APPENDIX TO "LEARNING MULTIVARIATE REGRESSION CHAIN GRAPHS UNDER FAITHFULNESS"

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Lemma 1. *Let G and H denote two Markov equivalent CGs that have exactly the minimal set of bidirected edges for their Markov equivalence class. Then, there is a sequence of feasible splits and mergings that transforms G into H.*

Proof. Note that G and H have the same connectivity components. Let O_G (resp. O_H) denote an ordering of the connectivity components of G (resp. H) st if $A \to B$ is in G (resp. H), then the connectivity component B belongs to is to the right of the connectivity component A belongs to. Assume that $O_G \neq O_H$ because, otherwise, $G = H$ and thus we are done. Let L denote the rightmost connectivity component in O_H st some connectivity component to the left of L in O_H is to the right of L in O_G. Let R denote the connectivity component immediately to the right of L in O_G. Note that R is to the left of L in O_H. We first show that merging L and R in G is feasible. Assume the contrary. Then, some of the conditions in Figure 2 cannot hold. Note that $de_G(L) \cap pa_G(R) = \emptyset$. Then, condition 3 in Figure 2 holds. This leaves us with the following three options.

- Condition 1 in Figure 2 does not hold because there are two nodes $A \in ch_G(L) \cap R$ and $B \in L$ st $B \to A$ is not in G and $B \leftrightarrow C \to A$ with $C \in L$ is an induced subgraph of G. Then, G does not have an unshielded collider between B and A over C. Then, neither does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $B \leftrightarrow C \leftarrow A$ is an induced subgraph of H. This is a contradiction.
- Condition 1 in Figure 2 does not hold because there are two nodes $A \in ch_G(L) \cap R$ and $B \in pa_G(L)$ st $B \to A$ is not in G. Note that the previous bullet allows us to assume without loss of generality that $L \subseteq pa_G(A)$. Then, $B \to C \to A$ with $C \in L$ is an induced subgraph of G. Then, G does not have an unshielded collider between B and A over C. Then, neither does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $B \to C \leftarrow A$ is an induced subgraph of H. Note that $B \to C$ is in H due to how L was selected. This is a contradiction.
- Condition 2 in Figure 2 does not hold because there are three nodes $B \in pa_G(R) \cap L$ and $A, C \in ch_G(B) \cap R$ st $A \leftrightarrow C$ is not in G. Then, $A \leftarrow B \to C$ is an induced subgraph of G. Then, G does not have an unshielded collider between B and A over C. Then, neither does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $A \to B \leftarrow C$ is an induced subgraph of H. This is a contradiction.

In summary, merging L and R in G is feasible. Let us perform the merging and call the resulting connectivity component $L \cup R$. We now show that splitting $L \cup R$ in G into R and L is feasible. Assume the contrary. Then, some of the conditions in Figure 1 cannot hold. We consider the three possible options below.

- Condition 1 in Figure 1 does not hold because there are two nodes $A \in sp_G(R) \cap L$ and $B \in R$ st $B \leftrightarrow A$ is not in G and $B \leftrightarrow C \leftrightarrow A$ with $C \in R$ is an induced subgraph

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of G. Then, G has an unshielded collider between B and A over C. Then, so does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $B \leftrightarrow C \rightarrow A$ is an induced subgraph of H. This is a contradiction.

- Condition 2 in Figure 1 does not hold because there are two nodes $A \in sp_G(R) \cap L$ and $B \in pa_G(R)$ st $B \rightarrow A$ is not in G. Note that the previous bullet allows us to assume without loss of generality that condition 1 in Figure 1 holds. Then, $B \rightarrow C \leftrightarrow A$ with $C \in R$ is an induced subgraph of G. Then, G has an unshielded collider between B and A over C. Then, so does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $B \rightarrow C \rightarrow A$ is an induced subgraph of H. This is a contradiction.
- Condition 3 in Figure 1 does not hold because there are three nodes $B \in sp_G(L) \cap R$ and $A, C \in sp_G(B) \cap L$ st $A \leftrightarrow C$ is not in G. Then, $A \leftrightarrow B \leftrightarrow C$ is an induced subgraph of G. Then, G has an unshielded collider between B and A over C. Then, so does H since G and H are Markov equivalent. However, R is to the left of L in O_H, which means that $A \leftarrow B \rightarrow C$ is an induced subgraph of H. This is a contradiction.

In summary, splitting $L \cup R$ in G into R and L is feasible. Let us perform the split and restart the proof. This iterative process will end when $O_G = O_H$, which means that $G = H$. □

Theorem 1. *Let G and H denote two Markov equivalent CGs. Then, there is a sequence of feasible splits and mergings that transforms G into H.*

Proof. Recall from Lemma 1 in the main text that G and H can be transformed via two sequences of feasible splits into two Markov equivalent CGs G' and H' that have exactly the minimal set of bidirected edges for their Markov equivalence class. Note that fesible splits and mergings are inverse operations and, thus, there is a sequence of feasible mergings that transforms H' into H. Now, if $G' = H'$ then we are done, else note that G' can be transformed into H' via a sequence of feasible splits and mergings by Lemma 1 above. □