LEARNING AMP CHAIN GRAPHS AND SOME MARGINAL MODELS THEREOF UNDER FAITHFULNESS: ADDENDUM

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A distinguished member of a class of triplex equivalent AMP CGs is the so-called essential graph G^* (Andersson and Perlman, 2006): An edge $A \to B$ is in G^* if and only if $A \leftarrow B$ is in no member of the class. Unfortunately, our learning algorithm in Table 1 in the main text does not output an essential graph, as we have shown in Section 3 with an example. However, it can easily be modified for this task, as we show below. It is worth mentioning that an algorithm for this task has been proposed before (Andersson and Perlman, 2004, Section 7). However, as far as we know, its correctness has not been proven. We do prove the correctness of our algorithm.

Table 1. Algorithm for constructing the essential graph in a class of triplex equivalent CGs.

Input: A CG G.
Output: The essential graph G^* in the class of triplex equivalent CGs containing G.

1 For each ordered pair of non-adjacent nodes A and B in G
2 Set $S_{AB} = S_{BA} = S$ such that $A \perp_G B|S$
3 Let G^* denote the undirected graph that has the same adjacencies as G
4 Apply the rules R1-R4 to G^* while possible
5 Replace every edge $A - B$ in every cycle in G^* that is of length greater than three, chordless, and without blocks with $A \mapsto B$
6 Apply the rules R2-R4 to G^* while possible
7 Replace every edge $A \vdash B$ (respectively $A \mapsto B$) in G^* with $A \to B$ (respectively $A - B$)

In line 2, any such S will do. For instance, if $co_G(A) = co_G(B)$ then $S = pa_G(A \cup ne_G(A)) \cup ne_G(A)$, otherwise $S = pa_G(A)$. To see this, check the proof of Lemma 2 in the main text. In line 5, that the cycle has no blocks means that the ends of the edges in the cycle have no blocks. Note that the rule R1 is not used in line 6, because it will never fire after its repeated application in line 4. Finally, note that G^* may have edges without blocks after line 6.

Lemma 2. *After line 3, G and G^* have the same adjacencies.*

Lemma 3. *After line 6, all the blocks in G^* are on edge ends that are not arrowheads in G.*

Proof. In Lemma 3 in the main text, we have proved that any of the rules R1-R4 only blocks edge ends that are not arrowheads in G. Of course, for this to hold, the blocks in the antecedent of the rule must be on edge ends that are not arrowheads in G. This implies that, after line 4, all the blocks in G^* are on edge ends that are not arrowheads in G, because G^* has no blocks before line 4. However, to prove that this result also holds after line 6, we have to prove that line 5 only blocks edge ends that are not arrowheads in G. To do so, consider any cycle ρ^* in G^* that is of length greater than three, chordless, and without blocks. Let ρ

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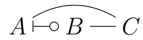
denote the cycle in G corresponding to the sequence of nodes in ρ^*. Note that no (undirected) edge in ρ^* can be directed in ρ because, otherwise, a subroute of the form $A \rightarrow B \circ\!\!- C$ must exist in ρ, which implies that G contains the triplex $A \rightarrow B \circ\!\!- C$ because A and C cannot be adjacent in G since ρ^* is chordless, which implies that $A \vdash\!\circ B \circ\!\dashv C$ is in G^* by R1 in line 4, which contradicts that ρ^* has no blocks. Therefore, every edge in ρ^* is undirected in ρ and, thus, line 5 only blocks edge ends that are not arrowheads in G. □

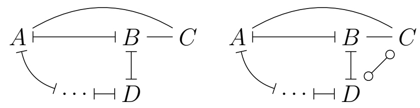
Lemma 4. *After line 7, G and G^* have the same triplexes. Moreover, G^* has all the immoralities that are in G.*

Proof. The proof is essentially the same as that of Lemma 4 in the main text. □

Lemma 5. *After line 6, G^* does not have any induced subgraph of the form* .

Proof. The proof is essentially the same as that of Lemma 5 in the main text. It just suffices to add the following case:

Case 0: Assume that $A \vdash\!\dashv B$ is in H due to line 5. Then, after line 5, H had an induced subgraph of one of the following forms, where possible additional edges between C and internal nodes of the route $A \vdash\!\dashv \ldots \vdash\!\dashv D$ are not shown:



case 0.1 case 0.2

Note that C cannot belong to the route $A \vdash\!\circ \ldots \vdash\!\circ D$ because, otherwise, the cycle $A \vdash\!\circ \ldots \vdash\!\circ D \vdash\!\circ B \multimap A$ would not have been chordless.

Case 0.1: If $B \notin S_{CD}$ then $B \dashv C$ is in H by R1, else $B \vdash C$ is in H by R2. Either case is a contradiction.

Case 0.2: Recall from line 5 that the cycle $A \vdash\!\dashv \ldots \vdash\!\dashv D \vdash\!\dashv B \vdash\!\dashv A$ is of length greater than three and chordless, which implies that there is no edge between A and D in H. Thus, if $C \notin S_{AD}$ then $A \vdash C$ is in H by R1, else $B \dashv C$ is in H by R4. Either case is a contradiction.

□

Lemma 6. *After line 6, every chordless cycle $\rho^* : V_1, \ldots, V_n = V_1$ in G^* that has an edge $V_i \vdash V_{i+1}$ also has an edge $V_j \dashv V_{j+1}$.*

Proof. The proof is essentially the same as that of Lemma 6 in the main text. □

Theorem 1. *After line 7, G^* is the essential graph in the class of triplex equivalent CGs containing G.*

Proof. The proof of Theorem 1 in the main text can be reused here to prove that, after line 7, G^* is triplex equivalent to G and it has no semidirected cycles. Moreover, the directed edges in G^* after line 7 must be directed in the essential graph in the class of triplex equivalent CGs containing G by Lemma 3. For the same reason, the undirected edges in G^* after line 7 that correspond to $A \vdash\!\dashv B$ edges when line 7 was to be executed must be undirected in the essential graph in the class of triplex equivalent CGs containing G. We show below that every other undirected edge in G^* after line 7 (i.e. those that correspond to edges without blocks when line 7 was to be executed) must also be undirected in the essential graph in the class of triplex equivalent CGs containing G.

Let H denote the graph that contains all and only the edges of G^* resulting from the replacements in line 7, and let U denote the graph that contains the rest of the edges of G^* after line 7. Note that all the edges in U are undirected and they had no blocks when line 7 was to be executed. Therefore, U has no cycle of length greater than three that is chordless by line 5. In other words, U is chordal. Then, we can orient all the edges in U without creating immoralities nor directed cycles by using, for instance, the maximum cardinality search (MCS) algorithm (Koller and Friedman, 2009, p. 312). Consider any such orientation of the edges in U and denote it D. Now, add all the edges in D to H. As we show below, this last step does not create any triplex or semidirected cycle in H:

- It does not create a triplex $(\{A, C\}, B)$ in H because, otherwise, $A - B \circ\!\!\dashv C$ must exist in G^* when line 7 was to be executed, which implies that $A \vdash\!\!\circ B$ or $A \circ\!\!\dashv B$ was in G^* by R1 or R2 when line 7 was to be executed, which contradicts that $A - B$ is in U.
- Assume to the contrary that it does create a semidirected cycle in H. Note that this cycle cannot have any $\vdash$ edge by Lemma 6 when line 7 was to be executed and, thus, it must have some $\vdash\!\dashv$ edge when line 7 was to be executed. However, this implies that $A - B \vdash\!\dashv C$ must exist in G^* when line 7 was to be executed, which implies that A and C are adjacent in G^* because, otherwise, $A \vdash\!\!\circ B$ or $A \circ\!\!\dashv B$ was in G^* by R1 or R2 when line 7 was to be executed, which contradicts that $A - B$ is in U. Then, $A \vdash\!\!\circ C$ or $A \circ\!\!\dashv C$ exists in G^* by Lemma 5 when line 7 was to be executed, which implies that $A \vdash\!\!\circ B$ or $A \circ\!\!\dashv B$ was in G^* by R3 when line 7 was to be executed, which contradicts that $A - B$ is in U.

Consequently, H is a CG that is triplex equivalent to G. Finally, let us recall how the MCS algorithm works. It first unmarks all the nodes in U and, then, iterates through the following step until all the nodes are marked: Select any of the unmarked nodes with the largest number of marked neighbors and mark it. Finally, the algorithm orients every edge in U away from the node that was marked earlier. Clearly, any node may get marked firstly by the algorithm because there is a tie among all the nodes in the first iteration, which implies that every edge may get oriented in any of the two directions in D and thus in H. Therefore, either orientation of every edge of U occurs in some CG H that is triplex equivalent to G. Then, every edge of U must be undirected in the essential graph in the class of triplex equivalent CGs containing G. □

References

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