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- In page 2, the definition of descending route should be replaced by the following:

  A route is called descending if $V_i \to V_{i+1}$, $V_i - V_{i+1}$ or $V_i \leftrightarrow V_{i+1}$ is in G for all $1 \leq i < n$.

- In page 2, the sentences

  "intersection $X \perp_p Y|Z \cup W \wedge X \perp_p W|Z \cup Y \Rightarrow X \perp_p Y \cup W|Z$. Moreover, M is called compositional graphoid if it is a graphoid that also satisfies the composition property $X \perp_M Y|Z \wedge X \perp_M W|Z \Rightarrow X \perp_M Y \cup W|Z$."

  should be replaced by

  "intersection $X \perp_M Y|Z \cup W \wedge X \perp_M W|Z \cup Y \Rightarrow X \perp_M Y \cup W|Z$. Moreover, M is called compositional graphoid if it is a graphoid that also satisfies the composition property $X \perp_M Y|Z \wedge X \perp_M W|Z \Rightarrow X \perp_M Y \cup W|Z$. Another property that M may satisfy is weak transitivity $X \perp_M Y|Z \wedge X \perp_M Y|Z \cup K \Rightarrow X \perp_M K|Z \vee K \perp_M Y|Z$ with $K \in V \setminus X \setminus Y \setminus Z$."

- In page 8, Corollary 3 should be replaced by the following (the old proof still applies):

  Corollary 3. *Any independence model represented by a MAMP CG is a compositional graphoid that satisfies weak transitivity.*

- In page 8, the definition of pairwise separation base should be replaced by the following:

  Specifically, we define the pairwise separation base of a MAMP CG G as the separations
  - $A \perp B|pa_G(A)$ for all $A, B \in V$ st $A \notin ad_G(B)$ and $B \notin de_G(A)$,
  - $A \perp B|ne_G(A) \cup pa_G(A \cup ne_G(A))$ for all $A, B \in V$ st $A \notin ad_G(B)$, $A \in de_G(B)$, $B \in de_G(A)$ and $uc_G(A) = uc_G(B)$, and
  - $A \perp B|pa_G(A)$ for all $A, B \in V$ st $A \notin ad_G(B)$, $A \in de_G(B)$, $B \in de_G(A)$ and $uc_G(A) \neq uc_G(B)$.

- In page 10, in the sentence

  "Note that for all $A, B \in K_i$ st $uc_G(A) = uc_G(B)$ and $A - B$ is not in G, $A \perp_G B|pa_G(K_i) \cup K_i \setminus A \setminus B$ and thus $(\Lambda^i_{uc_G(A), uc_G(A)})^{-1}_{A,B} = 0$ (Lauritzen, 1996, Proposition

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5.2)."

$A \perp_G B | pa_G(K_i) \cup K_i \setminus A \setminus B$ should be replaced by $A \perp_G B | pa_G(K_i) \cup uc_G(A) \setminus A \setminus B$.

- In page 20, the sentence

"On the other hand, if A and C are in different undirected connectivity components of G_1, then $A \notin de_{G_1}(C)$ or $C \notin de_{G_1}(A)$. Assume without loss of generality that $A \notin de_{G_1}(C)$. Then, $A \perp C | pa_{G_1}(C)$ holds for G_1 by Theorem 5 but it does not hold for G_2, which is a contradiction."

should be replaced by

"On the other hand, if A and C are in different undirected connectivity components of G_1, then $A \perp C | pa_{G_1}(C)$ or $A \perp C | pa_{G_1}(A)$ holds for G_1 by Theorem 5 but neither holds for G_2, which is a contradiction."

Likewise, the sentence

"On the other hand, if A and C are in different undirected connectivity components of G_1, then $A \notin de_{G_1}(C)$ or $C \notin de_{G_1}(A)$. Assume without loss of generality that $A \notin de_{G_1}(C)$. Then, $A \perp C | pa_{G_1}(C)$ holds for G_1 by Theorem 5. Note also that $B \notin pa_{G_1}(C)$ because, otherwise, G_1 would not have the triplex $(\{A, C\}, B)$. Then, $A \perp C | pa_{G_1}(C)$ does not hold for G_2, which is a contradiction."

should be replaced by

"On the other hand, if A and C are in different undirected connectivity components of G_1, then $A \perp C | pa_{G_1}(C)$ or $A \perp C | pa_{G_1}(A)$ holds for G_1 by Theorem 5. Note also that $B \notin pa_{G_1}(A)$ and $B \notin pa_{G_1}(C)$ because, otherwise, G_1 would not have the triplex $(\{A, C\}, B)$. Then, neither $A \perp C | pa_{G_1}(C)$ nor $A \perp C | pa_{G_1}(A)$ holds for G_2, which is a contradiction."

- In pages 17-20, the proofs of Theorems 5 and 6 should be replaced by the following:

Proof of Theorem 5. Since the independence model represented by G satisfies the compositional graphoid properties by Corollary 3, it suffices to prove that the pairwise separation base of G is a subset of the independence model represented by G. We prove this next. Let $A, B \in V$ st $A \notin ad_G(B)$. Consider the following cases.

Case 1: Assume that $B \notin de_G(A)$. Then, every path between A and B in G falls within one of the following cases.

Case 1.1: $A = V_1 \leftarrow V_2 \dots V_n = B$. Then, this path is not $pa_G(A)$-open.

Case 1.2: $A = V_1 \multimap V_2 \dots V_n = B$. Note that $V_2 \neq V_n$ because $A \notin ad_G(B)$. Note also that $V_2 \notin pa_G(A)$ due to the constraint C1. Then, $V_2 \rightarrow V_3$ must be in G for the path to be $pa_G(A)$-open. By repeating this reasoning, we can conclude that $A = V_1 \multimap V_2 \rightarrow V_3 \rightarrow \dots \rightarrow V_n = B$ is in G. However, this contradicts that $B \notin de_G(A)$.

Case 1.3: $A = V_1 - V_2 - \dots - V_m \leftarrow\!\circ\ V_{m+1} \dots V_n = B$. Note that $V_m \notin pa_G(A)$ due to the constraint C1. Then, this path is not $pa_G(A)$-open.

Case 1.4: $A = V_1 - V_2 - \ldots - V_m \to V_{m+1} \ldots V_n = B$. Note that $V_{m+1} \neq V_n$ because $B \notin de_G(A)$. Note also that $V_{m+1} \notin pa_G(A)$ due to the constraint C1. Then, $V_{m+1} \to V_{m+2}$ must be in G for the path to be $pa_G(A)$-open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 - \ldots - V_m \to V_{m+1} \to \ldots \to V_n = B$ is in G. However, this contradicts that $B \notin de_G(A)$.

Case 1.5: $A = V_1 - V_2 - \ldots - V_n = B$. This case contradicts the assumption that $B \notin de_G(A)$.

Case 2: Assume that $A \in de_G(B)$, $B \in de_G(A)$ and $uc_G(A) = uc_G(B)$. Then, there is an undirected path ρ between A and B in G. Then, every path between A and B in G falls within one of the following cases.

Case 2.1: $A = V_1 \leftarrow V_2 \ldots V_n = B$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open.

Case 2.2: $A = V_1 \multimap V_2 \ldots V_n = B$. Note that $V_2 \neq V_n$ because $A \notin ad_G(B)$. Note also that $V_2 \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraints C1 and C2. Then, $V_2 \to V_3$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open. By repeating this reasoning, we can conclude that $A = V_1 \multimap V_2 \to V_3 \to \ldots \to V_n = B$ is in G. However, this together with ρ violate the constraint C1.

Case 2.3: $A = V_1 - V_2 \leftarrow V_3 \ldots V_n = B$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open.

Case 2.4: $A = V_1 - V_2 \multimap V_3 \ldots V_n = B$. Note that $V_3 \neq V_n$ due to ρ and the constraints C1 and C2. Note also that $V_3 \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraints C1 and C2. Then, $V_3 \to V_4$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 \multimap V_3 \to \ldots \to V_n = B$ is in G. However, this together with ρ violate the constraint C1.

Case 2.5: $A = V_1 - V_2 - V_3 \ldots V_n = B$ st $sp_G(V_2) = \emptyset$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open.

Case 2.6: $A = V_1 - V_2 - \ldots - V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq n-1$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq n$ by the constraint C3. However, this contradicts that $A \notin ad_G(B)$.

Case 2.7: $A = V_1 - V_2 - \ldots - V_m - V_{m+1} - V_{m+2} \ldots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$ and $sp_G(V_{m+1}) = \emptyset$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq m+1$ by the constraint C3. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open.

Case 2.8: $A = V_1 - V_2 - \ldots - V_m - V_{m+1} \leftarrow V_{m+2} \ldots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq m+1$ by the constraint C3. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open.

Case 2.9: $A = V_1 - V_2 - \ldots - V_m - V_{m+1} \multimap V_{m+2} \ldots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$. Note that $V_{m+2} \neq V_n$ due to ρ and the constraints C1 and C2. Note also that $V_{m+2} \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraints C1 and C2. Then, $V_{m+2} \to V_{m+3}$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$-open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 - \ldots - V_m - V_{m+1} \multimap V_{m+2} \to \ldots \to V_n = B$ is in G. However, this together with ρ violate the constraint C1.

Case 3: Assume that $A \in de_G(B)$, $B \in de_G(A)$ and $uc_G(A) \neq uc_G(B)$. Then, every path between A and B in G falls within one of the following cases.

Case 3.1: $A = V_1 \leftarrow V_2 \ldots V_n = B$. Then, this path is not $pa_G(A)$-open.

Case 3.2: $A = V_1 \multimap V_2 \ldots V_n = B$. Note that $V_2 \neq V_n$ because $A \notin ad_G(B)$. Note also that $V_2 \notin pa_G(A)$ due to the constraint C1. Then, $V_2 \to V_3$ must be in G for the path to be $pa_G(A)$-open. By repeating this reasoning, we can conclude that $A = V_1 \multimap V_2 \to V_3 \to \ldots \to V_n = B$ is in G. However, this together with the assumption that $A \in de_G(B)$ contradict the constraint C1.

Case 3.3: $A = V_1 - V_2 - \ldots - V_m \leftarrow\!\!\circ\; V_{m+1} \ldots V_n = B$. Note that $V_m \notin pa_G(A)$ due to the constraint C1. Then, this path is not $pa_G(A)$-open.

Case 3.4: $A = V_1 - V_2 - \ldots - V_m \rightarrow V_{m+1} \ldots V_n = B$. Note that $V_{m+1} \neq V_n$ because, otherwise, this together with the assumption that $A \in de_G(B)$ contradict the constraint C1. Note also that $V_{m+1} \notin pa_G(A)$ due to the constraint C1. Then, $V_{m+1} \rightarrow V_{m+2}$ must be in G for the path to be $pa_G(A)$-open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 - \ldots - V_m \rightarrow V_{m+1} \rightarrow \ldots \rightarrow V_n = B$ is in G. However, this together with the assumption that $A \in de_G(B)$ contradict the constraint C1.

□

Lemma 3. *Let X and Y denote two nodes of a MAMP CG G with only one connectivity component. If $X \perp_G Y|Z$ and there is a node $C \in Z$ st $sp_G(C) \neq \varnothing$, then $X \perp_G Y|Z \setminus C$.*

Proof. Assume to the contrary that there is a $(Z \setminus C)$-open path ρ between X and Y in G. Note that C must occur in ρ because, otherwise, ρ is Z-open which contradicts that $X \perp_G Y|Z$. For the same reason, C must be a non-triplex node in ρ. Then, $D - C - E$ must be a subpath of ρ and, thus, the edge $D - E$ must be in G by the constraint C3, because $sp_G(C) \neq \varnothing$. Then, the path obtained from ρ by replacing the subpath $D - C - E$ with the edge $D - E$ is Z-open. However, this contradicts that $X \perp_G Y|Z$.

□

Lemma 4. *Let X and Y denote two nodes of a MAMP CG G with only one connectivity component. If $X \perp_G Y|Z$ then $X \perp_{cl(G)} Y|Z$.*

Proof. We prove the lemma by induction on $|Z|$. If $|Z| = 0$, then $uc_G(X) \neq uc_G(Y)$. Consequently, $X \perp_{cl(G)} Y$ follows from the pairwise separation base of G because $X \notin ad_G(Y)$. Assume as induction hypothesis that the lemma holds for $|Z| < l$. We now prove it for $|Z| = l$. Consider the following cases.

Case 1: Assume that $uc_G(X) = uc_G(Y)$. Consider the following cases.

Case 1.1: Assume that $Z \subseteq uc_G(X)$. Then, the pairwise separation base of G implies that $C \perp_{cl(G)} uc_G(X) \setminus C \setminus ne_G(C)|ne_G(C)$ for all $C \in uc_G(X)$ by repeated composition, which implies $X \perp_{cl(G)} Y|Z$ by the graphoid properties (Lauritzen, 1996, Theorem 3.7).

Case 1.2: Assume that there is some node $C \in Z \setminus uc_G(X)$ st $C \leftrightarrow D$ is in G with $D \in uc_G(X)$ and $X \not\perp_G C|Z \setminus C$. Then, $Y \perp_G C|Z \setminus C$. To see it, assume the contrary. Then, $X \not\perp_G Y|Z \setminus C$ by weak transitivity because $X \perp_G Y|Z$. However, this contradicts Lemma 3.
Now, note that $Y \perp_G C|Z \setminus C$ implies $Y \perp_{cl(G)} C|Z \setminus C$ by the induction hypothesis. Note also that $X \perp_G Y|Z \setminus C$ by Lemma 3 and, thus, $X \perp_{cl(G)} Y|Z \setminus C$ by the induction hypothesis. Then, $X \perp_{cl(G)} Y|Z$ by symmetry, composition and weak union.

Case 1.3: Assume that Cases 1.1 and 1.2 do not apply. Let $E \in Z \setminus uc_G(X)$. Such a node E exists because, otherwise, Case 1.1 applies. Moreover, $X \perp_G E|Z \setminus E$ because, otherwise, there is some node C that satisfies the conditions of Case 1.2. Note also that $X \perp_G Y|Z \setminus E$. To see it, assume the contrary. Then, there is a $(Z \setminus E)$-open path between X and Y in G. Note that E must occur in the path because, otherwise, the path is Z-open, which contradicts that $X \perp_G Y|Z$. However, this implies that $X \not\perp_G E|Z \setminus E$, which is a contradiction.

Now, note that $X \perp_G E|Z \setminus E$ and $X \perp_G Y|Z \setminus E$ imply $X \perp_{cl(G)} E|Z \setminus E$ and $X \perp_{cl(G)} Y|Z \setminus E$ by the induction hypothesis. Then, $X \perp_{cl(G)} Y|Z$ by composition and weak union.

Case 2: Assume that $uc_G(X) \neq uc_G(Y)$. Consider the following cases.

Case 2.1: Assume that there is some node $C \in Z$ st $C \leftrightarrow X$ is in G. Then, $Y \perp_G C|Z \setminus C$ because, otherwise, $X \not\perp_G Y|Z$. Then, $Y \perp_{cl(G)} C|Z \setminus C$ by the induction hypothesis. Note that $X \perp_G Y|Z \setminus C$ by Lemma 3 and, thus, $X \perp_{cl(G)} Y|Z \setminus C$ by the induction hypothesis. Then, $X \perp_{cl(G)} Y|Z$ by symmetry, composition and weak union.

Case 2.2: Assume that there is some node $C \in Z \cap uc_G(X)$ st $sp_G(C) \neq \varnothing$, and $X \perp_G C|Z \setminus C$. Then, $X \perp_{cl(G)} C|Z \setminus C$ by the induction hypothesis. Note that $X \perp_G Y|Z \setminus C$ by Lemma 3 and, thus, $X \perp_{cl(G)} Y|Z \setminus C$ by the induction hypothesis. Then, $X \perp_{cl(G)} Y|Z$ by composition and weak union.

Case 2.3: Assume that there is some node $C \in Z \cap uc_G(X)$ st $sp_G(C) \neq \varnothing$, and $X \not\perp_G C|Z \setminus C$. Then, $Y \perp_G C|Z \setminus C$. To see it, assume the contrary. Then, $X \not\perp_G Y|Z \setminus C$ by weak transitivity because $X \perp_G Y|Z$. However, this contradicts Lemma 3.

Now, note that $Y \perp_G C|Z \setminus C$ implies $Y \perp_{cl(G)} C|Z \setminus C$ by the induction hypothesis. Note also that $X \perp_G Y|Z \setminus C$ by Lemma 3 and, thus, $X \perp_{cl(G)} Y|Z \setminus C$ by the induction hypothesis. Then, $X \perp_{cl(G)} Y|Z$ by composition and weak union.

Case 2.4: Assume that Cases 2.1-2.3 do not apply. Let $V_1, \ldots, V_m$ be the nodes in $Z \cap uc_G(X)$. Let $W_1, \ldots, W_n$ be the nodes in $Z \setminus uc_G(X)$. Then,

(1) $X \perp_{cl(G)} Y$ follows from the pairwise separation base of G because $uc_G(X) \neq uc_G(Y)$ and $X \notin ad_G(Y)$. Moreover, for all $1 \leq i \leq m$

(2) $V_i \perp_{cl(G)} Y$ follows from the pairwise separation base of G because $V_i \notin uc_G(Y)$ and $V_i \notin ad_G(Y)$, since $sp_G(V_i) = \varnothing$ because, otherwise, Case 2.2 or 2.3 applies. Moreover, for all $1 \leq j \leq n$

(3) $X \perp_{cl(G)} W_j$ follows from the pairwise separation base of G because $W_j \notin uc_G(X)$ and $W_j \notin ad_G(X)$, since $W_j \leftrightarrow X$ is not in G because, otherwise, Case 2.1 applies. Moreover, for all $1 \leq i \leq m$ and $1 \leq j \leq n$

(4) $V_i \perp_{cl(G)} W_j$ follows from the pairwise separation base of G because $uc_G(V_i) \neq uc_G(W_j)$ and $V_i \notin ad_G(W_j)$, since $sp_G(V_i) = \varnothing$ because, otherwise, Case 2.2 or 2.3 applies. Then,

(5) $X \perp_{cl(G)} Y|Z$ by repeated symmetry, composition and weak union.

□

We sort the connectivity components of a MAMP CG G as $K_1, \ldots, K_n$ st if $X \to Y$ is in G, then $X \in K_i$ and $Y \in K_j$ with $i < j$. It is worth mentioning that, in the proofs below, we make use of the fact that the independence model represented by G satisfies weak transitivity by Corollary 3. Note, however, that this property is not used in the construction of $cl(G)$. In the expressions below, we give equal precedence to the operators set minus, set union and set intersection.

Lemma 5. *Let X and Y denote two nodes of a MAMP CG G st $X, Y \in K_m$, $X \perp_G Y|Z$ and $Z \cap (K_{m+1} \cup \ldots \cup K_n) = \varnothing$. Let H denote the subgraph of G induced by K_m. Let $W = Z \cap K_m$. Let W_1 denote a minimal (wrt set inclusion) subset of W st $X \perp_H W \setminus W_1|W_1$. Then, $X \perp_{cl(G)} Y|Z \cup pa_G(X \cup W_1)$.*

Proof. We define the restricted separation base of G as the following set of separations:
R1. $A \perp B|ne_G(A)$ for all $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) = uc_G(B)$, and
R2. $A \perp B$ for all $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) \neq uc_G(B)$.

We define the extended separation base of G as the following set of separations:

E1. $A \perp B | ne_G(A) \cup pa_G(K_m)$ for all $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) = uc_G(B)$, and

E2. $A \perp B | pa_G(K_m)$ for all $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) \neq uc_G(B)$.

Note that the separations E1 (resp. E2) are in one-to-one correspondence with the separations R1 (resp. R2) st the latter can be obtained from the former by adding $pa_G(K_m)$ to the conditioning sets. Let $W_2 = W \setminus W_1$. Then, $X \perp_H W_2 | W_1$ implies that $X \perp_{cl(H)} W_2 | W_1$ by Lemma 4. Note also that the pairwise separation base of H coincides with the restricted separation base of G. Then, $X \perp_{cl(H)} W_2 | W_1$ implies that $X \perp W_2 | W_1$ can be derived from the restricted separation base of G by applying the compositional graphoid properties. We can now reuse this derivation to derive $X \perp W_2 | W_1 \cup pa_G(K_m)$ from the extended separation base of G by applying the compositional graphoid properties: It suffices to apply the same sequence of properties but replacing any separation of the restricted separation base in the derivation with the corresponding separation of the extended separation base. In fact, $X \perp W_2 | W_1 \cup pa_G(K_m)$ is not only in the closure of the extended separation base of G but also in the closure of the pairwise separation base of G, i.e. $X \perp_{cl(G)} W_2 | W_1 \cup pa_G(K_m)$. To show it, it suffices to show that the extended separation base is in the closure of the pairwise separation base. Specifically, consider any $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) \neq uc_G(B)$. Then,

(1) $A \perp_{cl(G)} B | pa_G(A)$ follows from the pairwise separation base of G, and
(2) $A \perp_{cl(G)} pa_G(K_m) \setminus pa_G(A) | pa_G(A)$ follows from the pairwise separation base of G by repeated composition. Then,
(3) $A \perp_{cl(G)} B | pa_G(K_m)$ by composition on (1) and (2), and weak union.

Now, consider any $A, B \in K_m$ st $A \notin ad_G(B)$ and $uc_G(A) = uc_G(B)$. Then,

(4) $A \perp_{cl(G)} B | ne_G(A) \cup pa_G(A \cup ne_G(A))$ follows from the pairwise separation base of G. Moreover, for any $C \in A \cup ne_G(A)$
(5) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(C) | pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(6) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(A \cup ne_G(A)) | pa_G(A \cup ne_G(A))$ by weak union. Then,
(7) $A \perp_{cl(G)} pa_G(K_m) \setminus pa_G(A \cup ne_G(A)) | ne_G(A) \cup pa_G(A \cup ne_G(A))$ by repeated symmetry, composition and weak union. Then,
(8) $A \perp_{cl(G)} B | ne_G(A) \cup pa_G(K_m)$ by composition on (4) and (7), and weak union.

Note that $X \perp_H Y | W_1$ because, otherwise, $X \not\perp_G Y | Z$ which is a contradiction. Then, we can repeat the reasoning above to show that $X \perp_{cl(G)} Y | W_1 \cup pa_G(K_m)$. Then, $X \perp_{cl(G)} Y \cup W_2 | W_1 \cup pa_G(K_m)$ by composition on $X \perp_{cl(G)} W_2 | W_1 \cup pa_G(K_m)$. Finally, we show that this implies that $X \perp_{cl(G)} Y | Z \cup pa_G(X \cup W_1)$. Specifically,

(9) $X \perp_{cl(G)} Y \cup W_2 | W_1 \cup pa_G(K_m)$ as shown above. Moreover, for any $C \in X \cup W_1$
(10) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(C) | pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(11) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(X \cup W_1) | pa_G(X \cup W_1)$ by weak union. Then,
(12) $X \perp_{cl(G)} pa_G(K_m) \setminus pa_G(X \cup W_1) | W_1 \cup pa_G(X \cup W_1)$ by repeated symmetry, composition and weak union. Then,
(13) $X \perp_{cl(G)} Y \cup W_2 | W_1 \cup pa_G(X \cup W_1)$ by contraction on (9) and (12), and decomposition. Moreover, for any $C \in X \cup W_1$
(14) $C \perp_{cl(G)} Z \setminus W \cup pa_G(X \cup W_1) \setminus pa_G(C) | pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(15) $C \perp_{cl(G)} Z \setminus W \setminus pa_G(X \cup W_1) | pa_G(X \cup W_1)$ by weak union. Then,
(16) $X \perp_{cl(G)} Z \setminus W \setminus pa_G(X \cup W_1) | W_1 \cup pa_G(X \cup W_1)$ by repeated symmetry, composition and weak union. Then,
(17) $X \perp_{cl(G)} Y | Z \cup pa_G(X \cup W_1)$ by composition on (13) and (16), and weak union.

□

Lemma 6. *Let X and Y denote two nodes of a MAMP CG G st $Y \in K_1 \cup \ldots \cup K_m$, $X \in K_m$ and $X \perp_G Y|Z$. Let H denote the subgraph of G induced by K_m. Let $W = Z \cap K_m$. Let W_1 denote a minimal (wrt set inclusion) subset of W st $X \perp_H W \setminus W_1|W_1$. Then, $X \not\perp_G C|Z$ for all $C \in pa_G(X \cup W_1) \setminus Z$.*

Proof. Note that $X \not\perp_H D|W \setminus D$ for all $D \in W_1$. To see it, assume the contrary. Then, $X \perp_H D|W \setminus D$ and $X \perp_H W \setminus W_1|W_1$ imply $X \perp_H W \setminus W_1 \cup D|W_1 \setminus D$ by intersection, which contradicts the definition of W_1. Finally, note that $X \not\perp_H D|W \setminus D$ implies that there is a $(W \setminus D)$-open path between X and D in G whose all nodes are in K_m. Then, $X \not\perp_G C|Z$ for all $C \in pa_G(X \cup W_1) \setminus Z$.

□

Lemma 7. *Let X and Y denote two nodes of a MAMP CG G st $Y \in K_1 \cup \ldots \cup K_{m-1}$, $X \in K_m$, $X \perp_G Y|Z$ and $Z \cap (K_{m+1} \cup \ldots \cup K_n) = \varnothing$. Let H denote the subgraph of G induced by K_m. Let $W = Z \cap K_m$. Let W_1 denote a minimal (wrt set inclusion) subset of W st $X \perp_H W \setminus W_1|W_1$. Then, $X \perp_{cl(G)} Y|Z \cup pa_G(X \cup W_1)$.*

Proof. Let $W_2 = W \setminus W_1$. Note that $X \not\perp_G C|Z$ for all $C \in pa_G(X \cup W_1) \setminus Z$ by Lemma 6, because $Y \in K_1 \cup \ldots \cup K_{m-1}$, $X \in K_m$ and $X \perp_G Y|Z$. Then, $Y \notin pa_G(X \cup W_1)$ because, otherwise, $X \not\perp_G Y|Z$ which is a contradiction. Moreover, for any $C \in X \cup W_1$

(1) $C \perp_{cl(G)} Y \cup pa_G(K_m) \setminus pa_G(C)|pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(2) $C \perp_{cl(G)} Y|pa_G(K_m)$ by weak union. Then,
(3) $X \perp_{cl(G)} Y|W_1 \cup pa_G(K_m)$ by repeated symmetry, composition and weak union. Moreover,
(4) $X \perp_{cl(G)} W_2|W_1 \cup pa_G(K_m)$ as shown in the third paragraph of the proof of Lemma 5. Then,
(5) $X \perp_{cl(G)} Y \cup W_2|W_1 \cup pa_G(K_m)$ by composition on (3) and (4). Moreover, for any $C \in X \cup W_1$
(6) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(C)|pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(7) $C \perp_{cl(G)} pa_G(K_m) \setminus pa_G(X \cup W_1)|pa_G(X \cup W_1)$ by weak union. Then,
(8) $X \perp_{cl(G)} pa_G(K_m) \setminus pa_G(X \cup W_1)|W_1 \cup pa_G(X \cup W_1)$ by repeated symmetry, composition and weak union. Then,
(9) $X \perp_{cl(G)} Y \cup W_2|W_1 \cup pa_G(X \cup W_1)$ by contraction on (5) and (8), and decomposition. Moreover, for any $C \in X \cup W_1$
(10) $C \perp_{cl(G)} Z \setminus W \cup pa_G(X \cup W_1) \setminus pa_G(C)|pa_G(C)$ follows from the pairwise separation base of G by repeated composition. Then,
(11) $C \perp_{cl(G)} Z \setminus W \setminus pa_G(X \cup W_1)|pa_G(X \cup W_1)$ by weak union. Then,
(12) $X \perp_{cl(G)} Z \setminus W \setminus pa_G(X \cup W_1)|W_1 \cup pa_G(X \cup W_1)$ by repeated symmetry, composition and weak union. Then,
(13) $X \perp_{cl(G)} Y|Z \cup pa_G(X \cup W_1)$ by composition on (9) and (12), and weak union.

□

Proof of Theorem 6. Since the independence model induced by G satisfies the decomposition property and $cl(G)$ satisfies the composition property, it suffices to prove the theorem for $|X| = |Y| = 1$. Moreover, assume without loss of generality that $Y \in K_1 \cup \ldots \cup K_m$ and $X \in K_m$. We prove the theorem by induction on $|Z|$. The theorem holds for $|Z| = 0$ and $m = 1$ by Lemma 5, because $X, Y \in K_1$, $X \perp_G Y|Z$, $Z \cap (K_2 \cup \ldots \cup K_n) = \varnothing$ and $pa_G(X \cup W_1) \setminus Z = \varnothing$. Assume as induction hypothesis that the theorem holds for $|Z| = 0$ and $m < l$. We now prove it for $|Z| = 0$ and $m = l$. Consider the following cases.

Case 1: Assume that $Y \in K_1 \cup \ldots \cup K_{l-1}$. Then,

(1) $X \perp_{cl(G)} Y|Z \cup pa_G(X \cup W_1)$ by Lemma 7, because $Y \in K_1 \cup \ldots \cup K_{l-1}$, $X \in K_l$, $X \perp_G Y|Z$ and $Z \cap (K_{l+1} \cup \ldots \cup K_n) = \varnothing$. Moreover, for any $C \in pa_G(X \cup W_1) \smallsetminus Z$

(2) $X \not\perp_G C|Z$ by Lemma 6, because $Y \in K_1 \cup \ldots \cup K_{l-1}$, $X \in K_l$ and $X \perp_G Y|Z$. Then,

(3) $C \perp_G Y|Z$ because, otherwise, $X \not\perp_G Y|Z$ which is a contradiction. Then,

(4) $C \perp_{cl(G)} Y|Z$ by the induction hypothesis, because $C, Y \in K_1 \cup \ldots \cup K_{l-1}$. Then,

(5) $pa_G(X \cup W_1) \smallsetminus Z \perp_{cl(G)} Y|Z$ by repeated symmetry and composition. Then,

(6) $X \perp_{cl(G)} Y|Z$ by symmetry, contraction on (1) and (5), and decomposition.

Case 2: Assume that $Y \in K_l$. Then,

(1) $X \perp_{cl(G)} Y|Z \cup pa_G(X \cup W_1)$ by Lemma 5, because $X, Y \in K_l$, $X \perp_G Y|Z$ and $Z \cap (K_{l+1} \cup \ldots \cup K_n) = \varnothing$. Moreover, for any $D \in pa_G(X \cup W_1) \smallsetminus Z$

(2) $X \not\perp_G D|Z$ by Lemma 6, because $X, Y \in K_l$ and $X \perp_G Y|Z$. Then,

(3) $Y \perp_G D|Z$ because, otherwise, $X \not\perp_G Y|Z$ which is a contradiction. Then,

(4) $Y \perp_{cl(G)} D|Z$ by Case 1 replacing X with Y and Y with D, because $D \in K_1 \cup \ldots \cup K_{l-1}$, $Y \in K_l$ and (3). Then,

(5) $Y \perp_{cl(G)} pa_G(X \cup W_1) \smallsetminus Z|Z$ by repeated composition. Then,

(6) $X \perp_{cl(G)} Y|Z$ by symmetry, contraction on (1) and (5), and decomposition.

This ends the proof for $|Z| = 0$. Assume as induction hypothesis that the theorem holds for $|Z| < t$. We now prove it for $|Z| = t$ and $m = 1$. Let K_j be the connectivity component st $Z \cap K_j \neq \varnothing$ and $Z \cap (K_{j+1} \cup \ldots \cup K_n) = \varnothing$. Consider the following cases.

Case 3: Assume that $j = 1$. Then, $X \perp_{cl(G)} Y|Z$ holds by Lemma 5, because $X, Y \in K_1$, $X \perp_G Y|Z$, $Z \cap (K_2 \cup \ldots \cup K_n) = \varnothing$ and $pa_G(X \cup W_1) \smallsetminus Z = \varnothing$.

Case 4: Assume that $j > 1$ and $pa_G(Z \cap K_j) \smallsetminus Z = \varnothing$. Then, note that there is no $(Z \smallsetminus C)$-open path between X and any $C \in Z \cap K_j$. To see it, assume the contrary. Since $X \in K_1$ and $j > 1$, the path must reach K_j from one of its parents or children. However, the path cannot reach K_j from one of its children because, otherwise, the path has a triplex node outside Z since $X \in K_1$, $j > 1$ and $Z \cap (K_{j+1} \cup \ldots \cup K_n) = \varnothing$. This contradicts that the path is $(Z \smallsetminus C)$-open. Then, the path must reach K_j from one of its parents. However, this contradicts that the path is $(Z \smallsetminus C)$-open, because $pa_G(Z \cap K_j) \smallsetminus Z = \varnothing$. Then,

(1) $X \perp_G C|Z \smallsetminus C$ as shown above. Then,

(2) $X \perp_{cl(G)} C|Z \smallsetminus C$ by the induction hypothesis. Moreover,

(3) $X \perp_G Y|Z \smallsetminus C$ by contraction on $X \perp_G Y|Z$ and (1), and decomposition. Then,

(4) $X \perp_{cl(G)} Y|Z \smallsetminus C$ by the induction hypothesis. Then,

(5) $X \perp_{cl(G)} Y|Z$ by composition on (2) and (4), and weak union.

Case 5: Assume that $j > 1$ and $pa_G(C) \smallsetminus Z \neq \varnothing$ for some $C \in Z \cap K_j$. Then, note that there is no $(Z \smallsetminus C)$-open path between X and Y. To see it, assume the contrary. If C is not in the path, then $C \in pa_G(D)$ st $-D-$ is in the path and $D \in Z$ because, otherwise, the path is Z-open which contradicts that $X \perp_G Y|Z$. However, this implies a contradiction because $C \in K_j$ and thus $D \in K_{j+1} \cup \ldots \cup K_n$, but $Z \cap (K_{j+1} \cup \ldots \cup K_n) = \varnothing$. Therefore, C must be in the path. In fact, C must be a non-triplex node in the path because, otherwise, the path is not $(Z \smallsetminus C)$-open. Then, either (i) $-C-$, (ii) $\leftarrow C \multimap\!\!\!\circ$ or (iii) $\circ\!\!\!\multimap C \rightarrow$ is in the path. Case (i) implies that the path is Z-open, because $pa_G(C) \smallsetminus Z \neq \varnothing$. This contradicts that $X \perp_G Y|Z$. Cases (ii) and (iii) imply that the path has a directed subpath from C to (iv) X, (v) Y or (vi) a triplex node E in the path. Cases (iv) and (v) are impossible because $X, Y \in K_1$ but $C \in K_j$ with $j > 1$. Case (vi) contradicts that the path is $(Z \setminus C)$-open, because $C \in K_j$ and thus $E \in K_{j+1} \cup \ldots \cup K_n$, but $Z \cap (K_{j+1} \cup \ldots \cup K_n) = \varnothing$. Then,

(1) $X \perp_G Y | Z \setminus C$ as shown above. Then,
(2) $X \perp_{cl(G)} Y | Z \setminus C$ by the induction hypothesis. Moreover,
(3) $X \perp_G C | Z \setminus C$ or $C \perp_G Y | Z \setminus C$ by weak transitivity on $X \perp_G Y | Z$ and (1). Then,
(4) $X \perp_{cl(G)} C | Z \setminus C$ or $C \perp_{cl(G)} Y | Z \setminus C$ by the induction hypothesis. Then,
(5) $X \perp_{cl(G)} Y | Z$ by symmetry, composition on (2) and (4), and weak union.

This ends the proof for $|Z| = t$ and $m = 1$. Assume as induction hypothesis that the theorem holds for $|Z| = t$ and $m < l$. In order to prove it for $|Z| = t$ and $m = l$, it suffices to repeat Cases 1 and 2 if $Z \cap (K_{l+1} \cup \ldots \cup K_n) = \varnothing$, and Cases 4 and 5 replacing 1 with l otherwise.

□

- In page 24, the sentence

  "Note that ρ_1 cannot be Z-open because, otherwise, Part 2 would apply."

  should be replaced by

  "Note that ρ_1 cannot be Z-open st all its triplex nodes are in Z because, otherwise, Part 2 would apply."

References

Lauritzen, S. L. *Graphical Models*. Oxford University Press, 1996.