Tractable Subclasses of the Point-Interval Algebra: A Complete Classification

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Abstract

Several algebras have been proposed for reasoning about qualitative constraints over time. One of these algebras is Vilain's point-interval algebra, which can relate time points with time intervals. Apart from being a stand-alone qualitative algebra, it is also used as a subalgebra in Meiri's approach to temporal reasoning, which combines reasoning about quantitative and qualitative temporal constraints over both time points and time intervals. While the satsifiability problem for the full point-interval algebra is known to be NP-complete, not much has been known about its 4294967296 subclasses. We provide in this paper a complete classification of satisfiability for all these subclasses into polynomial and NP-complete respectively. We also identify all maximal tractable subalgebras—nine in total.

1 INTRODUCTION

Reasoning about temporal constraints is an important task in many areas of AI and elsewehere, such as planning (Allen, 1991), natural language processing (Song and Cohen, 1988), time serialization in archeology (Golumbic and Shamir, 1993) *etc.* In most applications, knowledge of temporal constraints is expressed in terms of collections of relations between time intervals or time points. Often we are only interested in qualitiative relations, *i.e.* the relative ordering of time points but not their exact occurrences in time. There are two archetypical examples of qualitative temporal reasoning: *Allen's algebra* ($\mathcal{A}$) (Allen, 1983) for reasoning about time intervals and the *point algebra* (PA) (Vilain, 1982) for reasoning about time points.

Attempts have been made to integrate reasoning about time intervals and time points. Meiri's (1991) approach to temporal reasoning makes it possible to reason about time points and time intervals with respect to both qualitative and quantitative time. This framework can be restricted to qualitiative time and the resulting fragment is known as the *qualitative algebra* (QA). In QA, a qualitative constraint between two objects O_i and O_j (each may be a point or an interval), is a disjunction of the form $(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$ where each one of the r_i's is a *basic relation* that may exist between two objects. There are three types of basic relations:

1. Interval-interval relations that can hold between pairs of intervals. These relations correspond to Allen's algebra.
2. Point-point relations that can hold between pairs of points. These relations correspond to the point algebra.
3. Point-interval and interval-point relations that can hold between a point and an interval and vice-versa. These relations were introduced by Vilain (1982). The point-interval and interval-point relations are symmetric so we will only consider the point-interval relations in the sequel.

The satisfiability problem for the point algebra is known to be tractable (Vilain et al., 1989) and the satisfiability problem for Allen's algebra is NP-complete (Vilain et al., 1989). However, a large number of tractable subclasses of Allen's algebra has been reported in the literature (van Beek and Cohen, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1996a). Clearly, QA suffers from computational difficulties since it subsumes the Allen algebra. Even worse, Meiri (1991) shows that the satisfiability problem is NP-complete even for point-interval relations. Besides this negative result, not very much is known about the computational properties of subclasses of the point-interval algebra. This is an unfortunate situation if we want to find tractable subclasses of the qualitative algebra since the point-interval and interval-point algebras are the glue that ties the world of time points together with the world of time intervals.

We also have reasons to believe that the point-interval algebra itself can be interesting in applications such as reasoning about action and change. In certain approaches to action and change, such as the Features and Fluents framework by Sandewall (1994), a clear distinction is made between observations and actions. Typically, observations occur at a single time point while actions occur over extended periods of time. Determining temporal relations between observations and actions in a given scenario seems to be a problem which can be addressed by reasoning in the point-interval algebra.

The main result of this paper is a complete classification of all subclasses of the point-interval algebra with respect to tractability. The classification makes it possible to determine whether a given subclass is tractable or not by a simple test that can be easily carried out by hand or automatically. We have thus gained a clear picture of the borderline between tractability and intractability in the point-interval algebra. In this process, we have also taken a small step towards a deeper understanding of the qualitative algebra.

A few words on methodology seem appropriate at this point. The proof of the main theorem relies on a quite extensive case analysis performed by a computer. The number of cases considered in this analysis was approximately 10^6. Naturally, such an analysis cannot be reproduced in a research paper or be verified manually. To allow for the verification of our results, we include a description of the program used in the analysis. Furthermore, the programs used can be obtained from the authors.

The rest of this paper is organized as follows: Section 2 defines the point-interval algebra and some auxiliary concepts. Section 3 contains the classification of subclasses. Section 4 is a brief discussion of the results and Section 5 concludes the paper. Most of the proofs are postponed to the appendix. Due to space limitations, we have not been able to give all proofs in their entirety. The full proofs can be found in the technical report (Jonsson et al., 1996).

2 POINT-INTERVAL RELATIONS

The point-interval approach to reasoning about time is based on the notions *time points*, *time intervals* and *binary relations* on them. A time point p is a variable interpreted over the set of real numbers $\mathbb{R}$. A time interval I is represented by a pair $\langle I^-, I^+ \rangle$ satisfying $I^- < I^+$ where I^- and I^+ are interpreted over $\mathbb{R}$. We assume that we have a fixed universe of variable names for time points and time intervals. Then, an *$\mathcal{V}$-interpretation* is a function that maps time point variables to $\mathbb{R}$ and time interval variables to $\mathbb{R} \times \mathbb{R}$ and satisfies the previously stated restrictions. We will frequently extend the notation by denoting the the first component of $\Im(I)$ by $\Im(I^-)$ and the second by $\Im(I^+)$.

Given an interpreted time point and an interpreted time interval, their relative positions can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic point-interval relations* where each basic relation can be defined in terms of its endpoint relations (see Table 1). A formula of the form pBI where p is a time point, I a time interval and $B \in \mathbf{B}$, is said to be satisfied by an $\mathcal{V}$-interpretation iff the interpretation of the points and intervals satisfies the endpoint relations specified in Table 1.

To express indefinite information, unions of the basic relations are used, written as sets of basic relations, leading to 2^5 binary *point-interval relations*. Naturally, a set of basic relations is to be interpreted as a disjunction of the basic relations. The set of all point-interval relations $2^{\mathbf{B}}$ is denoted by $\mathcal{V}$. Relations of special interest are the *null* relation $\varnothing$ (also denoted by $\bot$) and the *universal* relation $\mathbf{B}$ (also denoted $\top$). With the notation $\neg x$ we mean the relation $\mathbf{B} - \{x\}$, *e.g.* $\neg \mathtt{a} = \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}\}$.

A formula of the form $p\{B_1, \ldots, B_n\}I$ is called a *point-interval* formula. Such a formula is satisfied by an $\mathcal{V}$-interpretation $\Im$ iff pB_iI is satisfied by $\Im$ for some i, $1 \le i \le n$. A finite set Θ of point-interval formulae is said to be *$\mathcal{V}$-satisfiable* iff there exists an $\mathcal{V}$-interpretation $\Im$ that satisfies every formula of Θ. Such a satisfying $\mathcal{V}$-interpretation is called an *$\mathcal{V}$-model* of Θ. The reasoning problem we will study is the following:

INSTANCE: A finite set Θ of point-interval formulae.
QUESTION: Does there exist an $\mathcal{V}$-model of Θ?

We denote this problem $\mathcal{V}$-SAT. In the following, we often consider restricted versions of $\mathcal{V}$-SAT where the relations used in formulae in Θ are only from a subset $\mathcal{S}$ of $\mathcal{V}$. In this case we say that Θ is a set of formulae over $\mathcal{S}$ and we use a parameter in the problem description to denote the subclass under consideration, *e.g.* $\mathcal{V}$-SAT($\mathcal{S}$).

Meiri's extended definition of the point-interval algebra consists of $\mathcal{V}$ equipped with two binary operations *intersection* and *composition*. However, this definition does not constitute an algebra because it is not closed under composition. We replace the composition operation with an operation on $\mathcal{V}$ we call *cross-composition*. The reason for introducing the algebra is that it is needed for the introduction of a closure operation which will simplify the forthcoming proofs.

Definition 2.1 Let $\mathbf{B} = \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}, \mathtt{a}\}$. The *point-interval algebra* consists of the set $\mathcal{V} = 2^{\mathbf{B}}$ and the operations binary *intersection* (denoted by $\cap$) and ternary *cross-composition* (denoted by $\otimes$). Intersection is defined as $\forall p, I : p(R \cap S)I \Leftrightarrow pRI \wedge pSI$ while cross-composition is defined as $\forall p, I : p(R \otimes S \otimes T)I \Leftrightarrow \exists q, J : (qRJ \wedge qSI \wedge pTJ)$.

Table 1: The five basic relations of the $\mathcal{V}$-algebra. The endpoint relation $I^- < I^+$ that is valid for all relations has been omitted.

Basic relation	Symbol	Example	Endpoint rel.
p before I	b	p III	$p < I^-$
p starts I	s	p III	$p = I^-$
p during I	d	p III	$I^- < p < I^+$
p finishes I	f	p III	$p = I^+$
p after I	a	p III	$p > I^+$

It can easily be verified that $R \otimes S \otimes T = \bigcup\{B \otimes B' \otimes B'' | B \in R, B'' \in S, B''' \in T\}$, *i.e.* cross-composition is the union of the component-wise cross-composition of basic relations.

Next, we introduce a *closure* operation $\mathcal{C}_\mathcal{V}$ together with a *duality* operator $\mathcal{D}_\mathcal{V}$. Both $\mathcal{C}_\mathcal{V}$ and $\mathcal{D}_\mathcal{V}$ transform a given subclass of $\mathcal{V}$ to one that is polynomially equivalent to the original subclass wrt. satisfiability. The closure operation is similar to the closure operation for the Allen algebra introduced in (Nebel and Bürckert, 1995), and is defined as follows.

Definition 2.2 Let $\mathcal{S} \subseteq \mathcal{V}$. Then we denote by $\mathcal{C}_\mathcal{V}(\mathcal{S})$ the $\mathcal{V}$*-closure* of $\mathcal{S}$, defined as the least subalgebra containing $\mathcal{S}$ closed under intersection and cross-composition.

A program for computing $\mathcal{V}$-closures can be obtained from the authors.

Lemma 2.3 Let $\mathcal{S} \subseteq \mathcal{V}$. Then $\mathcal{V}$-SAT($\mathcal{C}_\mathcal{V}(\mathcal{S})$) can be polynomially transformed to $\mathcal{V}$-SAT(S).

Corollary 2.4 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{V}$-SAT($\mathcal{C}_\mathcal{V}(\mathcal{S})$) is polynomial. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-complete iff $\mathcal{V}$-SAT($\mathcal{C}_\mathcal{V}(\mathcal{S})$) is NP-complete.

Next we introduce the duality operator and show that it has the same transformational properties as the closure operation.

Definition 2.5 Let $R \in \mathcal{V}$. Define $\mathcal{D}_\mathcal{V}(R)$ as the set $\{\beta(r) \,|\, r \in R\}$ where $\beta(r)$ is defined as follows: $\beta(\mathtt{b}) = \mathtt{a}$, $\beta(\mathtt{s}) = \mathtt{f}$, $\beta(\mathtt{d}) = \mathtt{d}$, $\beta(\mathtt{f}) = \mathtt{s}$ and $\beta(\mathtt{a}) = \mathtt{b}$.

Let $\mathcal{S} \subseteq \mathcal{V}$. Define $\mathcal{D}_\mathcal{V}(\mathcal{S})$ as the set $\{\mathcal{D}_\mathcal{V}(R) \,|\, R \in \mathcal{S}\}$.

Lemma 2.6 Let $\mathcal{S} \subseteq \mathcal{V}$. Then $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) can be polynomially transformed to $\mathcal{V}$-SAT(S).

Proof sketch: Let Θ be an instance of the $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) problem that have a $\mathcal{V}$-model $\Im$. Construct the following $\mathcal{V}$-SAT(S) instance: $\Theta' = \{p\mathcal{D}_\mathcal{V}(R)I \,|\, pRI \in \Theta\}$. A $\mathcal{V}$-interpretation $\Im'$ of Θ' can be constructed as follows: Let $\Im'(p) = -\Im(p)$ for each time point p appearing in Θ and let $\Im'(I^-) = -\Im(I^+)$, $\Im'(I^+) = -\Im(I^-)$ for each time interval I appearing in Θ. Clearly, $\Im'$ is a $\mathcal{V}$-model of Θ'. Showing the converse direction is analogous. □

Corollary 2.7 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) is polynomial. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-complete iff $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) is NP-complete.

3 CLASSIFICATION OF $\mathcal{V}$

We begin this section by defining nine tractable subalgebras of the point-interval algebra. Later on, we show that these algebras are the only maximal tractable subalgebras of $\mathcal{V}$. Before we can define the algebras we need a definition concerning the point algebra.

Definition 3.1 A *PA formula* is an expression of the form xry where r is a member of $\{<, \leq, =, \neq, \geq, >, \perp, \top\}$ and x, y denote real-valued variables. The symbol $\perp$ denotes the relation $\varnothing$ which is unsatisfiable for every choice of $x, y \in \mathbb{R}$. Similarly, $\top$ denotes the relation $\mathbb{R} \times \mathbb{R}$ which is satisfiable for every choice of $x, y \in \mathbb{R}$.

Let Ω be a set of PA formulae and X the set of variables appearing in Ω. An assignment of real values to the variables in X is said to be an *PA-interpretation* of Ω. Furthermore, Ω is *satisfiable* iff there exists an PA-interpretation $\Im$ such that for each formula $xry \in \Omega$, $\Im(x) r \Im(y)$ holds. Such an PA-interpretation $\Im$ is said to be an *PA-model* of Ω.

The first algebra we will consider has a very close connection to PA. It is defined as follows.

Definition 3.2 The set $\mathcal{V}^{23}$ consists of those relations in $\mathcal{V}$ that can be expressed as one or more PA formulae over time points and endpoints of intervals.

The other eight subalgebras are defined in terms of the $\mathcal{C}_\mathcal{V}$ and $\mathcal{D}_\mathcal{V}$ operators.

Definition 3.3

$$v^{20}_{\mathtt{s}} = \{\{\mathtt{s}\}, \{\mathtt{b},\mathtt{s}\}, \{\mathtt{b},\mathtt{a}\}, \neg\mathtt{d}, \neg\mathtt{f}\},\ \mathcal{V}^{20}_{\mathtt{s}} = \mathcal{C}_\mathcal{V}(v^{20}_{\mathtt{s}})$$

$$\mathcal{V}^{20}_{\mathtt{f}} = \mathcal{D}_\mathcal{V}(\mathcal{V}^{20}_{\mathtt{s}})$$

$$v^{20}_{\mathtt{d}} = \{\{\mathtt{b},\mathtt{a}\}, \neg\mathtt{b}, \neg\mathtt{s}, \neg\mathtt{f}, \neg\mathtt{a}\},\ \mathcal{V}^{20}_{\mathtt{d}} = \mathcal{C}_\mathcal{V}(v^{20}_{\mathtt{d}})$$

$$v^{18}_{\neg\mathtt{a}} = \{\{\mathtt{d}\}, \{\mathtt{b},\mathtt{s},\mathtt{a}\}, \neg\mathtt{s}, \neg\mathtt{f}, \neg\mathtt{a}\},\ \mathcal{V}^{18}_{\neg\mathtt{a}} = \mathcal{C}_\mathcal{V}(v^{18}_{\neg\mathtt{a}})$$

$$\mathcal{V}^{18}_{\neg\mathtt{b}} = \mathcal{D}_\mathcal{V}(\mathcal{V}^{18}_{\neg\mathtt{a}})$$

$$v^{18}_{\neg\mathtt{d}} = \{\{\mathtt{d}\}, \{\mathtt{b},\mathtt{d}\}, \neg\mathtt{s}, \neg\mathtt{d}, \neg\mathtt{f}\},\ \mathcal{V}^{18}_{\neg\mathtt{d}} = \mathcal{C}_\mathcal{V}(v^{18}_{\neg\mathtt{d}})$$

$\mathcal{V}^{17}_{\mathtt{s}} = \{\bot\} \cup \{r \in \mathcal{V} \mid \{\mathtt{s}\} \subseteq r\}$

$\mathcal{V}^{17}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{18}_{\mathtt{s}})$

Given a subalgebra $\mathcal{V}^x_y$, x is the number of relations in the algebra and y is an element that is unique for $\mathcal{V}^x_y$ among the subalgebras of size x. For instance, $\mathcal{V}^{17}_{\mathtt{s}}$ is the only subalgebra of size 17 that contains $\{\mathtt{s}\}$. Let $\mathcal{V}_P$ be the set of all subalgebras in Definition 3.3 plus the algebra $\mathcal{V}^{23}$. The relations included in each of these algebras can be found in Table 2. Further, let $\mathcal{V}_{NP}$ denote the set of subalgebras listed in Table 3. We have the following theorem.

Theorem 3.4 If $V \in \mathcal{V}_P$ then $\mathcal{V}$-SAT(V) is polynomial. If $V \in \mathcal{V}_{NP}$ then $\mathcal{V}$-SAT(V) is NP-complete.

Proof: See Appendices A and B for the results concerning $\mathcal{V}_P$ and $\mathcal{V}_{NP}$, respectively. □

The main theorem can now be stated.

Theorem 3.5 For $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{V}$-SAT$(\mathcal{S})$ is polynomial iff $\mathcal{S}$ is a subset of some member of $\mathcal{V}_P$.

Proof: *if:* For each $V \in \mathcal{V}_P$, $\mathcal{V}$-SAT(V) is polynomial by Theorem 3.4.

only-if: Assume there exists a subclass $\mathcal{S} \subseteq \mathcal{V}$ such that $\mathcal{V}$-SAT$(\mathcal{S})$ is polynomial and $\mathcal{S}$ is not a subset of any algebra in $\mathcal{V}_P$. Without loss of generality, let $\mathcal{S}$ be such a class with the least number of elements. For each subalgebra V in $\mathcal{V}_P$, choose a relation x such that $x \in \mathcal{S}$ and $x \notin V$. This can always be done since $\mathcal{S} \not\subseteq V$. Let X be the set of these relations. The following holds for X:

1. $\mathcal{V}$-SAT(X) is polynomial since $X \subseteq \mathcal{S}$.
2. X is not a subset of any algebra in $\mathcal{V}_P$.

$\mathcal{S}$ is a minimal set satisfying (1) and (2) above. Hence, $|\mathcal{S}| \leq |X|$. Furthermore, $X \subseteq \mathcal{S}$ so $|\mathcal{S}| \leq |X|$ and $|\mathcal{S}| = |X|$. The set $\mathcal{V}_P$ contains nine algebras so by the construction of X, $|X| \leq 9$. As a consequence, $|\mathcal{S}| \leq 9$.

To show that $\mathcal{S}$ does not exist, a machine-assisted case analysis of the following form was performed:

1. Generate all subsets of $\mathcal{V}$ of size ≤ 9. There are $\sum_{i=0}^{9} \binom{32}{i} \approx 4.3 \times 10^6$ such subsets.
2. Let $\mathcal{T}$ be such a set. Test: $\mathcal{T}$ is a subset of some subalgebra in $\mathcal{V}_P$ or $D \subseteq \mathcal{C}_{\mathcal{V}}(\mathcal{T})$ for some $D \in \mathcal{V}_{NP}$.

The test succeeds for all $\mathcal{T}$. Hence, by Theorem 3.4, either $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-complete or $\mathcal{S}$ is a subset of some member of $\mathcal{V}_P$. Both cases contradict our initial assumptions so $\mathcal{S}$ cannot exist. □

4 DISCUSSION

We have only considered qualitative relations between time points and intervals in this paper. For certain applications this is satisfactory—for others we must have the ability to reason also about quantitative time. Previous research on reasoning about combined qualitative and quantitative time has proven this problem to be computationally hard. However, recent results show that tractable reasoning is possible in certain subclasses of Allen's algebra augmented with quite advanced quantitative information. The linear-programming approach by Jonsson and Bäckström (1996) offers a straightforward method for extending the ORD-Horn subclass with quantitative information. Several other subclasses of Allen's algebra with this property are exhibited in (Drakengren and Jonsson, 1996b). Almost certainly, these methods can be adapted to the point-interval algebra. This opens up for some interesting future research. Another interesting research direction is the study of tractable subclasses of Meiri's unrestricted approach, *i.e.*, allowing for time points and time intervals to be both qualitatively and quantitatively related.

The number of subclasses of $\mathcal{V}$ ($2^{32} \approx 4.3 \times 10^9$) is very small in comparison with the $2^{8192} \approx 10^{2466}$ subclasses of $\mathcal{A}$. In principle it would have been possible to enumerate all subclasses of $\mathcal{V}$ with the aid of a computer. Obviously, this is not the case with $\mathcal{A}$ (at least not with the computers available today). If we want to classify the subclasses of $\mathcal{A}$ with respect to tractability, we must use other methods. We are not pessimistic about the possibility of creating a complexity map of $\mathcal{A}$. Similar projects have been successfully performed in mathematics and computer science. A well-known example is the proof of the four-colour theorem (Appel and Haken, 1976) which combine theoretical studies of planar graphs with extensive machine-generated case analyses. It seems likely that we shall need methods that combine theoretical studies of the structure of $\mathcal{A}$ with brute-force computer methods. Here we can see a challenge for both theoreticians and practitioners in computer science.

5 CONCLUSIONS

We have studied computational properties of the point-interval algebra. All of the 2^{32} possible subclasses are classified with respect to whether their corresponding satisfiability problem is tractable or not. The classification reveals that there are exactly nine maximally tractable subclasses of the algebra.

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Table 2: The maximal tractable subalgebras of $\mathcal{V}$.

	$\mathcal{V}^{23}$	$\mathcal{V}^{20}_{\mathtt{s}}$	$\mathcal{V}^{20}_{\mathtt{f}}$	$\mathcal{V}^{20}_{\mathtt{d}}$	$\mathcal{V}^{18}_{\neg\mathtt{a}}$	$\mathcal{V}^{18}_{\neg\mathtt{b}}$	$\mathcal{V}^{18}_{\neg\mathtt{d}}$	$\mathcal{V}^{17}_{\mathtt{s}}$	$\mathcal{V}^{17}_{\mathtt{f}}$
$\perp$	•	•	•	•	•	•	•	•	•
{b}	•	•	•	•	•	•	•		
{s}	•	•						•	
{b, s}	•	•	•		•		•	•	
{d}	•			•	•	•	•		
{b, d}	•		•	•	•	•	•		
{s, d}	•			•		•		•	
{b, s, d}	•		•	•	•	•	•	•	
{f}	•		•						•
{b, f}			•						•
{s, f}								•	•
{b, s, f}			•					•	•
{d, f}	•			•	•				•
{b, d, f}	•		•	•	•				•
{s, d, f}	•			•				•	•
{b, s, d, f}	•		•	•	•			•	•
{a}	•	•	•	•	•	•	•		
{b, a}		•	•	•	•	•	•		
{s, a}		•						•	
{b, s, a}		•	•		•		•	•	
{d, a}	•	•		•	•	•	•		
{b, d, a}	•	•	•	•	•	•	•		
{s, d, a}	•	•		•		•		•	
{b, s, d, a}	•	•	•	•	•	•	•	•	
{f, a}	•	•	•			•	•		•
{b, f, a}		•	•			•	•		•
{s, f, a}		•						•	•
{b, s, f, a}		•	•				•	•	•
{d, f, a}	•	•		•	•	•	•		•
{b, d, f, a}	•	•	•	•	•	•	•		•
{s, d, f, a}	•	•		•		•		•	•
$\top$	•	•	•	•	•	•	•	•	•

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Appendix

This appendix collects the tractability and intractability proofs needed for the proof of Theorem 3.5. The former results can be found in part A and the latter results in part B. Due to space limitations, we have not been able to give all proofs in their entirety. Hence, certain proofs are only sketched or outlined. The full proofs can be found in the technical report (Jonsson et al., 1996).

A TRACTABILITY RESULTS

To make the forthcoming proofs less cumbersome, we need a few results of more general character. These can be found in Section A.1. The actual proofs of tractability can be found in Section A.2.

A.1 MODEL TRANSFORMATIONS

One of our main vehicles for showing computational properties of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem.

Definition A.1 Let T be a mapping on $\mathcal{V}$-interpretations. We say that T is a *model transformation*.

This definition is very general. To make it applicable in practice, we need a way of describing model transformations in greater detail.

Definition A.2 Let T be a model transformation. A function $f_T : \mathbf{B} \to 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{V}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and $p(b)J$ under $\Im$ then $p(f_T(b))J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$.

We can now provide two lemmata showing how model transformations can be used.

Lemma A.3 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{V}$ be such that $\mathcal{V}$-SAT($\mathcal{R}$) is polynomial. Let $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{V}$. Let T be a model transformation and f_T a description of T. If the following holds: (1) $r_k \subseteq r'_k$ for $1 \leq k \leq n$; and (2) $f_T(r'_k) \subseteq r_k$ for $1 \leq k \leq n$, then $\mathcal{V}$-SAT($\mathcal{R}'$) is polynomial.

Proof: Let Θ' be a instance of $\mathcal{V}$-SAT($\mathcal{R}'$). Let $\Theta = \{pr_k J \mid pr'_k J \in \Theta'\}$. Obviously, this is a polynomial transformation and Θ is an instance of $\mathcal{V}$-SAT($\mathcal{R}$). We show that Θ has a model iff Θ' has a model. Since it can be decided in polynomial time whether Θ has a model or not, the lemma follows.

only-if: Let $\Im$ be a model of Θ. Recall that $r_k \subseteq r'_k$, $1 \leq k \leq n$. Hence, $\Im$ is a model for Θ' since every relation $pr'I \in \Theta'$ is weaker than the corresponding relation $prI \in \Theta$.

if: Let $\Im'$ be a model of Θ'. We show that $T(\Im')$ is a model of Θ. Arbitrarily choose a formula $pr_k I$ in Θ. Clearly, there exists a formula $pr'_k I \in \Theta'$. Thus, we have that $pr'_k I$ under $\Im'$ which implies $pf_T(r'_k)I$ under $T(\Im')$ since f_T is a description of T. Furthermore, $f_T(r'_k) \subseteq r_k$ so $pr_k I$ under $T(\Im)$. Hence, $T(\Im)$ is a model of Θ. □

Lemma A.4 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{V}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{V}$ be such that $\mathcal{V}$-SAT($\mathcal{R}$) is NP-complete. If there exists a model transformation T with a description f_T such that $f_T(r'_k) \subseteq r_k$ for every $1 \leq k \leq n$ then $\mathcal{V}$-SAT($\mathcal{R}'$) is NP-complete.

Proof: Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($\mathcal{R}$). Let $\Theta' = \{pf_T(r)J \mid prJ \in \Theta\}$. Obviously, this is a polynomial transformation and Θ' is an instance of $\mathcal{V}$-SAT($\mathcal{R}'$). We show that Θ is satisfiable iff Θ' is satisfiable.

if: Let $\Im'$ be a model of Θ'. Recall that $f_T(r'_k) \subseteq r_k$, $1 \leq k \leq n$. Hence, $\Im'$ is a model for Θ since every relation $prI \in \Theta$ is weaker than the corresponding relation $pf_T(r)I \in \Theta'$.

only-if: Let $\Im$ be a model of Θ. We show that $T(\Im)$ is a model of Θ'. Arbitrarily choose a formula prI in Θ'. By the construction of Θ', there exists a formula $psI \in \Theta$ such that $r = f_T(s)$. Thus, we have that psI under $\Im$ which implies $pf_T(s)I$ under $T(\Im)$ since f_T is a description of T. Hence, IrJ under $T(\Im)$. □

Before we define a number of model transformations that we will use later on, we need an auxiliary definition.

Definition A.5 Let $S \subseteq \mathbb{R}$ be finite and denote the absolute value of x with abs(x) The *minimal distance in* S, MD(S), is defined as $\min\{\text{abs}(x-y) \mid x, y \in S \text{ and } x \neq y\}$.

Observe that $|S| \geq 2$ in order to make MD(S) defined. For all such S, MD(S) > 0. The definition of minimal distance can be extended to $\mathcal{V}$-interpretations in the following way: Let $\Im$ be an $\mathcal{V}$-interpretation that assigns values to a set of time points $\mathcal{P}$ and a set of intervals $\mathcal{I}$. Let $\text{MD}(\Im) = \text{MD}(\{\Im(p) \mid p \in \mathcal{P}\} \cup \{\Im(I^-), \Im(I^+) \mid I \in \mathcal{I}\})$.

In the forthcoming four propositions, let $\Im$ be an arbitrary $\mathcal{V}$-interpretation and let $\varepsilon = \text{MD}(\Im)$.

Proposition A.6 Define $T_1(\Im)$ as follows: (1) for each time point p let $T_1(\Im)(p) = \Im(p)$; (2) for each time interval I let $T_1(\Im)(I^-) = \Im(I^-) + \varepsilon$ and $T_1(\Im)(I^+) = \Im(I^+)$. The description of T_1, f_{T_1} is then: $f_{T_1}(\texttt{b}) = \{\texttt{b}\}$, $f_{T_1}(\texttt{s}) = \{\texttt{b}\}$, $f_{T_1}(\texttt{d}) = \{\texttt{d}\}$, $f_{T_1}(\texttt{f}) = \{\texttt{f}\}$ and $f_{T_1}(\texttt{a}) = \{\texttt{a}\}$.

Proposition A.7 Define $T_2(\Im)$ as follows: (1) for each time point p let $T_2(\Im)(p) = \Im(p)$; (2) for each time interval I let $T_2(\Im)(I^-) = \Im(I^-) + \varepsilon$ and $T_2(\Im)(I^+) = \Im(I^+) - \varepsilon$. The description of T_2, f_{T_2} is then: $f_{T_2}(\texttt{b}) = \{\texttt{b}\}$, $f_{T_2}(\texttt{s}) = \{\texttt{b}\}$, $f_{T_2}(\texttt{d}) = \{\texttt{d}\}$, $f_{T_2}(\texttt{f}) = \{\texttt{a}\}$ and $f_{T_2}(\texttt{a}) = \{\texttt{a}\}$.

Proposition A.8 Define $T_3(\Im)$ as follows: (1) for each time point p let $T_3(\Im)(p) = \Im(p)$; (2) for each time interval I let $T_3(\Im)(I^-) = \Im(I^-) - \varepsilon$ and $T_3(\Im)(I^+) = \Im(I^+)$. The description of T_3, f_{T_3} is then: $f_{T_3}(\texttt{b}) = \{\texttt{b}\}$, $f_{T_3}(\texttt{s}) = \{\texttt{d}\}$, $f_{T_3}(\texttt{d}) = \{\texttt{d}\}$, $f_{T_3}(\texttt{f}) = \{\texttt{f}\}$ and $f_{T_3}(\texttt{a}) = \{\texttt{a}\}$.

Proposition A.9 Define $T_4(\Im)$ as follows: (1) for each time point p let $T_4(\Im)(p) = \Im(p)$; (2) for each time interval I let $T_4(\Im)(I^-) = \Im(I^-)$ and $T_4(\Im)(I^+) = \Im(I^+) - \varepsilon$ The description of T_4, f_{T_4} is then: $f_{T_4}(\texttt{b}) = \{\texttt{b}\}$, $f_{T_4}(\texttt{s}) = \{\texttt{s}\}$, $f_{T_4}(\texttt{d}) = \{\texttt{d}\}$, $f_{T_4}(\texttt{f}) = \{\texttt{a}\}$ and $f_{T_4}(\texttt{a}) = \{\texttt{a}\}$.

A.2 PROOFS OF TRACTABILITY

Proving tractability of $\mathcal{V}$-SAT$(\mathcal{V}^{23})$ is straightforward.

Proposition A.10 Deciding satisfiability of a set of PA formulae is polynomial.

Proof: See (Vilain et al., 1989). □

Lemma A.11 $\mathcal{V}$-SAT$(\mathcal{V}^{23})$ is polynomial.

Proof: Follows immediately from the definition of $\mathcal{V}^{23}$ and the previous proposition. □

Lemma A.12 $\mathcal{V}$-SAT$(\mathcal{V}^{20}_{\texttt{s}})$ is polynomial.

Proof: By Corollary 2.4, it is sufficient to show that $\mathcal{V}$-SAT$(v^{20}_{\texttt{s}})$ is polynomial. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT$(v^{20}_{\texttt{s}})$. Define the function $\sigma : v^{20}_{\texttt{s}} \to \{\leq, =, \neq, \top\}$ as follows: $\sigma(\{\texttt{s}\}) =$ "$=$", $\sigma(\{\texttt{b},\texttt{s}\}) =$ "$\leq$", $\sigma(\{\texttt{b},\texttt{a}\}) =$ "$\neq$", $\sigma(\{\texttt{b},\texttt{s},\texttt{d},\texttt{a}\}) =$ "$\top$" and $\sigma(\{\texttt{b},\texttt{s},\texttt{f},\texttt{a}\}) =$ "$\top$".

Let $\Theta' = \{x_p\sigma(R)y_I \mid pRI \in \Theta\}$. In the sequel, y_I will be considered as the starting point of the interval I. By Proposition A.10, it is polynomial to decide R-satisfiability of Θ'. We show that Θ is $\mathcal{V}$-satisfiable iff Θ' is R-satisfiable.

if: Let $\Im'$ be an R-model of Θ'. Let $\varepsilon = \text{MD}(\Im')/2$. We construct an $\mathcal{V}$-interpretation $\Im$ of Θ as follows:

- For each variable x_p in Θ' let $\Im(p) = \Im'(x_p)$.
- For each variable y_I in Θ' let $\Im(I^-) = \Im'(y_I)$ and $\Im(I^+) = \Im'(y_I) + \varepsilon$

We show that $\Im$ is an $\mathcal{V}$-model of Θ. Let pRI be an arbitrary formula in Θ. We have five different cases.

1. $R = \{\texttt{s}\}$. Then $\Im'(x_p)\sigma(\{\texttt{s}\})\Im'(y_I) \Leftrightarrow \Im'(x_p) = \Im'(y_I) \Leftrightarrow \Im(p) = \Im(I^-)$. Consequently, $p\{\texttt{s}\}I$ under $\Im$.
2. $R = \{\texttt{b},\texttt{s}\}$. Then $\Im'(x_p)\sigma(\{\texttt{b},\texttt{s}\})\Im'(y_I) \Leftrightarrow \Im'(x_p) \leq \Im'(y_I) \Leftrightarrow \Im(p) \leq \Im(I^-)$. Consequently, $p\{\texttt{b},\texttt{s}\}I$ under $\Im$.
3. $R = \{\texttt{b},\texttt{a}\}$. Then $\Im'(x_p)\sigma(\{\texttt{b},\texttt{a}\})\Im'(y_I) \Leftrightarrow \Im'(x_p) \neq \Im'(y_I) \Leftrightarrow \Im(p) \neq \Im(I^-)$. If $\Im(p) < \Im(I^-)$ then $p\{\texttt{b}\}I$ under $\Im$. If $\Im(p) > \Im(I^-)$ then $\Im(p) > \Im(I^+)$ since $I^+ = I^- + \varepsilon$. In this case $p\{\texttt{a}\}I$ under $\Im$. Consequently, $p\{\texttt{b},\texttt{a}\}I$ under $\Im$.
4. $R = \{\texttt{b},\texttt{s},\texttt{f},\texttt{a}\}$. Assume $p\{\texttt{d}\}I$ under $\Im$. Then $\Im(I^-) < \Im(p) < \Im(I^+) \Leftrightarrow \Im(I^-) < \Im(p) < \Im(I^-) + \varepsilon \Leftrightarrow \Im'(y_I) < \Im'(x_p) < \Im'(y_I) + \varepsilon$ which contradicts the choice of ε. Hence, $p\{\texttt{b},\texttt{s},\texttt{f},\texttt{a}\}I$ under $\Im$
5. $R = \{\texttt{b},\texttt{s},\texttt{d},\texttt{a}\}$. This case is analogous to the previous case.

only-if: Let $\Im$ be an $\mathcal{V}$-model of Θ. We construct an R-interpretation $\Im'$ of Θ' as follows:

- For each time point p in Θ let $\Im'(x_p) = \Im(p)$.
- For each time interval I in Θ let $\Im'(y_I) = \Im(I^-)$.

Next, we show that $\Im'$ is an R-model of Θ'. Let x_pRy_I be an arbitrary formula in Θ'. We have four different cases:

1. $R =$ "$=$". Assume $\Im'(x_p) \neq \Im'(y_I)$. Then $\Im(p) \neq \Im(I^-)$ and $p\{\texttt{b},\texttt{d},\texttt{f},\texttt{a}\}I$ under $\Im$. But $x_p = y_I \in \Theta'$ iff $p\{\texttt{s}\}I \in \Theta$ which leads to a contradiction. Hence, $x_p = y_I$ under $\Im'$.
2. $R =$ "$\neq$". This case is analogous to the previous case.
3. $R =$ "$\leq$". Assume $\Im'(x_p) > \Im'(y_I)$. Then $\Im(p) > \Im(I^-)$ and $p\{\texttt{d},\texttt{f},\texttt{a}\}I$ under $\Im$. But $x_p \leq y_I \in \Theta'$ iff $p\{\texttt{b},\texttt{s}\}I \in \Theta$ which leads to a contradiction. Hence, $x_p \leq y_I$ under $\Im'$.
4. $R =$ "$\top$". This relation holds trivially in every R-model of Θ'. □

Lemma A.13 $\mathcal{V}$-SAT($\mathcal{V}^{20}_{\texttt{d}}$) is polynomial.

Proof sketch: By Corollary 2.4, it is sufficient to show that $\mathcal{V}$-SAT($v^{20}_{\texttt{d}}$) is polynomial. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($v^{20}_{\texttt{d}}$). Define the function $\sigma : v^{20}_{\texttt{d}} \to \{\leq, \geq, \neq, \top\}$ as follows: $\sigma(\{\texttt{b},\texttt{a}\}) =$ "$\neq$", $\sigma(\{\texttt{s},\texttt{d},\texttt{f},\texttt{a}\}) =$ "$\geq$", $\sigma(\{\texttt{b},\texttt{s},\texttt{d},\texttt{f}\}) =$ "$\leq$", $\sigma(\{\texttt{b},\texttt{s},\texttt{d},\texttt{a}\}) =$ "$\top$", and $\sigma(\{\texttt{b},\texttt{d},\texttt{f},\texttt{a}\}) =$ "$\top$".

Let $\Theta' = \{x_p \sigma(R) y_I \mid pRI \in \Theta\}$. By Proposition A.10, it is polynomial to decide R-satisfiability of Θ'. To show that Θ is $\mathcal{V}$-satisfiable iff Θ' is R-satisfiable is analogous to the proof of Lemma A.12. □

Lemma A.14 $\mathcal{V}$-SAT($\mathcal{V}^{18}_{\neg\texttt{a}}$) is polynomial.

Proof: By Corollary 2.4, it is sufficient to show that $\mathcal{V}$-SAT($v^{18}_{\neg\texttt{a}}$) is polynomial. Let $r'_1 = \{\texttt{d}\}$, $r'_2 = \{\texttt{b},\texttt{s},\texttt{a}\}$, $r'_3 = \neg\texttt{f}$, $r'_4 = \neg\texttt{a}$ and $r'_5 = \neg\texttt{s}$. Note that $v^{18}_{\neg\texttt{a}} = \bigcup_{i=1}^{5}\{r'_i\}$. Furthermore, let $r_i = r'_i$ for $i \in \{1,3,4,5\}$ and let $r_2 = \{\texttt{b},\texttt{a}\}$. It can easily be verified that $r_k \subseteq r'_k$ for $1 \leq k \leq 5$. Furthermore, $f_{T_1}(r'_k) \subseteq r_k$ for $1 \leq k \leq 5$. Hence, by Lemma A.3, the polynomiality of $\mathcal{V}$-SAT($\mathcal{V}^{18}_{\neg\texttt{a}}$) follows. □

Lemma A.15 $\mathcal{V}$-SAT($\mathcal{V}^{18}_{\neg\texttt{d}}$) is polynomial.

Proof: By Corollary 2.4, it is sufficient to show that $\mathcal{V}$-SAT($v^{18}_{\neg\texttt{d}}$) is polynomial. Let $r'_1 = \{\texttt{d}\}$, $r'_2 = \{\texttt{b},\texttt{d}\}$, $r'_3 = \neg\texttt{s}$, $r'_4 = \neg\texttt{d}$ and $r'_5 = \neg\texttt{f}$. Note that $v^{18}_{\neg\texttt{d}} = \bigcup_{i=1}^{5}\{r'_i\}$. Furthermore, let $r_i = r'_i$ for $i \in \{1,3,4,5\}$ and let $r_4 = \{\texttt{b},\texttt{a}\}$. It can easily be verified that $r_k \subseteq r'_k$ for $1 \leq k \leq 5$. Furthermore, $f_{T_2}(r'_k) \subseteq r_k$ for $1 \leq k \leq 5$. Consequently, by Lemma A.3, the polynomiality of $\mathcal{V}$-SAT($\mathcal{V}^{18}_{\neg\texttt{d}}$) follows. □

Lemma A.16 $\mathcal{V}$-SAT($\mathcal{V}^{17}_{\texttt{s}}$) is polynomial.

Proof: Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($\mathcal{V}^{17}_{\texttt{s}}$). If a formula of the form $p \bot I$ is in Θ then Θ is not satisfiable. Otherwise, consider the following $\mathcal{V}$-interpretation: $\Im(p) = 0$ for every time point p and $\Im(I^-) = 0$ and $\Im(I^+) = 1$ for every interval I. Let pRI be an arbitrary formula in Θ. By the definition of $\mathcal{V}^{17}_{\texttt{s}}$, $\texttt{s} \in R$. Obviously, $\Im$ satisfies pRI. Since it is polynomial to check whether $pRI \in \Theta$ or not, the lemma follows. □

B INTRACTABILITY RESULTS

This section provides proofs for the NP-complete subclasses of $\mathcal{V}$ presented in Table 3. The reductions are mostly made from different subalgebras of Allen's interval algebra. Consequently, we begin this section by recapitulating some results concerning Allen's algebra.

Table 3: NP-complete subclasses of $\mathcal{V}$.

Subclass	Relations	Proof
D_1	$\{\texttt{d}\},\{\texttt{s}\},\{\texttt{b},\texttt{a}\}$	Lemma B.7
D_2	$\{\texttt{d}\},\{\texttt{f}\},\{\texttt{b},\texttt{a}\}$	$D_2 = \mathcal{D}_{\mathcal{V}}(D_1)$
D_3	$\{\texttt{d}\},\{\texttt{s}\},\{\texttt{b},\texttt{f},\texttt{a}\}$	Lemma B.8
D_4	$\{\texttt{d}\},\{\texttt{f}\},\{\texttt{b},\texttt{s},\texttt{a}\}$	$D_4 = \mathcal{D}_{\mathcal{V}}(D_3)$
D_5	$\{\texttt{d}\},\{\texttt{s}\},\{\texttt{s},\texttt{a}\}$	Lemma B.9
D_6	$\{\texttt{d}\},\{\texttt{f}\},\{\texttt{b},\texttt{f}\}$	$D_6 = \mathcal{D}_{\mathcal{V}}(D_5)$
D_7	$\{\texttt{d}\},\{\texttt{s},\texttt{f}\}$	Lemma B.10
D_8	$\{\texttt{s}\},\{\texttt{b},\texttt{f}\}$	Lemma B.11
D_9	$\{\texttt{f}\},\{\texttt{s},\texttt{a}\}$	$D_9 = \mathcal{D}_{\mathcal{V}}(D_8)$
D_{10}	$\{\texttt{s}\},\{\texttt{b},\texttt{f},\texttt{a}\},\{\texttt{b},\texttt{s},\texttt{d},\texttt{f}\}$	$D_8 \subseteq \mathcal{C}_{\mathcal{V}}(D_{10})$
D_{11}	$\{\texttt{f}\},\{\texttt{b},\texttt{s},\texttt{a}\},\{\texttt{s},\texttt{d},\texttt{f},\texttt{a}\}$	$D_{11} = \mathcal{D}_{\mathcal{V}}(D_{10})$
D_{12}	$\{\texttt{d}\},\{\texttt{b},\texttt{f}\}$	Lemma B.12
D_{13}	$\{\texttt{d}\},\{\texttt{s},\texttt{a}\}$	$D_{13} = \mathcal{D}_{\mathcal{V}}(D_{12})$
D_{14}	$\{\texttt{d},\texttt{a}\},\{\texttt{b},\texttt{f}\}$	Lemma B.13
D_{15}	$\{\texttt{b},\texttt{d}\},\{\texttt{s},\texttt{a}\}$	$D_{15} = \mathcal{D}_{\mathcal{V}}(D_{14})$
D_{16}	$\{\texttt{d}\},\{\texttt{s},\texttt{f},\texttt{a}\}$	Lemma B.14
D_{17}	$\{\texttt{d}\},\{\texttt{b},\texttt{s},\texttt{f}\}$	$D_{17} = \mathcal{D}_{\mathcal{V}}(D_{16})$
D_{18}	$\{\texttt{s},\texttt{f}\},\{\texttt{b},\texttt{d}\}$	Lemma B.15
D_{19}	$\{\texttt{s},\texttt{f}\},\{\texttt{d},\texttt{a}\}$	$D_{19} = \mathcal{D}_{\mathcal{V}}(D_{18})$
D_{20}	$\{\texttt{s},\texttt{f}\},\{\texttt{b},\texttt{a}\}$	Lemma B.16
D_{21}	$\{\texttt{d},\texttt{a}\},\{\texttt{b},\texttt{s},\texttt{f}\}$	Lemma B.17
D_{22}	$\{\texttt{b},\texttt{d}\},\{\texttt{s},\texttt{f},\texttt{a}\}$	$D_{22} = \mathcal{D}_{\mathcal{V}}(D_{21})$
D_{23}	$\{\texttt{s},\texttt{d}\},\{\texttt{b},\texttt{f}\}$	Lemma B.18
D_{24}	$\{\texttt{d},\texttt{f}\},\{\texttt{s},\texttt{a}\}$	$D_{24} = \mathcal{D}_{\mathcal{V}}(D_{23})$
D_{25}	$\{\texttt{s},\texttt{d},\texttt{a}\},\{\texttt{b},\texttt{f}\}$	Lemma B.19
D_{26}	$\{\texttt{b},\texttt{d},\texttt{f}\},\{\texttt{s},\texttt{a}\}$	$D_{26} = \mathcal{D}_{\mathcal{V}}(D_{25})$
D_{27}	$\{\texttt{b},\texttt{f}\},\{\texttt{s},\texttt{a}\}$	Lemma B.20
D_{28}	$\{\texttt{s},\texttt{f}\},\{\texttt{b},\texttt{d},\texttt{a}\}$	Lemma B.21

B.1 ALLEN'S ALGEBRA

Allen's interval algebra (Allen, 1983) is based on the notion of *relations between pairs of intervals*. An interval X is represented as an ordered pair $\langle X^-, X^+\rangle$ of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*. Their exact definitions can be found in (Allen, 1983). Such disjunctions are represented as sets of basic relations. The algebra is provided with the operations of *converse, intersection and composition* on intervals. The exact definitions of these operations can be found in (Allen, 1983). By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. The reasoning problem we will consider is the problem of *satisfiability* ($\mathcal{A}$-SAT) of a set of interval variables with relations between them, *i.e.* deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals hold. Such an assignment is said to be a $\mathcal{A}$-*model* for the interval variables and relations. For $\mathcal{A}$, we have the following result.

Theorem B.1 Let $\mathcal{N}_2$ be the set $\{\{\prec, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{f}^{\smile}\}, \{\prec, \mathsf{d}, \mathsf{o}, \mathsf{m}, \mathsf{s}\}, \{\mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\}\}$ and let Δ_0 be the set $\{\{\prec, \succ\}, \{\equiv, \mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{m}, \mathsf{m}^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}\}$. $\mathcal{A}-$SAT($\mathcal{S}$) is NP-complete if $\mathcal{N}_2 \subseteq \mathcal{S}$ (Nebel and Bürckert, 1995), or $\Delta_0 \subseteq \mathcal{S}$ (Golumbic and Shamir, 1993).

To facilitate the forthcoming proofs we will use a closure operation for Allen's algebra which was defined in (Nebel and Bürckert, 1995).

Definition B.2 Let $S \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(S)$ the $\mathcal{A}$-*closure* of S under converse, intersection and composition, *i.e.* the least subalgebra containing S closed under the three operations.

The key result for $\mathcal{C}_{\mathcal{A}}$ is the following. The proof appears in (Nebel and Bürckert, 1995).

Proposition B.3 Let $\mathcal{S} \subseteq \mathcal{A}$. Then $\mathcal{A}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is and $\mathcal{A}$-SAT($\mathcal{S}$) is NP-complete iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is.

Next, we will define a number of subclasses of $\mathcal{A}$ and prove that $\mathcal{A}$-SAT for them is NP-complete. These subclasses will be used later on in the NP-completeness proofs for the subclasses in Table 3.

Definition B.4 Let $s_0, s_1, \ldots, s_7$ be defined as follows: $s_0 = \{\mathsf{d}, \mathsf{o}^{\smile}, \mathsf{f}\}$, $s_1 = \{\prec, \succ, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{f}^{\smile}\}$, $s_2 = \{\equiv, \succ, \mathsf{s}, \mathsf{s}^{\smile}\}$, $s_3 = \{\equiv, \mathsf{m}, \mathsf{m}^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}$, $s_4 = \{\mathsf{d}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}, \mathsf{f}\}$, $s_5 = \{\prec, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{m}, \mathsf{m}^{\smile}\}$, $s_6 = \{\equiv, \mathsf{s}, \mathsf{s}^{\smile}\}$ and $s_7 = \{\prec, \succ, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{m}, \mathsf{m}^{\smile}, \mathsf{s}^{\smile}, \mathsf{f}^{\smile}\}$. Denote by S_{ij} the set $\{s_i, s_j\}$.

Proposition B.5

Is_0J iff $J^- < I^- < J^+$;

Is_1J iff $I^- < J^- \vee I^- > J^+$;

Is_2J iff $I^- = J^- \vee I^- > J^+$;

Is_3J iff $(I^- = J^-) \vee (I^- = J^+) \vee (I^+ = J^-) \vee (I^+ = J^+)$;

Is_4J iff $(J^- < I^- < J^+) \vee (J^- < I^+ < J^+)$;

Is_5J iff $(I^- < J^-) \vee (I^- = J^+)$;

Is_6J iff $I^- = J^-$;

Is_7J iff $(I^- < J^-) \vee (I^- > J^+) \vee (I^+ < J^-) \vee (I^+ > J^+)$.

Lemma B.6 $\mathcal{A}$-SAT(S) is NP-complete for $S \in \{S_{01}, S_{02}, S_{34}, S_{56}, S_{37}\}$.

Proof: By Corollary B.3, we can study $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(S)$) instead of $\mathcal{A}$-SAT(S). It can be verified that $\mathcal{N}_2$ is a subset of $\mathcal{C}_{\mathcal{A}}(S_{01})$, $\mathcal{C}_{\mathcal{A}}(S_{02})$ and $\mathcal{C}_{\mathcal{A}}(S_{56})$. Likewise, it can be shown that Δ_0 is a subset of $\mathcal{C}_{\mathcal{A}}(S_{34})$ and $\mathcal{C}_{\mathcal{A}}(S_{37})$. Hence, NP-completeness follows from Theorem B.1. □

In the previous lemma, $\mathcal{C}_{\mathcal{A}}$ was computed by the utility `aclose` (Nebel and Bürckert, 1993).

B.2 NP-COMPLETE SUBCLASSES OF $\mathcal{V}$

Lemma B.7 $\mathcal{V}$-SAT(D_1) is NP-complete.

Proof: Reduction from $\mathcal{A}$-SAT(S_{01}) which is NP-complete by Lemma B.6. Let Θ be an instance of $\mathcal{A}$-SAT(S_{01}). We construct a set Θ' as follows.

1. For each formula of the type Is_0J in Θ, introduce a new time point $p_{I,J}$ and let $p_{I,J}\{\mathsf{s}\}I$ and $p_{I,J}\{\mathsf{d}\}J$ in Θ';
2. For each formula of the type Is_1J in Θ, introduce a new time point $q_{I,J}$ and let $q_{I,J}\{\mathsf{s}\}I$ and $q_{I,J}\{\mathsf{b}, \mathsf{a}\}J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_1) problem. We show that Θ is satisfiable iff Θ'.

only-if: Assume there exists a $\mathcal{A}$-model $\Im$ of Θ. We construct an $\mathcal{V}$-interpretation $\Im'$ of Θ' as follows: $\Im'(I^-) = \Im(I^-)$, $\Im'(I^+) = \Im(I^+)$, $\Im'(p_{I,J}) = \Im(I^-)$ and $\Im'(q_{I,J}) = \Im(I^-)$.

By the construction of Θ', three types of formulae can appear in Θ'. We consider them one at a time.

1. $p_{I,J}\{\mathsf{s}\}I$ and $q_{I,J}\{\mathsf{s}\}I$. Such formulae are trivially satisfied under $\Im'$.
2. $p_{I,J}\{\mathsf{d}\}J$. If $p_{I,J}\{\mathsf{d}\}J \in \Theta'$ then $Is_0J \in \Theta$. Since $\Im$ is a $\mathcal{A}$-model of Θ, $\Im(J^+) < \Im(I^-) < \Im(J^+)$ by Proposition B.5. By the construction of $\Im'$, $\Im'(p_{I,J}) = \Im(I^-)$, $\Im'(J^-) = \Im(J^-)$ and $\Im'(J^+) = \Im(J^+)$. Hence, $p_{I,J}\{\mathsf{d}\}J$ under $\Im'$.
3. $q_{I,J}\{\mathsf{b}, \mathsf{a}\}J$. If $p_{I,J}\{\mathsf{b}, \mathsf{a}\}J \in \Theta'$ then $Is_1J \in \Theta$. Since $\Im$ is a $\mathcal{A}$-model of Θ, $\Im(I^-) < \Im(J^-)$ or $\Im(I^-) > \Im(J^+)$ by Proposition B.5. By the construction of $\Im'$, $\Im'(q_{I,J}) = \Im(I^-)$, $\Im'(J^-) = \Im(J^-)$ and $\Im'(J^+) = \Im(J^+)$. Hence, $q_{I,J}\{\mathsf{b}, \mathsf{a}\}J$ under $\Im'$.

As a consequence, $\Im'$ is a $\mathcal{V}$-model of Θ'.

if: Assume there exists an $\mathcal{V}$-model $\Im'$ of Θ'. We construct an $\mathcal{A}$-model $\Im$ of Θ as follows: $\Im'(I^-) = \Im(I^-)$ and $\Im'(I^+) = \Im(I^+)$.

Arbitrarily choose a formula of the form Is_0J in Θ. Then there exists a time point $p_{I,J}$ in Θ' such that $p_{I,J}\{s\}I$ and $p_{I,J}\{d\}J$. It follows that $\Im'(J^-) < \Im'(I^+) < \Im'(J^+)$ and, by the construction of $\Im$, $\Im(J^-) < \Im(I^+) < \Im(J^+)$. Hence, Is_0J under $\Im$.

Arbitrarily choose a formula of the form Is_1J in Θ. Then there exists a time point $q_{I,J}$ in Θ' such that $q_{I,J}\{\mathsf{s}\}I$ and $q_{I,J}\{\mathsf{b}, \mathsf{a}\}J$. It follows that $\Im'(I^-) <$

$\Im'(J^-)$ or $\Im'(I^-) > \Im'(J^+)$. By the construction of $\Im$, $\Im(I^-) < \Im(J^-)$ or $\Im(I^-) > \Im(J^+)$. Hence, Is_1J under $\Im$ by Proposition B.5.

Consequently, $\Im$ is a $\mathcal{A}$-model of Θ. We have thus shown that Θ is satisfiable iff Θ' is satisfiable. NP-completeness of $\mathcal{V}$-SAT(D_1) follows immediately. □

Lemma B.8 $\mathcal{V}$-SAT(D_3) is NP-complete.

Proof: Let $D_3 = \{\{\mathtt{d}\}, \{\mathtt{s}\}, \{\mathtt{b}, \mathtt{f}, \mathtt{a}\}\} = \{r'_1, r'_2, r'_3\}$. Observe that $f_{T_4}(r'_1) = \{\mathtt{d}\}$, $f_{T_4}(r'_2) = \{\mathtt{s}\}$ and $f_{T_4}(r'_3) = \{\mathtt{b}, \mathtt{a}\}$. By Lemma B.7, $D_1 = \{\{\mathtt{d}\}, \{\mathtt{s}\}, \{\mathtt{b}, \mathtt{a}\}\}$ and $\mathcal{V}$-SAT(D_1) is NP-complete. Hence, $\mathcal{V}$-SAT(D_3) is NP-complete by Lemma A.4. □

Lemma B.9 $\mathcal{V}$-SAT(D_5) is NP-complete.

Proof sketch: Reduction from $\mathcal{A}$-SAT(S_{02}) which is NP-complete by Lemma B.6. Let Θ be an instance of $\mathcal{A}$-SAT(S_{02}). We construct a set Θ' as follows.

1. For each formula of the type Is_0J in Θ, introduce a new time point $p_{I,J}$ and let $p_{I,J}\{\mathtt{s}\}I$ and $p_{I,J}\{\mathtt{d}\}J$ in Θ';
2. For each formula of the type Is_2J in Θ, introduce a new time point $q_{I,J}$ and let let $q_{I,J}\{\mathtt{s}\}I$ and $q_{I,J}\{\mathtt{s}, \mathtt{a}\}J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_5) problem. Show that Θ is satisfiable iff Θ' is similar to the proof of Lemma B.7. □

Lemma B.10 $\mathcal{V}$-SAT(D_7) is NP-complete.

Proof: Reduction from $\mathcal{A}$-SAT(S_{34}) which is NP-complete by Lemma B.6. Let Θ be an instance of $\mathcal{A}$-SAT(S_{34}). We construct a set Θ' as follows.

1. For each formula of the type Is_3J in Θ, introduce a new time point $p_{I,J}$ and let $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ and $p_{I,J}\{\mathtt{s}, \mathtt{f}\}J$ in Θ';
2. For each formula of the type Is_3J in Θ, introduce a new time point $q_{I,J}$ and let $q_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ and $q_{I,J}\{\mathtt{d}\}J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_7) problem. We show that Θ is satisfiable iff Θ'.

Assume there exists a $\mathcal{A}$-model $\Im$ of Θ. We construct an $\mathcal{V}$-interpretation $\Im'$ of Θ' as follows:

1. $\Im'(I^-) = \Im(I^-)$;
2. $\Im'(I^+) = \Im(I^+)$;
3. $\Im'(p_{I,J}) = \Im(I^-)$ if $\Im(I^-) = \Im(J^-)$ or $\Im(I^-) = \Im(J^+)$;
4. $\Im'(p_{I,J}) = \Im(I^+)$ otherwise. In this case $\Im(I^+) = \Im(J^-)$ or $\Im(I^+) = \Im(J^+)$;
5. $\Im'(q_{I,J}) = \Im(I^-)$ if $\Im(J^-) < \Im(I^-) < \Im(J^+)$;
6. $\Im'(q_{I,J}) = \Im(I^+)$ otherwise. In this case $\Im(J^-) < \Im(I^+) < \Im(J^+)$.

By the construction of Θ', four types of formulae can appear in Θ'. We consider them one at a time.

1. $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I$. If $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I \in \Theta'$ then $Is_3J \in \Theta$. By Proposition B.5, $(I^- = J^-)$ or $(I^- = J^+)$ or $(I^+ = J^-)$ or $(I^+ = J^+)$. If $\Im(I^-) = \Im(J^-)$ or $\Im(I^-) = \Im(J^-)$ then $\Im'(p_{I,J}) = \Im(I^-)$ and, consequently, $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ under $\Im'$. Otherwise, $\Im(I^+) = \Im(J^-)$ or $\Im(I^+) = \Im(J^-)$ and $\Im'(p_{I,J}) = \Im(I^+)$. Hence $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ under $\Im'$.
2. $p_{I,J}\{\mathtt{s}, \mathtt{f}\}J$. This case is analogous to the previous case.
3. $q_{I,J}\{\mathtt{s}, \mathtt{f}\}I$. If $q_{I,J}\{\mathtt{s}, \mathtt{f}\}I \in \Theta'$ then $Is_4J \in \Theta$. By Proposition B.5, $(\Im(J^-) < \Im(I^-) < \Im(J^+)) \vee (\Im(J^-) < \Im(I^+) < \Im(J^+))$. If $\Im(J^-) < \Im(I^-) < \Im(J^+$ then $\Im'(q_{I,J}) = \Im(I^-)$ and $q_{I,J}\{\mathtt{s}\}I$ under $\Im$. Otherwise, $\Im(J^-) < \Im(I^+) < \Im(J^+$, $\Im'(q_{I,J}) = \Im(I^+)$ and $q_{I,J}\{\mathtt{f}\}I$ under $\Im$.
4. $q_{I,J}\{\mathtt{d}\}J$. If $q_{I,J}\{\mathtt{d}\}J \in \Theta'$ then $Is_4J \in \Theta$. By Proposition B.5, $(\Im(J^-) < \Im(I^-) < \Im(J^+)) \vee (\Im(J^-) < \Im(I^+) < \Im(J^+))$. If $\Im(J^-) < \Im(I^-) < \Im(J^+$ then $\Im'(q_{I,J}) = \Im(I^-)$ and $q_{I,J}\{\mathtt{d}\}I$ under $\Im$. Otherwise, $\Im(J^-) < \Im(I^+) < \Im(J^+$, $\Im'(q_{I,J}) = \Im(I^+)$ and $q_{I,J}\{\mathtt{d}\}I$ under $\Im$.

As a consequence, $\Im'$ is a $\mathcal{V}$-model of Θ'.

Now, assume there exists a $\mathcal{V}$-model $\Im'$ of Θ'. We construct an $\mathcal{A}$-interpretation $\Im$ of Θ as follows: $\Im(I^-) = \Im'(I^-)$ and $\Im(I^+) = \Im'(I^+)$.

Arbitrarily choose a formula of the form Is_3J in Θ. Then there exists a time point $p_{I,J}$ in Θ' such that $p_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ and $p_{I,J}\{\mathtt{s}, \mathtt{f}\}J$ under $\Im'$. This implies that one endpoint in I equals one endpoint in J and, by Proposition B.5, Is_3J under $\Im$.

Arbitrarily choose a formula of the form Is_4J in Θ. Then there exists a time point $q_{I,J}$ in Θ' such that $q_{I,J}\{\mathtt{s}, \mathtt{f}\}I$ and $q_{I,J}\{\mathtt{d}\}J$ under $\Im'$. Assume $q_{I,J}\{\mathtt{s}\}I$ under $\Im'$. Then $\Im(J^-) < \Im(I^-) < \Im(J^+)$. Assume to the contrary that $p_{I,J}\{\mathtt{f}\}I$ under $\Im'$. It follows that $\Im(J^-) < \Im(I^+) < \Im(J^+)$. Hence, by Proposition B.5, Is_4J under $\Im$. □

Lemma B.11 $\mathcal{V}$-SAT(D_8) is NP-complete.

Proof sketch: Reduction from $\mathcal{A}$-SAT(S_{56}) which is NP-complete by Lemma B.6. Let Θ be an instance of $\mathcal{A}$-SAT(S_{56}). We construct a set Θ' as follows.

1. For each formula of the type Is_5J in Θ, introduce a new time point $p_{I,J}$ and let $p_{I,J}\{\mathtt{s}\}I$ and $p_{I,J}\{\mathtt{b}, \mathtt{f}\}J$ in Θ';

2. For each formula of the type Is_6J in Θ, introduce a new time point $q_{I,J}$ and let $q_{I,J}\{\mathtt{s}\}I$ and $q_{I,J}\{\mathtt{s}\}J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_8) problem. It is a routine verification to show that Θ is satisfiable iff Θ' is satisfiable. □

Lemma B.12 $\mathcal{V}$-SAT(D_{12}) is NP-complete.

Proof: By Lemma B.9, $\mathcal{V}$-SAT(D_5) is NP-complete and $\mathcal{V}$-SAT(D_6) is NP-complete since $D_6 = \mathcal{D}_{\mathcal{V}}(D_5)$. Let $E = \{\{\mathtt{b}\}, \{\mathtt{d}\}, \{\mathtt{b},\mathtt{f}\}\}$. It can be verified that $D_6 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and, hence, $\mathcal{V}$-SAT(E) is NP-complete. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem. We show how to construct an instance Θ' of the $\mathcal{V}$-SAT(D_{12}) problem that is satisfiable iff Θ is satisfiable.

We begin showing how to relate a point p_1 and an interval I_1 such as $p_1\{\mathtt{b}\}I_1$ by only using the relations in D_{12}. We introduce two fresh time points p_2 and p_3 together with two fresh time intervals I_2 and I_3. Consider the following construction: $p_1\{\mathtt{b},\mathtt{f}\}I_1$, $p_1\{\mathtt{b},\mathtt{f}\}I_2$, $p_1\{\mathtt{b},\mathtt{f}\}I_3$, $p_2\{\mathtt{b},\mathtt{f}\}I_1$, $p_2\{\mathtt{d}\}I_2$, $p_2\{\mathtt{b},\mathtt{f}\}I_3$, $p_3\{\mathtt{b},\mathtt{f}\}I_1$, $p_3\{\mathtt{b},\mathtt{f}\}I_2$ and $p_3\{\mathtt{d}\}I_3$.

We denote this set of relations with Ω. Let $\Im$ be a $\mathcal{V}$-model of Ω. For the sake of brevity we identify the time points and time intervals with their values when interpreted by $\Im$. Hence, instead of writing $\Im(p_1) < \Im(I_1^-)$, we simply write $p_1 < I_1^-$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\delta = (I_1^- - p_1)/5$. Consider the following assignment of values: $I_3^- = p_1 + \delta$, $p_3 = p_1 + 2\delta$, $I_2^- = p_1 + 3\delta$, $p_2 = p_1 + 4\delta$, $I_3^+ = p_1 + 4\delta$ and $I_2^+ = I_1^+$. It is not hard to see that this assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. By relation (4), we can see that $p_2 < I_1^-$ or $p_2 = I_1^+$. By assumption, $p_1 = I_1^+$. Hence, either $p_2 < I_1^-$ or $p_2 = p_1$. If $p_2 = p_1$ then relation (2) is equivalent to $p_2\{\mathtt{b},\mathtt{f}\}I_2$ which clearly contradicts relation (5). Thus, $p_2 < I_1^-$ and $p_2\{\mathtt{b}\}I_1$. By analogous reasoning one can see that $p_3 < I_1^-$ and $p_3\{\mathtt{b}\}I_1$.

Next, observe that relations (2) and (3) implies $p_1 \leq I_2^+$ and $p_1 \leq I_3^+$. Furthermore, $p_2 < I_1^-$ and $p_3 < I_1^-$ which implies $p_2 < I_1^+$ and $p_3 < I_1^-$. By our initial assumption $p_1 = I_1^+$ we get $p_2 < I_1^+ = p_1 \leq I_3^+$ and $p_3 < I_1^+ = p_1 \leq I_2^+$.

Consequently, $p_2 < I_3^+$ and $p_3 < I_2^+$. Observe that $p_2\{\mathtt{b},\mathtt{f}\}I_3$ and $p_3\{\mathtt{b},\mathtt{f}\}I_2$ by relations (6) and (8), respectively. Hence, $p_2 < I_3^-$ and $p_3 < I_2^-$.

By relations (5) and (9), $I_2^- < p_2 < I_2^+$ and $I_3^- < p_3 < I_3^+$. Hence, $p_2 < I_3^- < p_3 < I_2^- < p_2$ which is a contradiction. Consequently, every $\mathcal{V}$-model of Ω satisfies $p_1 < I_1^-$.

We have thus shown how to express the relation $\{\mathtt{b}\}$ by only using $\{\mathtt{d}\}$ and $\{\mathtt{b},\mathtt{f}\}$. Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(D_{12}) problem. NP-completeness of $\mathcal{V}$-SAT(D_{12}) follows immediately. □

Lemma B.13 $\mathcal{V}$-SAT(D_{14}) is NP-complete.

Proof sketch: By Lemma B.9, $\mathcal{V}$-SAT(D_5) is NP-complete and $\mathcal{V}$-SAT(D_6) is NP-complete since $D_6 = \mathcal{D}_{\mathcal{V}}(D_5)$. Let $E = \{\{\mathtt{b}\}, \{\mathtt{d},\mathtt{a}\}, \{\mathtt{b},\mathtt{f}\}\}$. It can be verified that $D_6 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and, hence, $\mathcal{V}$-SAT(E) is NP-complete. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem. We show how to construct an instance Θ' of the $\mathcal{V}$-SAT(D_{14}) problem that is satisfiable iff Θ is satisfiable.

The proof boils down to showing how to relate a point p_1 and an interval I_1 such as $p_1\{\mathtt{b}\}I_1$ by only using the relations in D_{14}. We introduce two fresh time points p_2 and p_3 together with two fresh time intervals I_2 and I_3. Consider the following construction: $p_1\{\mathtt{b},\mathtt{f}\}I_1$, $p_1\{\mathtt{b},\mathtt{f}\}I_2$, $p_1\{\mathtt{b},\mathtt{f}\}I_3$, $p_2\{\mathtt{b},\mathtt{f}\}I_1$, $p_2\{\mathtt{d},\mathtt{a}\}I_2$, $p_2\{\mathtt{b},\mathtt{f}\}I_3$, $p_3\{\mathtt{b},\mathtt{f}\}I_1$, $p_3\{\mathtt{b},\mathtt{f}\}I_2$ and $p_3\{\mathtt{d},\mathtt{a}\}I_3$. It is fairly straightforward to show that p_1 can only be related to I_1 with the relation $\mathtt{b}$. Hence, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(D_{14}) problem. NP-completeness of $\mathcal{V}$-SAT(D_{14}) follows immediately. □

Lemma B.14 $\mathcal{V}$-SAT(D_{16}) is NP-complete.

Proof: Reduction from $\mathcal{V}$-SAT(D_{13}). Use the model transformation T_4 and apply Lemma A.4. □

Lemma B.15 $\mathcal{V}$-SAT(D_{18}) is NP-complete.

Proof sketch: By Lemma B.10, $\mathcal{V}$-SAT(D_7) is NP-complete. Let $E = \{\{\mathtt{s}\}, \{\mathtt{s},\mathtt{f}\}, \{\mathtt{b},\mathtt{d}\}\}$. It can be verified that $D_7 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and, hence, $\mathcal{V}$-SAT(E) is NP-complete. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem.

We show how to relate a time point p_1 and an interval I_1 with the relation $\mathtt{s}$. Introduce two fresh time points p_2 and p_3 together with a fresh time interval I_2. Consider the following construction: $p_1\{\mathtt{s},\mathtt{f}\}I_1$, $p_1\{\mathtt{b},\mathtt{d}\}I_2$, $p_2\{\mathtt{b},\mathtt{d}\}I_1$, $p_2\{\mathtt{s},\mathtt{f}\}I_2$, $p_3\{\mathtt{s},\mathtt{f}\}I_1$ and $p_3\{\mathtt{s},\mathtt{f}\}I_2$.

It is not hard to show that p_1 can only be related to I_1 with the relation $\mathtt{s}$. Consequently, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(D_{18}) problem. NP-completeness of $\mathcal{V}$-SAT(D_{18}) follows immediately. □

Lemma B.16 $\mathcal{V}$-SAT(D_{20}) is NP-complete.

Proof sketch: Reduction from $\mathcal{A}$-SAT(S_{37}) which is NP-complete by Lemma B.6. Let Θ be an instance of $\mathcal{A}$-SAT(S_{37}). We construct a set Θ' as follows.

1. For each formula of the type $I s_3 J$ in Θ, introduce a new time point $p_{I,J}$ and let $p_{I,J}\{\mathtt{s},\mathtt{f}\}I$ and $p_{I,J}\{\mathtt{s},\mathtt{f}\}J$ in Θ';
2. For each formula of the type $I s_3 J$ in Θ, introduce a new time point $q_{I,J}$ and let $q_{I,J}\{\mathtt{s},\mathtt{f}\}I$ and $q_{I,J}\{\mathtt{b},\mathtt{a}\}J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_{20}) problem. Proving that Θ is satisfiable iff Θ' is similar to the proof of Lemma B.10. □

Lemma B.17 $\mathcal{V}$-SAT(D_{21}) is NP-complete.

Proof: Reduction from $\mathcal{V}$-SAT(D_{14}). Use the model transformation T_1 and apply Lemma A.4. □

Lemma B.18 $\mathcal{V}$-SAT(D_{23}) is NP-complete.

Proof: Reduction from $\mathcal{V}$-SAT(D_{12}). Use the model transformation T_3 and apply Lemma A.4. □

Lemma B.19 $\mathcal{V}$-SAT(D_{25}) is NP-complete.

Proof: Reduction from $\mathcal{V}$-SAT(D_{14}). Use the model transformation T_3 and apply Lemma A.4. □

Lemma B.20 $\mathcal{V}$-SAT(D_{27}) is NP-complete.

Proof sketch: By Lemma B.10, $\mathcal{V}$-SAT(D_7) is NP-complete. Let $E = \{\{\mathtt{b}\},\{\mathtt{b},\mathtt{f}\},\{\mathtt{s},\mathtt{a}\}\}$. It can be verified that $D_5 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and, hence, $\mathcal{V}$-SAT(E) is NP-complete. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT(E). We show how to construct an instance Θ' of $\mathcal{V}$-SAT(D_{27}) that is satisfiable iff Θ is satisfiable.

We begin by showing how to relate a point p_1 and an interval I_1 such as $p_1\{\mathtt{b}\}I_1$ by only using the relations in D_{27}. We introduce two fresh time points p_2 and p_3 together with a fresh time interval I_2. Consider the following construction: $p_1\{\mathtt{b},\mathtt{f}\}I_1$, $p_1\{\mathtt{b},\mathtt{f}\}I_2$, $p_2\{\mathtt{b},\mathtt{f}\}I_1$, $p_2\{\mathtt{s},\mathtt{a}\}I_2$, $p_3\{\mathtt{s},\mathtt{a}\}I_1$ and $p_3\{\mathtt{b},\mathtt{f}\}I_2$.

It is fairly straightforward to show that p_1 can only be related to I_1 with the relation $\mathtt{b}$. Hence, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(D_{27}) problem. NP-completeness of $\mathcal{V}$-SAT(D_{27}) follows immediately. □

Lemma B.21 $\mathcal{V}$-SAT(D_{28}) is NP-complete.

Proof: Reduction from Graph 3-Colourability, which is NP-complete. Let $G = \langle V, E\rangle$ be an arbitrary undirected graph.

In the proof we will make repeated use of the concept of a *separator*, a construction which forces two points to have distinct values in all models. Given two points p, q we construct a separator by introducing a new interval I and adding the relations $p\{\mathtt{s},\mathtt{f}\}I$ and $q\{\mathtt{b},\mathtt{d},\mathtt{a}\}I$. Clearly, all models $\Im$ must satisfy $\Im(p) \neq \Im(q)$.

We now construct the set of relations stepwise. First, we construct a *paint-box* by introducing two points p_1 and p_2, two intervals I_1 and I_2 plus the relations

$$p_1\{\mathtt{s},\mathtt{f}\}I_1,\ p_1\{\mathtt{s},\mathtt{f}\}I_2,\ p_2\{\mathtt{s},\mathtt{f}\}I_1,\ p_2\{\mathtt{b},\mathtt{d},\mathtt{a}\}I_2$$

over these. Note that the interval I_2 acts as a separator for p_1 and p_2, which are thus forced to take on different values. Further, the intervals I_1 and I_2 must have some common end-point, coinciding with p_1. We use the constant r to denote this value. Hence, the remaining end-point of I_1 must coincide with p_2 and the remaining end-point of I_2 must be distinct from both p_1 and p_2. We denote the values of these two remaining end-points g and b respectively. We can think of the values r, g and b as colours, constituting our palette. Of course, the actual denotations of these three values differ between models, but the important thing is only that they denote three distinct values in each and every model.

Now, for each vertex $v_i \in V$, we construct a *selector* consisting of three points q_i^0, q_i^1 and q_i^2 plus two intervals J_i^0 and $J_i^{1,2}$, connected as follows. First introduce a separator for q_i^1 and q_i^2, using interval $J_i^{1,2}$, *i.e.* introduce the relations $q_i^1\{\mathtt{s},\mathtt{f}\}J_i^{1,2}$, $q_i^2\{\mathtt{b},\mathtt{d},\mathtt{a}\}J_i^{1,2}$. Then connect the points to the remaining interval by adding the relations $q_i^0\{\mathtt{s},\mathtt{f}\}J_i^0$, $q_i^1\{\mathtt{s},\mathtt{f}\}J_i^0$, $q_i^2\{\mathtt{s},\mathtt{f}\}J_i^0$. Finally, connect this whole gadget to the paint-box by adding the relations $q_i^1\{\mathtt{s},\mathtt{f}\}I_1$, $q_i^2\{\mathtt{s},\mathtt{f}\}I_2$. The selector works as follows. The endpoints of I_1 correspond to the colours r and g, so q_i^1 is forced to have either of these values. Similarly, q_i^2 must have either of the values r and b. Now, q_i^1 and q_i^2 are separated, so together they select a subpalette of two colours, assigning one colour each to the end-points of J_i^0. Finally, q_i^0 selects one of these two colours. So far, there are no further constraints, so q_i^0 may be freely assigned any of the three colours from our palette.

Finally, for each edge $\{v_i, v_j\} \in E$ we introduce a separator, consisting of the new interval $K_{i,j}$ and the two relations $q_i^0\{\mathtt{s},\mathtt{f}\}K_{i,j}$, $q_j^0\{\mathtt{b},\mathtt{d},\mathtt{a}\}K_{i,j}$, preventing q_i^0 and q_j^0 to have the same value whenever there is an edge between the vertices v_i and v_j.

It is obvious that G is 3-colourable iff the network just constructed is satisfiable, so NP-completeness follows. □