present a polynomial-time algorithm for HORNDLRSAT in Algorithm 2.10. In order to understand it, some auxiliary concepts are needed.

**Definition 2.5** A linear relation $\alpha R \beta$ is said to be *convex* if $R$ is not the relation $\neq$.

Let $\gamma$ be a DLR. We let $\mathcal{C}(\gamma)$ denote the DLR where all nonconvex relations in $\gamma$ have been removed, and $\mathcal{NC}(\gamma)$ the DLR where all convex relations in $\gamma$ have been removed.

We say that $\gamma$ is *convex* if $\mathcal{NC}(\gamma) = \emptyset$, and that $\gamma$ is *disequational* if $\mathcal{C}(\gamma) = \emptyset$. If $\gamma$ is convex or disequational we say that $\gamma$ is *homogeneous*, and otherwise it is said to be *heterogeneous*. We extend these definitions to sets of relations in the obvious way; for example, if $\Gamma$ is a set of DLRs and all $\gamma \in \Gamma$ are Horn, then $\Gamma$ is Horn. □

The algorithm for deciding satisfiability of Horn DLRs is based on linear programming techniques, so we begin by providing the basic facts for that. The linear programming problem is defined as follows.

**Definition 2.6** Let $A$ be an arbitrary $m \times n$ matrix of rational numbers and let $x = (x_1, \ldots, x_n)$ be an $n$-vector of variables over the real numbers. Then an instance of the *linear programming* (LP) problem is defined by $\{\min\ c^{\mathrm{T}}x$ subject to $Ax \leq b\}$, where $b$ is an $m$-vector of rational numbers, and $c$ an $n$-vector of rational numbers. The computational problem is as follows:

1. Find an assignment to the variables $x_1, \ldots, x_n$ such that the condition $Ax \leq b$ holds, and $c^{\mathrm{T}}x$ is minimial subject to these conditions, or

2. Report that there is no such assignment, or

3. Report that there is no lower bound for $c^{\mathrm{T}}x$ under the conditions.

□

Analogously, we can define an LP problem where the objective is to maximize $c^{\mathrm{T}}x$ under the condition $Ax \leq b$. We have the following theorem.

**Theorem 2.7** The linear programming problem is solvable in polynomial time.
**Proof:** Several polynomial-time algorithms have been developed for solving LP. Well-known examples are the algorithms by Khachiyan [1979] and Karmarkar [1984]. □

**Definition 2.8** Let $A$ be a satisfiable set of DLRs and let $\gamma$ be a DLR. We say that $\gamma$ *blocks* $A$ if $A \cup \{d\}$ is unsatisfiable for any $d \in \mathcal{NC}(\gamma)$. □

**Lemma 2.9** Let $A$ be an arbitrary $m \times n$ matrix of rational numbers, $b$ an $m$-vector of rational numbers and $x = (x_1, \ldots, x_n)$ an $n$-vector of variables over the real numbers. Let $\alpha$ be a linear polynomial over $x_1, \ldots, x_n$ and $c$ a rational number. Deciding whether the system $S = \{Ax \leq b, \alpha \neq c\}$ is satisfiable or not is a polynomial-time problem.
**Proof:** Consider the following instances of LP:

LP1= $\{\min\ \alpha$ subject to $Ax \leq b\}$

LP2= $\{\max\ \alpha$ subject to $Ax \leq b\}$

**Algorithm 2.10 (Alg-hornDLRsat($\Gamma$))**

**input** Set $\Gamma$ of DLRs

1 $A \leftarrow \{\gamma \mid \gamma \in \Gamma \text{ is convex}\}$
2 **if** $A$ is not satisfiable **then**
3    **reject**
4 **if** $\exists \beta \in \Gamma$ that blocks $A$ **then**
5    **if** $\beta$ is disequational **then**
6       **reject**
7    **else**
8       Alg-hornDLRsat($(\Gamma - \{\beta\}) \cup \mathcal{C}(\beta)$)
9 **accept**

□

If either LP1 or LP2 has no solutions, then $S$ is not satisfiable. If both LP1 and LP2 yield the same optimal value $c$, then $S$ is not satisfiable, since every solution $y$ to LP1 and LP2 satisfies $\alpha(y) = c$. Otherwise $S$ is obviously satisfiable. Since we can solve the LP problem in polynomial time by Theorem 2.7, the result follows. □

**Theorem 2.11** Algorithm 2.10 correctly solves hornDLRsat in polynomial time.

**Proof:** The test in line 2 can be performed in polynomial time using linear programming, and the test in line 4 can be performed in polynomial time by Lemma 2.9. Thus, the algorithm runs in polynomial time. The correctness proof can be found in [Jonsson and Bäckström, 1998]. □

## 2.3 Subsumed Formalisms

Several formalisms can easily be expressed as DLRs, but more importantly, most proposed tractable temporal formalisms are subsumed by the Horn DLR formalism.

For the following definitions, let $x, y$ be real-valued variables, $c, d$ rational numbers, and $\mathcal{A}$ Allen's algebra [Allen, 1983] (see Section 3 for its definition).

It is trivial to see that the DLR language subsumes Allen's algebra. Furthermore, it subsumes the universal temporal language by Kautz and Ladkin, defined as follows.

**Definition 2.12 (Universal temporal language)** The *universal temporal language* [Kautz and Ladkin, 1991] consists of $\mathcal{A}$, augmented with formulae of the form $-cr_1(x-y)r_2d$, where $r_1, r_2 \in \{<, \leq\}$, and $x, y$ are endpoints of intervals. □

DLRs also subsume the *qualitative algebra* (QA) by Meiri [1996]. In QA, a qualitative constraint between two objects $O_i$ and $O_j$ (each may be a point or an interval), is a disjunction of the form

$$(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$$

where each one of the $r_i's$ is a *basic relation* that may exist between two objects. There are three types of basic relations.

1. *Interval-interval* relations that can hold between a pair of intervals. These relations correspond to Allen's algebra.

2. *Point-point* relations that can hold between a pair of points. These relations correspond to the point algebra [Vilain, 1982].

3. *Point-interval* and *interval-point* relations that can hold between a point and an interval and vice-versa. These relations were introduced by Vilain [1982].

Obviously, DLRs subsume QA. Meiri also considers QA extended with metric constraints of the following two forms, $x_1, \ldots, x_n$ being time points or endpoints of intervals.

1. $(c_1 \leq x_1 \leq d_1) \vee \ldots \vee (c_1 \leq x_n \leq d_1)$;

2. $(c_1 \leq x_n - x_1 \leq d_1) \vee \ldots \vee (c_1 \leq x_n - x_{n-1} \leq d_1)$.

Also this extension to QA can easily be expressed as DLRs. It has been shown that the satisfiability problems for all of these formalisms are NP-complete [Vilain *et al.*, 1989; Kautz and Ladkin, 1991; Meiri, 1996]. In retrospect, the different restrictions imposed on these formalisms seem quite artificial when compared to DLRs, especially since they do not reduce the computational complexity of the problem.

Next, we review some of the formalisms that are subsumed by Horn DLRs.

**Definition 2.13 (Point algebra formulae, pointisable algebra)** A *point algebra formula* [Vilain, 1982] is an expression $xRy$, where $x$ and $y$ are variables, and $R$ is one of the relations $<$, $\leq$, $=$, $\neq$, $\geq$ and $>$.

The *pointisable algebra* [van Beek and Cohen, 1990] is the set of relations in $\mathcal{A}$ which can be expressed as point algebra formulae. $\Box$

The satisfiability problem for point algebra formulae will be denoted PAsat$(H)$, for a set $H$ of point algebra formulae.

**Definition 2.14 (Continuous endpoint formula, continuous endpoint algebra)** A *continuous endpoint formula* [Vilain *et al.*, 1989] is a point algebra formula $xRy$ where $R$ is not the relation $\neq$.

The *continuous endpoint algebra* [Vilain *et al.*, 1989] is the set of relations in $\mathcal{A}$ which can be expressed as continuous endpoint formulae. $\Box$

The following formalism subsumes those of the previous two definitions.

**Definition 2.15 (ORD-Horn algebra)** An *ORD clause* is a disjunction of relations of the form $xRy$, where $R \in \{\leq, =, \neq\}$. The *ORD-Horn* subclass $\mathcal{H}$ [Nebel and Bürckert, 1995] is the set of relations in $\mathcal{A}$ that can be written as ORD clauses containing only disjunctions, with at most one relation of the form $x = y$ or $x \leq y$, and an arbitrary number of relations of the form $x \neq y$. $\Box$

**Definition 2.16 (Koubarakis formula)** Let $R \in \{\leq, \geq, \neq\}$. A *Koubarakis formula* [Koubarakis, 1992] is a formula of one of the following forms:

1. $(x - y)Rc$
2. $xRc$
3. A disjunction of formulae of the form $(x - y) \neq c$ or $x \neq c$.

$\Box$

**Definition 2.17 (Simple temporal constraint)** A *simple temporal constraint* [Dechter *et al.*, 1991] is a formula on the form $c \leq (x - y) \leq d$. $\square$

**Definition 2.18 (Simple metric constraint)** A *simple metric constraint* [Kautz and Ladkin, 1991] is a formula on the form $-cR_1(x - y)R_2d$ where $R_1, R_2 \in \{<, \leq\}$. $\square$

**Definition 2.19 (PA/single-interval formula)** A *PA/single-interval* formula [Meiri, 1996] is a formula on one of the following forms:

1. $cR_1(x - y)R_2\,d$, where $R_1, R_2 \in \{<, \leq\}$
2. $xRy$ where $R \in \{<, \leq, =, \neq, \geq, >\}$

$\square$

**Definition 2.20 (TG-II formula)** A *TG-II* formula [Gerevini *et al.*, 1993] is a formula on one of the following forms:

1. $c \leq x \leq d$,
2. $c \leq x - y \leq d$
3. $xRy$ where $R \in \{<, \leq, =, \neq, \geq, >\}$

$\square$

Besides these classes, other temporal classes that can be expressed as Horn DLRs have been identified by different authors. Examples include the approach by Barber [1993], the subclass $\mathcal{V}^{23}$ for relating points and intervals [Jonsson *et al.*, 1999] (see Section 4), and the temporal part of TMM by Dean and Boddy [1988].

Not all known tractable classes can be modeled as Horn DLRs (in any obvious way[2]), however. Examples of this are Golumbic and Shamir [1993] and Drakengren and Jonsson [1997a, 1997b].

[2]Linear programming is a P-complete problem, so in principle, all polynomial-time computable problems can be transformed into Horn DLRs.

| Basic relation | | Example | Endpoints |
|---|---|---|---|
| $x$ before $y$ | $\prec$ | `xxx` | $x^+ < y^-$ |
| $y$ after $x$ | $\succ$ | `    yyy` | |
| $x$ meets $y$ | $\mathsf{m}$ | `xxxx` | $x^+ = y^-$ |
| $y$ met-by $x$ | $\mathsf{m}^{-1}$ | `    yyyy` | |
| $x$ overlaps $y$ | $\mathsf{o}$ | `xxxx` | $x^- < y^- < x^+$, |
| $y$ overl.-by $x$ | $\mathsf{o}^{-1}$ | `  yyyy` | $x^+ < y^+$ |
| $x$ during $y$ | $\mathsf{d}$ | `  xxx` | $x^- > y^-$, |
| $y$ includes $x$ | $\mathsf{d}^{-1}$ | `yyyyyyy` | $x^+ < y^+$ |
| $x$ starts $y$ | $\mathsf{s}$ | `xxx` | $x^- = y^-$, |
| $y$ started by $x$ | $\mathsf{s}^{-1}$ | `yyyyyyy` | $x^+ < y^+$ |
| $x$ finishes $y$ | $\mathsf{f}$ | `    xxx` | $x^+ = y^+$, |
| $y$ finished by $x$ | $\mathsf{f}^{-1}$ | `yyyyyyy` | $x^- > y^-$ |
| $x$ equals $y$ | $\equiv$ | `xxxx` | $x^- = y^-$, |
| | | `yyyy` | $x^+ = y^+$ |

Table 1: The thirteen basic relations. The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.

# 3 Interval-Interval Relations: Allen's Algebra

## 3.1 Definitions

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval $x$ is represented as a tuple $\langle x^-, x^+\rangle$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1. Denote the set of basic interval relations $\mathbf{B}$. Such disjunctions are represented as *sets* of basic relations, but using a notation such that, for example, the disjunction of the basic intervals $\prec$, $\mathsf{m}$ and $\mathsf{f}^{-1}$ is written $(\prec\ \mathsf{m}\ \mathsf{f}^{-1})$. Thus, we have that $(\prec\ \mathsf{f}^{-1}) \subseteq (\prec\ \mathsf{m}\ \mathsf{f}^{-1})$. The disjunction of all basic relations is written $\top$, and the empty relation is written $\bot$ (this is also used for relations between interval endpoints, denoting "always satisfiable" and "unsatisfiable", respectively). The algebra is provided with the operations of *converse, intersection and composition* on intervals, but we shall need only the converse operation explicitly. The converse operation[3] takes an interval relation $i$ to its converse $i^{-1}$, obtained by inverting each basic relation in $i$, *i.e.*, exchanging $x$ and $y$ in the endpoint relations shown in Table 1.

By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$. There are $2^{8192} \approx 10^{2466}$ such subclasses. Classes that are closed under the operations of intersection, converse and composition are said to be *algebras*.

The problem of *satisfiability* (ISAT) of a set of interval variables with relations between them is that of deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. This is defined as follows.

**Definition 3.1** **(ISAT($\mathcal{I}$))** Let $\mathcal{I} \subseteq \mathcal{A}$ be a set of interval relations. An instance of ISAT($\mathcal{I}$) is a labelled directed graph $G = \langle V, E \rangle$, where the nodes in $V$ are interval variables and $E$ is a subset of $V \times \mathcal{I} \times V$. A labelled edge $\langle u, r, v \rangle \in E$ means that $u$ and $v$ are related by $r$.

A function $M$ taking an interval variable $v$ to its interval representation $M(v) = \langle x^-, x^+ \rangle$ with $x^-, x^+ \in \mathcal{R}$ and $x^- < x^+$, is said to be an *interpretation* of $G$.

An instance $G = \langle V, E \rangle$ is said to be *satisfiable* if there exists an interpretation $M$ such that for each $\langle u, r, v \rangle \in E$, $M(u) r M(v)$ holds, *i.e.*, the endpoint relations required by $r$ (see Table 1) are satisfied by the assignments of $u$ and $v$. Then $M$ is said to be a *model* of $G$.

We refer to the *size* of an instance $G$ as $|V| + |E|$. $\square$

## 3.2 Complexity Results

A complete classification of the computational complexity of ISAT($X$) has been presented by Krokhin *et al.* [2001b]. The classification provides no new tractable subclasses; interestingly, it turns out that all existing tractable

[3]The notation varies for this operation. However, we believe that the standard notation for inverse relations is the best and simplest choice.

subclasses of Allen's algebra had been published in earlier papers [Nebel and Bürckert, 1995; Drakengren and Jonsson, 1997b; Drakengren and Jonsson, 1997a]. For the complete classification, the lengthy proof uses results from a number of earlier publications, *cf.* [Krokhin *et al.*, 2001a; Drakengren and Jonsson, 1998; Nebel and Bürckert, 1995].

Next, we present the main result and the tractable subclasses; after that we present the polynomial-time algorithms for the tractable subclasses.

**Theorem 3.2** Let $X$ be a subset of $\mathcal{A}$. Then $\textsc{Isat}(X)$ is tractable iff $X$ is a subset of the ORD-Horn algebra (Definition 2.15), or of one of the 17 subalgebras defined below. Otherwise, $\textsc{Isat}(X)$ is NP-complete.
**Proof:** See Krokhin *et al.* [2001b]. □

**Definition 3.3 (Subclasses** $A(r, b)$ [Drakengren and Jonsson, 1997b]**)** Let $b \in \{\mathsf{s}, \mathsf{s}^{-1}, \mathsf{f}, \mathsf{f}^{-1}\}$, and $r$ one of the relations

$(\prec\ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{-1})$
$(\prec\ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{-1}\ \mathsf{f}^{-1})$
$(\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f})$
$(\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{-1})$.

containing $b$. First define the subclasses $A_1(b)$, $A_2(r, b)$ and $A_3(r, b)$ by

$$A_1(b) = \{r' \cup (b\ b^{-1}) | r' \in \mathcal{A}\},$$

$$A_2(r, b) = \{r' \cup (b) | r' \subseteq r\}$$

and

$$A_3(r, b) = \{r' \cup (\equiv) | r' \in A_2(r, b)\} \cup \{(\equiv)\}.$$

Then set

$$B = A_1(b) \cup A_2(r, b) \cup A_3(r, b)$$

and finally define the subclass $A(r, b)$ by

$$A(r, b) = B \cup \{x^{-1} | x \in B\} \cup \{(\ )\}.$$

□

For an explicit enumeration of the sets $A(r, b)$, see Drakengren and Jonsson [1997b].

**Definition 3.4 (Subclass $A_{\equiv}$ [Drakengren and Jonsson, 1997b])** Define the subclass $A_{\equiv}$ to contain every relation that contains $\equiv$, and the empty relation ( ). $\square$

**Definition 3.5 (Subclasses $S(b)$ and $E(b)$ [Drakengren and Jonsson, 1997a])** Set $r_s = (\succ\ \mathsf{d}\ \mathsf{o}^{-1}\ \mathsf{m}^{-1}\ \mathsf{f})$, and $r_e = (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})$. Note that $r_s$ contains all basic relations $b$ such that whenever $IbJ$ for interval variables $I$, $J$, $I^- > J^-$ has to hold in any model, and symmetrically, $r_e$ is equivalent to $I^+ < J^+$ holding in any model.

First, for $b \in \{\succ, \mathsf{d}, \mathsf{o}^{-1}\}$, define $S(b)$ to be the set of relations $r$, such that either of the following holds:

$$\begin{array}{rcccl} (b\ b^{-1}) & \subseteq & r & & \\ (b) & \subseteq & r & \subseteq & r_s \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) \\ (b^{-1}) & \subseteq & r & \subseteq & {r_s}^{-1} \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}) \\ & & r & \subseteq & (\equiv\ \mathsf{s}\ \mathsf{s}^{-1}). \end{array}$$

Then, by switching the starting and ending points of intervals, $E(b)$ is defined, for $b \in \{\prec, \mathsf{d}, \mathsf{o}\}$, to be the set of relations $r$, such that either of the following holds:

$$\begin{array}{rcccl} (b\ b^{-1}) & \subseteq & r & & \\ (b) & \subseteq & r & \subseteq & r_e \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}) \\ (b^{-1}) & \subseteq & r & \subseteq & {r_e}^{-1} \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}) \\ & & r & \subseteq & (\equiv\ \mathsf{f}\ \mathsf{f}^{-1}). \end{array}$$

$\square$

**Definition 3.6 (Subclasses $S^*$ and $E^*$ [Drakengren and Jonsson, 1997a])** Let $r_s$ and $r_e$ be as in Definition 3.5, and define $S^*$ to be the set of relations $r$, such that either of the following holds:

$$\begin{array}{rcrcl}
(\equiv\ \mathsf{f}\,\mathsf{f}^{-1}) & \subseteq & r & & \\
(\mathsf{f}\,\mathsf{f}^{-1}) & \subseteq & r & \subseteq & r_s \cup {r_s}^{-1} \\
(\equiv\ \mathsf{f}) & \subseteq & r & \subseteq & r_s \cup (\equiv\ \mathsf{s}\,\mathsf{s}^{-1}) \\
(\equiv\ \mathsf{f}^{-1}) & \subseteq & r & \subseteq & {r_s}^{-1} \cup (\equiv\ \mathsf{s}\,\mathsf{s}^{-1}) \\
(\mathsf{f}) & \subseteq & r & \subseteq & r_s \\
(\mathsf{f}^{-1}) & \subseteq & r & \subseteq & {r_s}^{-1} \\
(\equiv) & \subseteq & r & \subseteq & (\equiv\ \mathsf{s}\,\mathsf{s}^{-1}) \\
 & & r & = & \perp
\end{array}$$

Symmetrically, replacing $\mathsf{f}$ by $\mathsf{s}$ (and their inverses), $(\equiv\ \mathsf{s}\,\mathsf{s}^{-1})$ by $(\equiv\ \mathsf{f}\,\mathsf{f}^{-1})$, and $r_s$ by $r_e$, we get the subclass $E^*$. $\square$

## 3.3 Algorithms

We will now present the tractable algorithms for the subclasses of Allen's algebra presented in the previous section. The proofs of the following claims can be found in [Drakengren and Jonsson, 1997a; Drakengren and Jonsson, 1997b].

- Algorithm 3.8 correctly solves $\textsc{Isat}(A(r,b))$ in polynomial time.
- Algorithm 3.9 correctly solves $\textsc{Isat}(A_{\equiv})$ in polynomial time.
- Algorithm 3.12 correctly solves $\textsc{Isat}(S(b))$ and $\textsc{Isat}(S^*)$ in polynomial time, and exchanging starting and ending points in the algorithm, also $\textsc{Isat}(E(b))$ and $\textsc{Isat}(E^*)$ can be solved in polynomial time.

A definition is needed to understand Algorithm 3.8.

**Definition 3.7 (Strong component)** A subgraph $C$ of a graph $G$ is said to be a *strong component* of $G$ if it is maximal such that for any nodes $a$, $b$ in $C$, there is always a path in $G$ from $a$ to $b$. $\square$

A few definitions are needed for Algorithm 3.12. The observant reader might notice that some of the definitions differ slightly from the original ones [Drakengren and Jonsson, 1997a]. However, the changes were only done in order to improve the presentation; it is easy to see that they are equivalent (and cleaner).

**Algorithm 3.8** **(Alg-**$\textsc{Isat}(A(r,b))$**)**

```
input Instance G = ⟨V, E⟩ of ISAT(𝒜)

Redirect the arcs of G so that all relations
are in A1(b) ∪ A2(r, b) ∪ A3(r, b)
Let G′ be the graph obtained from G by removing arcs which are not
labelled by some relation in A2(r, b) ∪ A3(r, b)
Find all strong components C in G′
for every arc e in G whose relation
   does not contain ≡
   if e connects two nodes in some C then
      reject
accept
```

□

**Definition 3.10** **(**$\mathit{sprel}(r)$**,** $\mathit{eprel}(r)$**,** $\mathit{sprel}^{+}(r)$**,** $\mathit{eprel}^{-}(r)$**)** Take the relation $r \in \mathcal{A}$, let $u$ and $v$ be interval variables, and consider the instance $S$ of $\textsc{Isat}(\{r\})$ which relates $u$ and $v$ with the relation $r$ only. Define the relation $\mathit{sprel}(r)$ on real numbers to be the implied relation between the starting points of $u$ and $v$. That is, for basic relations, we define (the quotation marks are only to avoid notational confusion; the actual relations are intended)

$$\begin{array}{rcl}
\mathit{sprel}(\equiv) & = & \text{“}=\text{”} \\
\mathit{sprel}(\prec) & = & \text{“}<\text{”} \\
\mathit{sprel}(\mathsf{d}) & = & \text{“}>\text{”} \\
\mathit{sprel}(\mathsf{o}) & = & \text{“}<\text{”} \\
\mathit{sprel}(\mathsf{m}) & = & \text{“}<\text{”} \\
\mathit{sprel}(\mathsf{s}) & = & \text{“}=\text{”} \\
\mathit{sprel}(\mathsf{f}) & = & \text{“}>\text{”} \\
\mathit{sprel}(r^{-1}) & = & (\mathit{sprel}(r))^{-1},
\end{array}$$

and for disjunctions, $\mathit{sprel}(r)$ is the relation corresponding to $\bigvee_{b\in r} \mathit{sprel}(b)$. For example, $\mathit{sprel}((\prec\ \succ)) = \text{“}\neq\text{”}$. Symmetrically, we define $\mathit{eprel}(r)$ to be the implied relation between ending points given $r$. Note that $\mathit{sprel}(r)$ and $\mathit{eprel}(r)$ have to be either of $<$, $\leq$, $=$, $\geq$, $>$, $\neq$, $\top$ or $\bot$.

**Algorithm 3.9 (Alg-$\textsc{Isat}(A_{\equiv})$)**

**input** Instance $G = \langle V, E\rangle$ of $\textsc{Isat}(\mathcal{A})$

```
if some arc in G is labelled by ( ) then
    reject
else
    accept
```

□

Further, we define specialisations of these, by $\mathit{sprel}^{+}(r) = \mathit{sprel}(r \cap (\equiv\ \mathsf{f}\,\mathsf{f}^{-1}))$ and $\mathit{eprel}^{-}(r) = \mathit{eprel}(r \cap (\equiv\ \mathsf{s}\,\mathsf{s}^{-1}))$, *i.e.*, the implied relations on starting (ending) points by $r$, given that the ending (starting) points are known to be equal. □

**Definition 3.11 (Explicit starting (ending) point relations)** Let $\mathcal{I} \subseteq \mathcal{A}$, and define the function $\mathit{expl}^{-}$ on instances $G = \langle V, E\rangle$ of $\textsc{Isat}(\mathcal{I})$ by setting

$$\mathit{expl}^{-}(G) = \{u^{-}\mathit{sprel}(r)v^{-} \mid \langle u, r, v\rangle \in E\}.$$

$\mathit{expl}^{-}(G)$ is said to be obtained from $G$ by *making starting point relations explicit.*

Symmetrically, using *eprel* and ending points instead of *sprel* and starting points, $\mathit{expl}^{+}(G)$ is said to be obtained from $G$ by *making ending points explicit.* □

## 3.4 Computing Entailed Relations

Given an instance $\Theta$ of $\textsc{Isat}(\mathcal{I})$ and two distinguished nodes $X$ and $Y$, we define an instance of the *entailed relation problem* ($\textsc{Ient}$) to be the triple $\langle \Theta, X, Y\rangle$, and the computational task as follows: find the smallest set $R$ of basic relations such that $\Theta \cup X(\mathbf{B} - R)Y$ is not satisfiable[4]. $\textsc{Ient}$ is

[4]An equivalent definition of the computational task is the following: find the largest set $R$ of basic relations such that $XRY$ holds in all models of $\Theta$. This is the standard notion of entailment.

**Algorithm 3.12 (Alg-Isat($S(b)$), Alg-Isat($S^*$))**

**input** Instance $G = \langle V, E \rangle$ of Isat($\mathcal{A}$)

$H \leftarrow \mathit{expl}^-(G)$<br>
**if** not PAsat($H$) **then**<br>
    **reject**<br>
$K \leftarrow \emptyset$<br>
**for** each $\langle u, r, v \rangle \in E$<br>
    **if** not PAsat($H \cup \{u^- \neq v^-\}$) **then**<br>
        $K \leftarrow K \cup \{u^- = v^-\}$<br>
    **else**<br>
        $K \leftarrow K \cup \{u^- \neq v^-\}$<br>
$P \leftarrow \{u^+ \mathit{eprel}^-(r) v^+ \mid \langle u, r, v \rangle \in E \wedge u^- = v^- \in H \cup K\}$<br>
**if** not PAsat($P$) **then**<br>
    **reject**<br>
**accept**

□

polynomially equivalent to a number of other computational problems such as the *minimum labelling problem*[5] (MLP) where one computes the entailed relation between *all* pairs of variables.

For the ORD-Horn algebra, it turns out that computing entailed relations is a polynomial-time problem, as proved by Nebel and Bürckert [1995]. We state the simple proof here.

**Theorem 3.13** Ient($\mathcal{H}$) is solvable in polynomial time.
**Proof:** Let $\langle \Theta, X, Y \rangle$ be an instance of Ient($\mathcal{H}$). Using a polynomial-time algorithm for Isat($\mathcal{H}$), one can check whether $\Theta \cup (X(B_i)Y)$ is satisfiable for each $B_i \in \mathbf{B}$. The set of basic relations for which the test succeeds is the relation between $X$ and $Y$ which is entailed by $\Theta$. □

It is easy to see that if Ient($\mathcal{I}$) can be solved in polynomial time, then Isat($\mathcal{I}$) is a polynomial-time problem. Next, we show that the converse

[5] This problem is denoted ISI in [Nebel and Bürckert, 1995].

does not hold in general. Let $r_1 = (\mathsf{m}\ \mathsf{m}^{-1}\ \mathsf{s}\ \mathsf{s}^{-1}\ \mathsf{f}\ \mathsf{f}^{-1})$ and $r_2 = \mathcal{B} - \{\equiv\}$.

**Lemma 3.14** Let $A, B, X$ be intervals such that

1. $A(\prec)B$;
2. $Xr_1A$; and
3. $Xr_2B$.

Then, in any model $I$,

$$[I(X^-), I(X^+)] \in \Big\{ [I(A^-), I(B^-)], [I(A^-), I(B^+)],$$

$$[I(A^+), I(B^-)], [I(A^+), I(B^+)] \Big\}.$$

**Proof:** Easy exercise. □

**Theorem 3.15** If $\mathcal{S}$ is a subclass containing $r_1$ and $r_2$, then $\textsc{Ient}(\mathcal{S})$ is NP-complete.

**Proof:** Polynomial-time reduction from the NP-complete problem 4-Colourability. Let $G = \langle V, E \rangle$ be an arbitrary graph, and construct a set of interval formulae as follows:

1. Introduce two auxiliary interval variables $A$ and $B$;
2. For each $w \in V$, introduce an interval variable $W$ and the relations $Wr_1A, Wr_2B$;
3. For each $(w_1, w_2) \in E$, add the relation $W_1r_2W_2$.

Let $r$ be the entailed relation between $A$ and $B$ in $\Theta$. We claim that $\prec \in r$ iff $G$ is 4-colourable.

*if:* Let $f : V \to \{1, 2, 3, 4\}$ be a legal colouring of the vertices in $G$. We incrementally construct a model $I$ of $\Theta$ such that $A(\prec)B$. First, arbitrarily choose $I$ such that $I(A)(\prec)I(B)$. For each $w \in V$, let

1. $I(W) = [I(A^-), I(B^-)]$ iff $f(w) = 1$;
2. $I(W) = [I(A^-), I(B^+)]$ iff $f(w) = 2$;

3. $I(W) = [I(A^+), I(B^-)]$ iff $f(w) = 3$;

4. $I(W) = [I(A^+), I(B^+)]$ iff $f(w) = 4$.

It is easy to see that $I$ is a model of $\Theta$.

*only-if:* Let $I$ be a model of $\Theta$ such that $I(A)(\prec)I(B)$. By Lemma 3.14 and the construction of $\Theta$, we know that for each $w \in V$,

$$I(W) \in \{[I(A^-), I(B^-), [I(A^-), I(B^+)], [I(A^+), I(B^-)], [I(A^+), I(B^+)]\}.$$

Furthermore, if $(w_1, w_2) \in E$, then $I(W_1) \neq I(W_2)$, and thus $G$ is 4-colourable. □

**Corollary 3.16** Define $A(r, b)$ as in Definition 3.3. Then $\textsc{Ient}(A(r, b))$ is NP-complete.
**Proof:** $r_1, r_2 \in A(r, b)$ for all possible choices of $r$ and $b$. □

# 4 Point-Interval Relations: Vilain's Point-Interval Algebra

The point-interval algebra [Vilain, 1982] is based on the notions of *points*, *intervals* and *binary relations* on these. Where Allen's algebra is used for expressing relations between intervals, and the point algebra is used for expressing relations between points, the point-interval algebra allows points to be related to intervals. Thus, the relations in this algebra relate objects of different types, making it useful for combining the world of points with the world of intervals. That is exactly how it is used in Meiri's [1996] qualitative algebra.

## 4.1 Definitions

A point $p$ is a variable interpreted over the set of real numbers $\mathcal{R}$. An interval $I$ is represented by a pair $\langle I^-, I^+\rangle$ satisfying $I^- < I^+$, where $I^-$ and $I^+$ are interpreted over $\mathcal{R}$. We assume that we have a fixed universe of variable names for points and intervals. Then, a $\mathcal{V}$-*interpretation* is a function $M$ that maps point variables to $\mathcal{R}$ and interval variables to $\mathcal{R} \times \mathcal{R}$, and which satisfies the previously stated restrictions. We extend the notation by denoting the first component of $M(I)$ by $M(I^-)$ and the second by $M(I^+)$.

Given an interpreted point and an interpreted interval, their relative positions can be described by exactly one of five *basic point-interval relations*, where each basic relation can be defined in terms of its endpoint relations (see Table 2). A formula of the form $pBI$, where $p$ is a point, $I$ an interval and $B$ is a basic point-interval relation, is said to be satisfied by a $\mathcal{V}$-interpretation if the interpretation of the points and intervals satisfies the endpoint relations specified in Table 2.

To express indefinite information, unions of the basic relations are used, yielding $2^5$ distinct binary *point-interval* relations. Naturally, a set of basic relations is to be interpreted as a disjunction of its member relations. A point-interval relation is written as a list of its members, *e.g.*, (`b d a`). The set of all point-interval relations is denoted by $\mathcal{V}$. We denote the empty relation $\perp$ and the universal relation $\top$.

A formula of the form $p(B_1, \ldots, B_n)I$ is said to be a *point-interval* formula. Such a formula is said to be satisfied by a $\mathcal{V}$-interpretation $M$ if $pB_iI$ is satisfied by $M$ for some $i$, $1 \leq i \leq n$. A set $\Theta$ of point-interval formulae is said to be $\mathcal{V}$-*satisfiable* if there exists an $\mathcal{V}$-interpretation $M$ that satisfies every formula of $\Theta$. Such a satisfying $\mathcal{V}$-interpretation is called a $\mathcal{V}$-*model* of $\Theta$. The reasoning problem we will study is the following:

INSTANCE: A finite set $\Theta$ of point-interval formulae.
QUESTION: Does there exist a $\mathcal{V}$-model of $\Theta$?

We denote this problem $\mathcal{V}$-SAT. In the following, we often consider restricted versions of $\mathcal{V}$-SAT, where relations used in the formulae in $\Theta$ are taken only from a subset $\mathcal{S}$ of $\mathcal{V}$. In this case we say that $\Theta$ is a set of formulae over $\mathcal{S}$, and use a parameter in the problem description to denote the subclass under consideration, *e.g.* $\mathcal{V}$-SAT($\mathcal{S}$).

## 4.2 Complexity Results

The restriction of expressiveness only to allow relations between points and intervals does not reduce computational complexity when compared to Allen's algebra.

| Basic relation | | Example | Endpoints |
|---|---|---|---|
| $p$ before $I$ | `b` | `p`<br>`  III` | $p < I^-$ |
| $p$ starts $I$ | `s` | `p`<br>`III` | $p = I^-$ |
| $p$ during $I$ | `d` | `p`<br>`III` | $I^- < p < I^+$ |
| $p$ finishes $I$ | `f` | `p`<br>`III` | $p = I^+$ |
| $p$ after $I$ | `a` | `p`<br>`III` | $p > I^+$ |

Table 2: The five basic relations of the $\mathcal{V}$-algebra. The endpoint relation $I^- < I^+$ that is required for all relations has been omitted.

**Theorem 4.1** Deciding satisfiability in the point-interval algebra is NP-complete.
**Proof:** See Meiri [1996]. □

However, the reduction of expressiveness makes it easier to completely classify which subclasses are tractable and which are not: a complete classification of tractability in the point-interval algebra was done by Jonsson *et al.* [1999]. It turns out that there are only five maximal tractable subclasses, named $\mathcal{V}^{23}$, $\mathcal{V}^{20}_{\mathtt{s}}$, $\mathcal{V}^{20}_{\mathtt{f}}$, $\mathcal{V}^{17}_{\mathtt{s}}$ and $\mathcal{V}^{17}_{\mathtt{f}}$. See Table 3 for a presentation of these subclasses.

## 4.3 Algorithms

We will now present the tractable algorithms for the subclasses presented in the previous section; the correctness proofs and complexity analyses can be found in [Jonsson *et al.*, 1999].

- Algorithm 4.2 correctly solves satisfiability for $\mathcal{V}^{23}$ in polynomial time[6].
- Algorithm 4.3 correctly solves satisfiability for $\mathcal{V}^{20}_{\mathtt{s}}$ in polynomial time.
- Algorithm 4.3, exchanging starting and ending points of intervals, correctly solves satisfiability for $\mathcal{V}^{20}_{\mathtt{f}}$ in polynomial time.

[6] The set $\mathcal{V}^{23}$ is exactly the set of relations which can be expressed in the point-algebra, so line 1 can be performed in linear time.

| | $\mathcal{V}^{23}$ | $\mathcal{V}^{20}_{\mathtt{s}}$ | $\mathcal{V}^{20}_{\mathtt{f}}$ | $\mathcal{V}^{17}_{\mathtt{s}}$ | $\mathcal{V}^{17}_{\mathtt{f}}$ |
|---|---|---|---|---|---|
| $\perp$ | • | • | • | • | • |
| (b) | • | • | • | | |
| (s) | • | • | | • | |
| (b s) | • | • | • | • | |
| (d) | • | | | | |
| (b d) | • | | • | | |
| (s d) | • | | | • | |
| (b s d) | • | | • | • | |
| (f) | • | | • | | • |
| (b f) | | | • | | • |
| (s f) | | | | • | • |
| (b s f) | | | • | • | • |
| (d f) | • | | | | • |
| (b d f) | • | | • | | • |
| (s d f) | • | | | • | • |
| (b s d f) | • | | • | • | • |
| (a) | • | • | • | | |
| (b a) | | • | • | | |
| (s a) | | • | | • | |
| (b s a) | | • | • | • | |
| (d a) | • | • | | | |
| (b d a) | • | • | • | | |
| (s d a) | • | • | | • | |
| (b s d a) | • | • | • | • | |
| (f a) | • | • | • | | • |
| (b f a) | | • | • | | • |
| (s f a) | | • | | • | • |
| (b s f a) | | • | • | • | • |
| (d f a) | • | • | | | • |
| (b d f a) | • | • | • | | • |
| (s d f a) | • | • | | • | • |
| $\top$ | • | • | • | • | • |

Table 3: The maximal subclasses of $\mathcal{V}$ which have a polynomial-time satisfiability problem.

- Algorithm 4.4 correctly solves satisfiability for $\mathcal{V}^{17}_{\mathtt{s}}$ in polynomial time.
- Algorithm 4.4 correctly solves satisfiability for $\mathcal{V}^{17}_{\mathtt{f}}$ in polynomial time.

**Algorithm 4.2 (Alg-**PIAsat$(\mathcal{V}^{23})$**)**

**input** Instance $G = \langle V, E\rangle$ of PIASAT$(\mathcal{V}^{23})$

Transform $G$ into an equivalent set $P$ of point-algebra formulae
**if** PASAT$(P)$ **then**
  **accept**
**else**
  **reject**

$\square$

**Algorithm 4.3 (Alg-**PIASAT$(\mathcal{V}^{20}_{\mathtt{s}})$**)**

**input** Instance $G = \langle V, E\rangle$ of PIASAT$(\mathcal{V}^{20}_{\mathtt{s}})$

Define $f : \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}, \mathtt{a}\} \rightarrow \{<, =, >\}$ such that $f(\mathtt{b}) =$ " $<$",
$f(\mathtt{s}) =$ " $=$" and $f(\mathtt{d}) = f(\mathtt{f}) = f(\mathtt{a}) =$ " $>$".
Let $P = \{v(\bigcup_{r\in R} f(r))w \mid (v, R, w) \in E\}$.
**if** PASAT$(P)$ **then**
  **accept**
**else**
  **reject**

$\square$

# 5 Formalisms with Metric Time

We will now examine known tractable formalisms allowing for metric time, and which are not subsumed by the Horn-DLR framework. By formalisms

**Algorithm 4.4 (Alg-PIAsat($\mathcal{V}_{\mathsf{s}}^{17}$))**

**input** Instance $G = \langle V, E \rangle$ of PIAsat($\mathcal{V}_{\mathsf{s}}^{17}$)

**if** $G$ contains $\perp$ **then**
    **reject**
**else**
    **accept**

$\square$

allowing metric time, we mean formalisms with the ability to express statements such as "$X$ happened at time point 100" or "$X$ happened at least 50 time units before $Y$". Note that Allen's algebra cannot express this, while the Horn DLRs can.

The first example is an extension to the continuous endpoint formulae, and the second is a method for expressing metric time in the subalgebras $S(\cdot)$, $E(\cdot)$, $S^*$ and $E^*$.

## 5.1 Definitions

**Definition 5.1 (Augmented (continuous) endpoint formula)** An *augmented (continuous) endpoint formula* [Meiri, 1996] is

1. a (continuous) point algebra formula; or
2. a formula of the type $x \in \{[d_1^-, d_1^+], \ldots, [d_n^-, d_n^+]\}$ where $d_1^-, \ldots, d_n^-, d_1^+, d_n^- \in \mathcal{Q}$ and $d_i^- \leq d_i^+$, $1 \leq i \leq n$.

$\square$

If there is a need for unbounded intervals, $\mathcal{Q}$ can be replaced by $\mathcal{Q} \cup \{-\infty, +\infty\}$ in the previous definition. Note that the definition allows for discrete domains by setting the left and right endpoint of the intervals equal. A set $\Gamma$ of augmented endpoint formulae is satisfiable if there exists an assignment $I$ to the variables that

1. satisfies (in the ordinary sense) the point algebra formulae; and

2. if $x \in \{[d_1^-, d_1^+], \ldots, [d_n^-, d_n^+]\} \in \Gamma$, then $I(x) \in \bigcup\{[d_1^-, d_1^+], \ldots, [d_n^-, d_n^+]\}$.

We will now turn back to the interval satisfiability problem (Definition 3.1), and extend it to allow for metric information on starting points of intervals.

**Definition 5.2** **($M$-ISAT($\mathcal{I}$))** Let $\langle V, E\rangle$ be an instance of ISAT($\mathcal{I}$) and $H$ a finite set of DLRs over the set $\{v^+, v^- \mid v \in V\}$ of variables, $v^-$ representing starting points and $v^+$ ending points of intervals $v$.

An instance of the problem of *interval satisfiability with metric information* for a set $\mathcal{I}$ of interval relations, denoted $M$-ISAT($\mathcal{I}$), is a tuple $Q = \langle V, E, H\rangle$.

An *interpretation* $M$ for $Q$ is an interpretation for $\langle V, E\rangle$. Since we now need to refer to starting and ending points of intervals, we extend the notation such that $M(v^-)$ obtains the starting point of the interval $M(v)$, and similarly for $M(v^+)$.

An instance $Q$ is said to be *satisfiable* if there exists a model $M$ of $\langle V, E\rangle$ such that the DLRs in $H$ are satisfied, with values for all $v^-$ and $v^+$ by $M(v^-)$ and $M(v^+)$, respectively. $\square$

In order to obtain tractability, the following restrictions are imposed (the definitions differ slightly from the original ones).

**Definition 5.3** **($M_s$-ISAT($\mathcal{I}$), $M_e$-ISAT($\mathcal{I}$))** Let $\langle V, E, H\rangle$ be an instance of $M$-ISAT($\mathcal{I}$) where the DLRs of $H$ are restricted in two ways: first, $H$ may only contain Horn DLRs and second, $H$ may not contain any variables $v^+$, where $v \in V$, *i.e.*, it may only relate starting points of intervals. The set of such instances is denoted $M_s$-ISAT($\mathcal{I}$), and is said to be the problem of *interval satisfiability with metric information on starting points.*

Symmetrically, by exchanging starting and ending points, we get the problem of *interval satisfiability with metric information on ending points*, denoted $M_e$-ISAT($\mathcal{I}$). $\square$

## 5.2 Complexity Results

**Theorem 5.4** Deciding the satisfiability of augmented endpoint formulae is NP-complete, while deciding satisfiability of augmented continuous endpoint formulae is a polynomial-time task.

**Proof:** See Meiri [1996]. A set of augmented continuous endpoint formulae is satisfiable iff it is arc and path consistent; explicit algorithms can be found in Meiri's paper. □

**Theorem 5.5** $M_s$-ISAT$(S(b))$, $M_e$-ISAT$(E(b))$, $M_s$-ISAT$(S^*)$ and $M_e$-ISAT$(E^*)$ are polynomial-time problems, for $b \in \{\succ, \mathsf{d}, \mathsf{o}^{-1}\}$.
**Proof:** See Drakengren and Jonsson [1997a]. A polynomial-time algorithm is presented in Algorithm 5.7 for the case of $M_s$-ISAT; an algorithm for the case of $M_e$-ISAT is easily obtained by exchanging starting and ending points of intervals. □

The restriction that we cannot express starting and ending point information at the same time is essential for obtaining tractability, once we want to go outside the ORD-Horn algebra.

**Proposition 5.6** Let $S \subseteq \mathcal{A}$ such that $S$ is not a subset of the ORD-Horn algebra, and let $SE$ be the set of instances $Q = \langle V, E, H \rangle$ of $M$-ISAT$(S)$, where $H$ may contain only DLRs $u^+ = v^-$ for some $u, v \in V$. Then the satisfiability problem for $SE$ is NP-complete.
**Proof:** See Drakengren and Jonsson [1997a]. □

# 6 Other Approaches to Temporal Constraint Reasoning

## 6.1 Unit Intervals and Omitting $\top$

Most results on Allen's algebra that we have presented so far rely on two underlying assumptions:

1. The top relation is always included in any subalgebra[7]; and
2. Any interval model is regarded as a valid model of a set of Allen relations.

[7] In other words, we allow variables that are not explicitly constrained by any relation.

**Algorithm 5.7 (Alg-$M_s$-Isat($\mathcal{I}$))**

**input** Instance $Q = \langle V, E, H \rangle$ of $M_s$-Isat($\mathcal{A}$)

$H' \leftarrow H \cup \mathit{expl}^-(\langle V, E \rangle)$<br>
**if** not hornDLRsat($H'$) **then**<br>
    **reject**<br>
$K \leftarrow \emptyset$<br>
**for** each $\langle u, r, v \rangle \in E$<br>
    **if** not hornDLRsat($H' \cup \{u^- \neq v^-\}$) **then**<br>
        $K \leftarrow K \cup \{u^- = v^-\}$<br>
    **else**<br>
        $K \leftarrow K \cup \{u^- \neq v^-\}$<br>
$P \leftarrow \{u^+ \mathit{eprel}^-(r) v^+ \mid \langle u, r, v \rangle \in E \wedge u^- = v^- \in H' \cup K\}$<br>
**if** not PAsat($P$) **then**<br>
    **reject**<br>
**accept**

□

These assumptions are not always appropriate. For instance, there are examples of graph-theoretic applications where there is no need to use the top relations, *e.g.*, interval graph recognition [Golumbic and Shamir, 1993]. Similarly, there are scheduling and physical mapping applications where it is required that the intervals must be of length 1 [Pe'er and Shamir, 1997].

The implications of such "non-standard" assumptions have not been studied in any greater detail in the literature. However, for a subclass known as $\mathcal{A}_3$ (defined by Golumbic and Shamir [1993]), the picture is very clear, as we will see.

Let $\cap$ denote the Allen relation ($\equiv$ $\mathsf{d}\ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{m}\ \mathsf{m}^{-1}\ \mathsf{s}\ \mathsf{s}^{-1}\ \mathsf{f}\ \mathsf{f}^{-1}$), that is, the relation stating that two intervals have at least one point in common (they have a nonempty intersection). Let $\mathcal{A}_3$ denote the following set of Allen

| | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ |
|---|---|---|---|
| $\bot$ | $\bullet$ | $\bullet$ | $\bullet$ |
| $(\prec)$ | $\bullet$ | $\bullet$ | $\bullet$ |
| $(\succ)$ | $\bullet$ | $\bullet$ | $\bullet$ |
| $(\prec\succ)$ | | $\bullet$ | $\bullet$ |
| $(\cap)$ | $\bullet$ | $\bullet$ | |
| $(\prec\cap)$ | $\bullet$ | | |
| $(\succ\cap)$ | $\bullet$ | | |
| $\top$ | $\bullet$ | | $\bullet$ |

Table 4: Maximal tractable subclasses of $\mathcal{A}_3$.

relations[8]:

$$\{\bot, (\prec), (\succ), (\prec\ \succ), (\cap), (\prec\ \ \cap), (\succ\ \ \cap), \top\}.$$

The maximal tractable subclasses of $\mathcal{A}_3$ have been identified by Golumbic and Shamir [1993] and Webber [1995], and they are presented in Table 4. Note that $\top$ is not a member of $\Delta_2$. The maximal tractable subclasses of $\mathcal{A}_3$ under the additional assumption that all intervals are of unit length have been identified by Pe'er and Shamir [1997]. These subclasses can be found in Table 5[9].

Some of the maximal tractable subclasses of $\mathcal{A}_3$ are related to the tractable subclasses presented in Sections 2 and 3. For instance, $\Delta_1' \subset \Delta_1 \subset \mathcal{H}$ and $\Delta_3 \subset S(\succ)$. It should be noted that satisfiability in the ORD-Horn-algebra can be decided in polynomial time even under the unit interval assumption. Given a set of ORD-Horn relations, convert them to Horn DLRs and add constraints of the type $x^+ - x^- = 1$ for each interval $I = [x^-, x^+]$. The resulting set of formulae is also a set of Horn DLRs, and thus the satisfiability can be decided in polynomial time.

[8]Here, the relations are to be viewed as "macro relations", so that $(\prec\ \ \cap)$ denotes the Allen relation $(\prec \equiv\ \mathsf{d}\ \mathsf{d}^{-1}\ \mathsf{o}\ \mathsf{o}^{-1}\ \mathsf{m}\ \mathsf{m}^{-1}\ \mathsf{s}\ \mathsf{s}^{-1}\ \mathsf{f}\ \mathsf{f}^{-1})$.

[9]Golumbic and Shamir [1993] and Pe'er and Shamir [1997] agree on the definition of $\Delta_2$ and $\Delta_3$ but they define $\Delta_1$ differently. By $\Delta_1$, we mean $\Delta_1$ in the sense of Golumbic and Shamir [1993] and by $\Delta_1'$, we mean $\Delta_1$ in the sense of Pe'er and Shamir [1997].

| | $\Delta'_1$ | $\Delta_2$ | $\Delta_3$ |
|---|---|---|---|
| $\perp$ | • | • | • |
| $(\prec)$ | | • | • |
| $(\succ)$ | | • | • |
| $(\prec\succ)$ | | • | • |
| $(\cap)$ | • | • | |
| $(\prec \cap)$ | • | | |
| $(\succ \cap)$ | • | | |
| $\top$ | • | | • |

Table 5: Maximal tractable subclasses of $\mathcal{A}_3$ under the unit interval assumption.

## 6.2 Point-Duration Relations

Reasoning about durations has recently obtained a certain amount of interest, *cf.* [Condotta, 2000; Pujari and Sattar, 1999; Wetprasit and Sattar, 1998; Navarrete and Marin, 1997].

We will present the framework by Navarrete and Marin [1997] due to its appealing simplicity, and since many of the other methods build on it. Navarrete and Marin have proposed a formalism for reasoning about durations in the point algebra, and they have provided certain tractability results. Below, we present their approach and slightly generalize their tractability result.

**Definition 6.1** A *point-duration network* (PDN) is a tuple $\Sigma = \langle N_P, N_D \rangle$ where

1. $N_P$ is a set of PA formulae over a set $P = \{x_1, \ldots, x_n\}$ of *point variables*;
2. $N_D$ is a set of PA formulae over a set $D = \{d_{ij} \mid 1 \leq i < j \leq n\}$ of *duration variables*;

$\square$

A PDN $\Sigma = \langle N_P, N_D \rangle$ is satisfiable if there exists a assignment $I$ to the variables in $N_P$ such that

1. $I(x_i) r I(x_j)$ whenever $x_i r x_j \in N_P$; and

2. $|I(x_i) - I(x_j)| r |I(x_k) - I(x_m)|$ whenever $d_{ij} r d_{km} \in N_D$.

**Theorem 6.2** Deciding whether a PDN is satisfiable or not is NP-complete.

**Proof:** See Navarrete and Marin [1997]. □

In order to obtain tractability, Navarrete and Marin [1997] define a restriction of a PDN.

**Definition 6.3 (Simple PDN** [Navarrete and Marin, 1997]**)** A PDN is said to be *simple* if the following holds:

- Only the relations $<$, $>$ or $=$ are allowed in $N_P$ and $N_D$;
- For each $x_i, x_j \in P$, $x_i r x_j \in N_P$ for some $r$; and
- For each $d_i, d_j \in D$, $d_i r d_j \in N_D$ for some $r$.

□

It is important to note that this defintion does not allow two variables to be unrelated. Furthermore, they show that deciding the satisfiability of simple PDNs is a polynomial-time problem. We now intend to weaken their restriction in two steps, still obtaining tractability. The tool for this will be the Horn DLRs.

**Definition 6.4 (Point-simple PDN)** A PDN is said to be *point-simple* if the following holds:

- Only the relations $<$, $>$ or $=$ are allowed in $N_P$; and
- For each $x_i, x_j \in P$, $x_i r x_j \in N_P$ for some $r$.

□

Note that there are no requirements on the formulae in $N_D$; thus durations may be related with arbitrary PA relations, including the $\top$ relation.

We now show how the satisfiability problem for point-simple PDNs can be solved in polynomial time, by a straightforward reduction to Horn DLRs.

Let $\Sigma = \langle N_P, N_D \rangle$ be a point-simple PDN. Construct a set $\Theta$ of Horn DLR formulae incrementally as follows: Check whether $N_P$ is satisfiable or not. If it is not satisfiable, report that $\Sigma$ is not satisfiable. Otherwise, let $\Theta$ initially equal $N_P$.

For each formula $d_{ij} r d_{km} \in N_D$, check whether $x_i < x_j$, $x_i > x_j$ or $x_i = x_j$ is in $N_P$. Since $\Sigma$ is point-simple, at least one of these relations is in $N_P$. By observing that $N_P$ is satisfiable, exactly one of the relations is in $N_P$. Note the following:

1. if $x_i < x_j \in N_P$, then $d_{ij} = |x_i - x_j| = x_j - x_i$;
2. if $x_i > x_j \in N_P$, then $d_{ij} = |x_i - x_j| = x_i - x_j$;
3. if $x_i = x_j \in N_P$, then $d_{ij} = |x_i - x_j| = 0$;

Continue by checking whether $x_k < x_m$, $x_k > x_m$ or $x_k = x_m$, and decide the value of $d_{km}$ as above. Now, it is easy to convert the relation $d_{ij} r d_{km}$ to a Horn DLR.

As an example, assume that $d_{ij} < d_{km}$, $x_i > x_j$ and $x_k < x_m$. The corresponding Horn DLR then will be $x_i - x_j < x_m - x_k$. Add the Horn DLR to $\Theta$ and note that $\Sigma$ is satisfiable iff $\Theta$ is satisfiable. The transformation from point-simple PDNs to Horn DLRs can easily be performed in polynomial time, and thus we have shown that deciding satisfiability of point-simple PDNs is a polynomial-time solvable problem.

We are in the position to make one more generalization, still retaining tractability.

**Definition 6.5 (Horn-simple PDN)** We say that $\Sigma = \langle N_P, N_D \rangle$ *Horn-simple* if $\Sigma$ satisfies all the requirements for being point-simple, except that $N_D$ is allowed to contain arbitrary Horn DLRs over $D$, instead of requiring PA formulae. $\Box$

**Theorem 6.6** Deciding whether a Horn-simple PDN is satisfiable or not is a polynomial-time problem.

**Proof:** The above transformation from $N_D$ relations to Horn DLR point relations simply replaces duration variables $d_{ij}$ by either $x_i - x_j$, $x_j - x_i$ or 0. If a Horn DLR $\phi$ is in $N_D$, then the transformed formula will obviously be a Horn DLR too, but now over the point variables. $\Box$

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