Computational Complexity of Relating Time Points with Intervals

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Abstract

Several algebras have been proposed for reasoning about qualitative constraints over the time line. One of these algebras is Vilain's point-interval algebra, which can relate time points with time intervals. Apart from being a stand-alone qualitative algebra, it is also used as a subalgebra in Meiri's approach to temporal reasoning, which combines reasoning about metric and qualitative temporal constraints over both time points and time intervals. While the satsifiability problem for the full point-interval algebra is known to be NP-complete, not much is known about its 4 294 967 296 subclasses. This article completely determines the computational complexity of these subclasses and it identifies all of the maximal tractable subalgebras—five in total.

Keywords: Temporal reasoning, constraint satisfaction, point-interval algebra, algorithms, complexity.

1 Introduction

Reasoning about temporal constraints is an important task in many areas of AI and elsewhere, such as planning [2], natural language processing [16], time serialization in archeology [8] *etc.* In most applications, knowledge of temporal constraints is expressed in terms of collections of relations between time intervals or time points. Often we are only interested in qualitative relations, *i.e.*, the relative ordering of time points but not their exact occurrences in time. There are two archetypical examples of qualitative temporal reasoning: *Allen's algebra* ($\mathcal{A}$) [1] for reasoning about time intervals and the *point algebra* (PA) [18] for reasoning about time points.

Attempts have been made to integrate reasoning about time intervals and time points. Meiri's [13] approach to temporal reasoning makes it possible to reason about time points and time intervals with respect to both qualitative and metric time. This framework can be restricted to qualitative time and the resulting fragment is known as the *qualitative algebra* (QA). In QA, a qualitative constraint between two objects O_i and O_j (each may be a point or an interval), is a disjunction of the form $(O_i r_1 O_j) \vee \ldots \vee (O_i r_k O_j)$ where each one of the r_i's is a *basic relation* that may exist between two objects. There are three types of basic relations:

1. Interval-interval relations that can hold between pairs of intervals. Such relations correspond to Allen's algebra.

2. Point-point relations that can hold between pairs of points. These relations correspond to the point algebra.

3. Point-interval and interval-point relations that can hold between a point and an interval and vice-versa. These relations were introduced by Vilain [18]. The point-interval and interval-point relations are symmetric so we will only consider the point-interval relations in the sequel.

The satisfiability problem for the point algebra is known to be tractable [19] and the satisfiability problem for Allen's algebra is NP-complete [19]. However, a large number of tractable subclasses of Allen's algebra has been reported in the literature [6, 8, 15, 17]. Clearly, QA suffers from computational difficulties since it subsumes the Allen algebra. Even worse, Meiri [13] shows that the satisfiability problem is NP-complete even for point-interval relations. Besides this negative result, not much is known about the computational properties of subclasses of the point-interval algebra. This is an unfortunate situation if we want to find tractable subclasses of the qualitative algebra since the point-interval and interval-point algebras provide the glue that ties the world of time points together with the world of time intervals.

The main result of this article is a complete classification of all subclasses of the point-interval algebra with respect to tractability. The classification reveals that there exists five maximal tractable[1] subclasses of the point-interval algebra, denoted $\mathcal{V}^{23}$, $\mathcal{V}^{20}_{\mathtt{s}}$, $\mathcal{V}^{20}_{\mathtt{f}}$, $\mathcal{V}^{17}_{\mathtt{s}}$ and $\mathcal{V}^{17}_{\mathtt{f}}$ where the superscripts tell how many relations there are in the subclasses. The classification makes it possible to determine whether a given subclass is tractable or not by a simple test that can be easily carried out by hand or automatically. We have thus gained a clear picture of the borderline between tractability and intractability in the point-interval algebra. In this process, we have also taken a small step towards a deeper understanding of the qualitative algebra.

The tractable subclasses roughly fall into two classes: three of them ($\mathcal{V}^{23}$, $\mathcal{V}^{20}_{\mathtt{s}}$ and $\mathcal{V}^{20}_{\mathtt{f}}$) are very closely related to the aforementioned point algebra. The subclass $\mathcal{V}^{23}$ consists of the point-interval relations that can be directly expressed as point algebra formulae over the endpoints. One should note that this is the only tractable subclass that contain all basic relations. Consequently, $\mathcal{V}^{23}$ resembles the ORD-Horn algebra [15] which is the unique maximal tractable subclass of Allen's interval algebra containing all basic relations. The subclasses $\mathcal{V}^{20}_{\mathtt{s}}$ and $\mathcal{V}^{20}_{\mathtt{f}}$ can be transformed to the point algebra since they exhibit a special property: any solution can be transformed to a new solution where the intervals are of equal and arbitrarily small length. These classes contains only three basic relations each but they both contain the interesting (`b a`) relation which states that a point comes either before or after an interval. This relation is not a member of $\mathcal{V}^{23}$. The remaining two subclasses ($\mathcal{V}^{17}_{\mathtt{s}}$ and $\mathcal{V}^{17}_{\mathtt{f}}$) are trivial in the sense that an instance of these problems is satisfiable iff the empty relation (which is always unsatisfiable) does not appear in the instance.

A few words on methodology seem appropriate at this point. The proof of the main theorem relies on an extensive case analysis performed by a computer. The number of cases considered in this analysis was approximately 10^5. Naturally, such an analysis cannot be reproduced in an article or be verified manually. To allow for the verification of our results, we include a description of the program used in the analysis. Furthermore, the programs used can be obtained from the authors.

The rest of this article is organized as follows: Section 2 defines the point-interval algebra and some auxiliary concepts. Section 3 contains the classification of subclasses. Section 4 is a brief discussion of the results and Section 5 concludes the article. Most of the proofs are postponed to the appendix. This article is an extended and corrected version of an earlier paper [10].

[1] A maximal tractable subclass X has the following property: the satisfiability problem is tractable for X but intractable for all strict supersets of X.

2 Point-Interval Relations

The point-interval algebra is based on the notions of *points*, *intervals* and *binary relations* on these. A point p is a variable interpreted over the set of real numbers $\mathbb{R}$. An interval I is represented by a pair $\langle I^-, I^+\rangle$ satisfying $I^- < I^+$ where I^- and I^+ are interpreted over $\mathbb{R}$. We assume that we have a fixed universe of variable names for points and intervals. Then, a $\mathcal{V}$*-interpretation* is a function $\Im$ that maps point variables to $\mathbb{R}$ and interval variables to $\mathbb{R} \times \mathbb{R}$ and satisfies the previously stated restrictions. We will frequently extend the notation by denoting the first component of $\Im(I)$ by $\Im(I^-)$ and the second by $\Im(I^+)$.

Given an interpreted point and an interpreted interval, their relative positions can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic point-interval relations* where each basic relation can be defined in terms of its endpoint relations (see Table 1). A formula of the form pBI where p is a point, I an interval and $B \in \mathbf{B}$, is said to be satisfied by a $\mathcal{V}$-interpretation iff the interpretation of the points and intervals satisfies the endpoint relations specified in Table 1.

To express indefinite information, unions of the basic relations are used which lead to 2^5 distinct binary *point-interval relations*. Naturally, a set of basic relations is to be interpreted as a disjunction of its member relations. A point-interval relation is written as a list of its members, *e.g.*, (`b d a`). The set of all point-interval relations $2^{\mathbf{B}}$ is denoted by $\mathcal{V}$. Relations of special interest are the *null* relation $\emptyset$ (also denoted by $\perp$) and the *universal* relation $\mathbf{B}$ (also denoted $\top$).

A formula of the form $p(B_1, \ldots, B_n)I$ is called a *point-interval* formula. Such a formula is satisfied by a $\mathcal{V}$-interpretation $\Im$ iff pB_iI is satisfied by $\Im$ for some i, $1 \leq i \leq n$. A set Θ of point-interval formulae is said to be $\mathcal{V}$-*satisfiable* iff there exists an $\mathcal{V}$-interpretation $\Im$ that satisfies every formula of Θ. Such a satisfying $\mathcal{V}$-interpretation is called a $\mathcal{V}$*-model* of Θ. The decision problem we will study is the following:

INSTANCE: A finite set Θ of point-interval formulae.
QUESTION: Does there exist a $\mathcal{V}$-model of Θ?

We denote this problem $\mathcal{V}$-SAT. In the following, we often consider restricted versions of $\mathcal{V}$-SAT where relations used in the formulae in Θ are only from a subset $\mathcal{S}$ of $\mathcal{V}$. In this case we say that Θ is a set of formulae over $\mathcal{S}$ and use a parameter in the problem description to denote the subclass under consideration, *e.g.*, $\mathcal{V}$-SAT($\mathcal{S}$).

Meiri's extended definition of the point-interval algebra consists of $\mathcal{V}$ equipped with the two binary operations *intersection* and *composition*. However, such a definition does not constitute an algebra because it is not closed

Basic relation	Symbol	Example	Endpoint relation
p before I	`b`	`p` `    III`	$p < I^-$
p starts I	`s`	`    p` `    III`	$p = I^-$
p during I	`d`	`     p` `    III`	$I^- < p < I^+$
p finishes I	`f`	`      p` `    III`	$p = I^+$
p after I	`a`	`         p` `    III`	$p > I^+$

Table 1: The five basic relations of the $\mathcal{V}$-algebra. The endpoint relation $I^- < I^+$ that is required for all relations has been omitted.

under composition. We replace the composition operation with an operation on $\mathcal{V}$ which we will refer to as *3-composition*.

Definition 2.1 Let $\mathbf{B} = \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}, \mathtt{a}\}$. The *point-interval algebra* consists of the set $\mathcal{V} = 2^{\mathbf{B}}$ and the operations binary *intersection* (denoted by $\cap$) and ternary *3-composition* (denoted by $\otimes$). They are defined as follows:

$$\forall p, I : p(R \cap S)I \Leftrightarrow pRI \wedge pSI;$$

$$\forall p, I : p(R \otimes S \otimes T)I \Leftrightarrow \exists q, J : (qRI \wedge qSJ \wedge pTJ).$$

It can easily be verified that

$$R \otimes S \otimes T = \bigcup\{B \otimes B' \otimes B'' | B \in R, B' \in S, B'' \in T\},$$

i.e., 3-composition is the union of the component-wise 3-composition of basic relations. In Table 2, we present the tables for 3-composition of basic relations. We can see that, for instance, $(\mathtt{f}) \otimes (\mathtt{b}) \otimes (\mathtt{s}) = (\mathtt{a})$.

Next, we introduce a *closure* operation $\mathcal{C}_\mathcal{V}$. This operation will simplify some of the following proofs.

Definition 2.2 Let $\mathcal{S} \subseteq \mathcal{V}$. Then we denote by $\mathcal{C}_\mathcal{V}(\mathcal{S})$ the $\mathcal{V}$-*closure* of $\mathcal{S}$, defined as the least subset of $\mathcal{V}$ containing $\mathcal{S}$ which is closed under intersection and 3-composition.

Given a set $\mathcal{S} \subseteq \mathcal{V}$, we can easily compute $\mathcal{C}_\mathcal{V}(\mathcal{S})$ by defining a function $\Psi : 2^\mathcal{V} \rightarrow 2^\mathcal{V}$ such that

$$\Psi(X) = X \cup \{x \cap y \mid x, y \in X\} \cup \{x \otimes y \otimes z \mid x, y, z \in X\}.$$

$R =$ b					
	b	s	d	f	a
b	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)
s	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
d	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
f	(b)	(b)	(b)	(b)	(b s d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ s					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b)	(s)	(d f a)	(d f a)	(d f a)
d	(b)	(b)	(b s d f a)	(d f a)	(d f a)
f	(b)	(b)	(b)	(s)	(d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ d					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b s d)	(d)	(d f a)	(d f a)	(d f a)
d	(b s d)	(b s d)	(b s d f a)	(d f a)	(d f a)
f	(b s d)	(b s d)	(b s d)	(d)	(d f a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ f					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d)	(f)	(a)	(a)	(a)
d	(b s d)	(b s d)	(b s d f a)	(a)	(a)
f	(b s d)	(b s d)	(b s d)	(f)	(a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ a					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d f a)	(a)	(a)	(a)	(a)
d	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
f	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
a	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)

Table 2: 3-composition in the $\mathcal{V}$-algebra. Each subtable represents one of the five possible choices of the R relation. Within each subtable, the vertical axis represents the S relation and the horizontal axis the T relation.

Since $\Psi^i(\mathcal{S}) \subseteq \Psi^{i+1}(\mathcal{S})$ for all i and $|\mathcal{V}| = 32$, there exists a $k \leq 32$ such that $\Psi^k(\mathcal{S}) = \Psi^{k+1}(\mathcal{S})$. Clearly, $\Psi^k(\mathcal{S}) = \mathcal{C}_\mathcal{V}(\mathcal{S})$. A program for computing $\mathcal{V}$-closures can be obtained from the authors.

The proof of the following result is omitted since proofs of analogous results can be found in Nebel and Bürckert [15].

Lemma 2.3 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT($\mathcal{S}$) is in P iff $\mathcal{V}$-SAT($\mathcal{C}_\mathcal{V}(\mathcal{S})$) is in P. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-hard iff $\mathcal{V}$-SAT($\mathcal{C}_\mathcal{V}(\mathcal{S})$) is NP-hard.

It should be noted that these results would not hold if the $\mathcal{V}$-SAT problem were defined somewhat differently. Temporal reasoning problems are sometimes defined such that each pair of objects (*e.g.*, intervals) has to be related by some relation different from the universal relation $\top$ (*cf.* Golumbic and Shamir [8]). If $\mathcal{V}$-SAT were defined in this way, then Lemma 2.3 would not be valid since new points and intervals are added in the reduction but certain combinations of them are not related to each other. By our way of defining $\mathcal{V}$-SAT, this is not a problem since we always allow points and intervals to be unrelated.

We continue by defining a duality operation on $\mathcal{V}$.

Definition 2.4 Let $R \in \mathcal{V}$. Define $\mathcal{D}_\mathcal{V}(R)$ as the set $\{\beta(r) \mid r \in R\}$ where $\beta(r)$ is defined as follows:

- $\beta(\mathtt{b}) = \mathtt{a}$;
- $\beta(\mathtt{s}) = \mathtt{f}$;
- $\beta(\mathtt{d}) = \mathtt{d}$;
- $\beta(\mathtt{f}) = \mathtt{s}$;
- $\beta(\mathtt{a}) = \mathtt{b}$.

Let $\mathcal{S} \subseteq \mathcal{V}$. Define $\mathcal{D}_\mathcal{V}(\mathcal{S})$ as the set $\{\mathcal{D}_\mathcal{V}(R) \mid R \in \mathcal{S}\}$.

Lemma 2.5 Let $\mathcal{S} \subseteq \mathcal{V}$. There is a polynomial-time reduction from $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) to to $\mathcal{V}$-SAT($\mathcal{S}$) and vice versa.

Proof: We show the reduction from $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) to $\mathcal{V}$-SAT($\mathcal{S}$); the other reduction is analogous. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT($\mathcal{D}_\mathcal{V}(\mathcal{S})$) problem. Let $\Theta' = \{p\mathcal{D}_\mathcal{V}(R)I \mid pRI \in \Theta\}$. Obviously, Θ' is an instance of the $\mathcal{V}$-SAT(S) problem. Assume Θ has a $\mathcal{V}$-model $\Im$. Construct a new $\mathcal{V}$-interpretation $\Im'$ as follows:

$\Im'(p) = -\Im(p)$ for each point p appearing in Θ;

$\Im'(I^-) = -\Im(I^+)$ and $\Im'(I^+) = -\Im(I^-)$ for each interval I appearing in Θ.

It is not hard to see that $\Im'$ is a $\mathcal{V}$-model of Θ'. □

Corollary 2.6 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT($\mathcal{S}$) is in P iff $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) is in P. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-hard iff $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) is NP-hard.

3 Classification of $\mathcal{V}$

We begin this section by defining five subalgebras of the point-interval algebra having a polynomial-time $\mathcal{V}$-SAT problem. Later on, we show that these algebras are the only maximal subalgebras of $\mathcal{V}$ with this property. Before we can define the algebras we need a definition concerning the point algebra.

Definition 3.1 A *point algebra (PA) formula* is an expression of the form xry where r is a member of $\{<, \leq, =, \neq, \geq, >, \perp, \top\}$ and x, y denote real-valued variables. The symbols $<, \leq, =, \neq, \geq, >$ denote the relations "strictly less than", "less than" and so on. The symbol $\perp$ denotes the relation $\emptyset$ which is unsatisfiable for every choice of $x, y \in \mathbb{R}$ and $\top$ denotes the relation $\mathbb{R} \times \mathbb{R}$ which is satisfiable for every choice of $x, y \in \mathbb{R}$.

Let Ω be a set of PA formulae and X the set of variables appearing in Ω. An assignment of real values to the variables in X is said to be a *PA-interpretation* of Ω. Furthermore, Ω is *PA-satisfiable* iff there exists a PA-interpretation $\Im$ such that for each formula $xry \in \Omega$, $\Im(x) r \Im(y)$ holds. Such an PA-interpretation $\Im$ is said to be a *PA-model* of Ω.

The first point-interval subalgebra we will consider has a very close connection to PA.

Definition 3.2 The set $\mathcal{V}^{23}$ consists of those relations in $\mathcal{V}$ that can be expressed as one or more PA formulae over points and endpoints of intervals.

Alternatively, $\mathcal{V}^{23}$ can be characterized in the following two ways:

$$\mathcal{V}^{23} = \mathcal{C}_{\mathcal{V}}(\{(\mathtt{s}), (\mathtt{f\ a}), (\mathtt{b\ d\ f})\}); \text{ or}$$

$$\mathcal{V}^{23} = \{r \in \mathcal{V} \mid (\mathtt{d}) \subseteq r \text{ or } r \subseteq (\mathtt{b\ s}) \text{ or } r \subseteq (\mathtt{f\ a})\}.$$

The remaining four tractable subalgebras can be defined as follows:

Definition 3.3

$$\mathcal{V}^{20}_{\mathtt{s}} = \{r \in \mathcal{V} \mid (\mathtt{a}) \subseteq r \text{ or } r \subseteq (\mathtt{b\ s})\}$$

$$\mathcal{V}^{20}_{\texttt{f}} = \{r \in \mathcal{V} \mid (\texttt{b}) \subseteq r \text{ or } r \subseteq (\texttt{f a})\}$$

$$\mathcal{V}^{17}_{\texttt{s}} = \{\bot\} \cup \{r \in \mathcal{V} \mid (\texttt{s}) \subseteq r\}$$

$$\mathcal{V}^{17}_{\texttt{f}} = \{\bot\} \cup \{r \in \mathcal{V} \mid (\texttt{f}) \subseteq r\}$$

Given a subalgebra $\mathcal{V}^x_y$, x denotes the number of relations in the algebra. Let $\mathcal{V}_P$ be the set of all subalgebras in Definition 3.3 together with the algebra $\mathcal{V}^{23}$. The relations included in each of these algebras can be found in Table 3.

By studying Table 3, one can see that $\mathcal{V}^{23}$ is the unique maximal tractable subalgebra containing all basic relations. Thus, there is a similarity with Nebel and Bürckerts famous ORD-Horn algebra which is the unique maximal tractable subalgebra of Allen's algebra containing all basic relations.

Let $\mathcal{V}_{NP}$ denote the set of subalgebras listed in Table 4. We have the following theorem.

Theorem 3.4 If $V \in \mathcal{V}_P$ then $\mathcal{V}$-SAT(V) is in P. If $V \in \mathcal{V}_{NP}$ then $\mathcal{V}$-SAT(V) is NP-hard.

Proof: See Appendices A and B for the results concerning $\mathcal{V}_P$ and $\mathcal{V}_{NP}$, respectively. □

The main theorem can now be stated.

Theorem 3.5 For $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{V}$-SAT$(\mathcal{S})$ is in P iff $\mathcal{S}$ is a subset of some member of $\mathcal{V}_P$. Otherwise, $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-complete.

Proof: *if:* For each $C \in \mathcal{V}_P$, $\mathcal{V}$-SAT(C) is in P by Theorem 3.4.
only-if: Choose $\mathcal{S} \subseteq \mathcal{V}$ such that $\mathcal{S}$ is not a subset of any algebra in $\mathcal{V}_P$. For each subalgebra C in $\mathcal{V}_P$, choose a relation x such that $x \in \mathcal{S}$ and $x \notin C$. This can always be done since $\mathcal{S} \not\subseteq C$. Let X be the set of these relations. We make three observations about X:

1. $|X| \leq 5$ (by construction);
2. X is not a subset of any algebra in $\mathcal{V}_P$ (by construction);
3. $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard if $\mathcal{V}$-SAT(X) is NP-hard since $X \subseteq \mathcal{S}$.

To show that $\mathcal{V}$-SAT$(\mathcal{S})$ has to be NP-hard, a machine-assisted case analysis of the following form was performed:

1. Generate all subsets of $\mathcal{V}$ of size ≤ 5. There are $\sum_{i=0}^{5} \binom{32}{i} \approx 2.4 \times 10^5$ such subsets.
2. Let $\mathcal{T}$ be such a set. Test: $\mathcal{T}$ is a subset of some subalgebra in $\mathcal{V}_P$ or $D \subseteq \mathcal{C}_{\mathcal{V}}(\mathcal{T})$ for some $D \in \mathcal{V}_{NP}$.

The test succeeds for all $\mathcal{T}$. Hence, $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard. Since $\mathcal{V}$-SAT is in NP [13], NP-completeness follows. □

	$\mathcal{V}^{23}$	$\mathcal{V}^{20}_{\mathtt{s}}$	$\mathcal{V}^{20}_{\mathtt{f}}$	$\mathcal{V}^{17}_{\mathtt{s}}$	$\mathcal{V}^{17}_{\mathtt{f}}$
$\perp$	•	•	•	•	•
(b)	•	•	•		
(s)	•	•		•	
(b s)	•	•	•	•	
(d)	•				
(b d)	•		•		
(s d)	•			•	
(b s d)	•		•	•	
(f)	•		•		•
(b f)			•		•
(s f)				•	•
(b s f)			•	•	•
(d f)	•				•
(b d f)	•		•		•
(s d f)	•			•	•
(b s d f)	•		•	•	•
(a)	•	•	•		
(b a)		•	•		
(s a)		•		•	
(b s a)		•	•	•	
(d a)	•	•			
(b d a)	•	•	•		
(s d a)	•	•		•	
(b s d a)	•	•	•	•	
(f a)	•	•	•		•
(b f a)		•	•		•
(s f a)		•		•	•
(b s f a)		•	•	•	•
(d f a)	•	•			•
(b d f a)	•	•	•		•
(s d f a)	•	•		•	•
$\top$	•	•	•	•	•

Table 3: The maximal subalgebras of $\mathcal{V}$ which have a polynomial-time satisfiability problem.

Subclass	Relations	Proof of NP-hardness
A_0	(d), (b a)	Lemma B.6
A_1	(d), (b s a)	Lemma B.7
A_2	(d), (b f a)	Lemma B.7
A_3	(s d), (b a)	Lemma B.7
A_4	(s d), (b f a)	Lemma B.7
A_5	(d f), (b a)	Lemma B.7
A_6	(d f), (b s a)	Lemma B.7
A_7	(s d f), (b a)	Lemma B.7
A_8	(d), (b s f a)	Lemma B.7
B_0	(d), (b f)	Lemma B.8
B_0^D	(d), (s a)	$B_0^D = \mathcal{D}_{\mathcal{V}}(B_0)$
B_1	(d), (b s f)	Lemma B.9
B_1^D	(d), (s f a)	$B_1^D = \mathcal{D}_{\mathcal{V}}(B_1)$
B_2	(s d), (b f)	Lemma B.9
B_2^D	(d f), (s a)	$B_2^D = \mathcal{D}_{\mathcal{V}}(B_2)$
B_3	(d a), (b f)	Lemma B.10
B_3^D	(b d), (s a)	$B_3^D = \mathcal{D}_{\mathcal{V}}(B_3)$
B_4	(s d a), (b f)	Lemma B.11
B_4^D	(b d f), (s a)	$B_4^D = \mathcal{D}_{\mathcal{V}}(B_4)$
B_5	(d a), (b s f)	Lemma B.11
B_5^D	(b d), (s f a)	$B_5^D = \mathcal{D}_{\mathcal{V}}(B_5)$
B_6	(b f), (s a)	Lemma B.12
C_0	(s), (b f)	Lemma B.13
C_0^D	(f), (s a)	$C_0^D = \mathcal{D}_{\mathcal{V}}(C_0)$
D_0	(s f), (b d a)	Lemma B.14
D_1	(s f), (b a)	Corollary B.15
D_2	(s f), (b d)	Corollary B.15
D_3	(s f), (d a)	Corollary B.15
D_4	(s f), (d)	Corollary B.15

Table 4: NP-hard subclasses of $\mathcal{V}$.

4 Discussion

We have only considered qualitative relations between time points and intervals in this article. For certain applications this is satisfactory—for others we must have the ability to reason also about metric time. Previous research on reasoning about combined qualitative and metric time has proven this problem to be computationally hard. However, recent results show that tractable reasoning is possible in certain subclasses of Allen's algebra augmented with quite advanced metric information. The linear-programming approach by Jonsson and Bäckström [9] and Koubarakis [11] offers a straightforward method for extending the ORD-Horn subclass with metric constraints. Several other subclasses of Allen's algebra with this property are exhibited in Drakengren and Jonsson [5]. Almost certainly, these methods can be adapted to the point-interval algebra which opens up for some interesting future research. Another interesting research direction is the study of tractable subclasses of Meiri's unrestricted approach, *i.e.*, allowing for time points and time intervals to be both qualitatively and metrically related.

The number of subclasses of $\mathcal{V}$ ($2^{32} \approx 4.3 \times 10^9$) is very small in comparison with the $2^{8192} \approx 10^{2466}$ subclasses of $\mathcal{A}$. In principle it would have been possible to enumerate all subclasses of $\mathcal{V}$ with the aid of a computer. This is not obviously the case with $\mathcal{A}$ (at least not with the computers available today). If we want to classify the subclasses of $\mathcal{A}$ with respect to tractability, we must use other methods. We are not pessimistic about the possibility of creating a complexity map of $\mathcal{A}$, especially not in the light of Ligozat's [12] recent results. By using algebraic techniques, he provides succinct proofs of some central complexity results on Allen's algebra which previously had only been proved by computerized enumeration methods. Furthermore, similar projects have been successfully performed in mathematics and computer science. A well-known example is the proof of the four-colour theorem [3] which combine theoretical studies of planar graphs with extensive machine-generated case analyses. It seems likely that we shall need methods that combine theoretical studies of the structure of $\mathcal{A}$ with brute-force computer methods. Here we can see a challenge for both theoreticians and practitioners in computer science.

5 Conclusion

We have studied computational properties of the point-interval algebra. All of the 2^{32} possible subclasses are classified with respect to whether their corresponding satisfiability problem is in P or NP-complete. The classification reveals that there are exactly five maximal subclasses having a polynomial-time solvable satisfiability problem.

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Appendix

This appendix collects the complexity results needed for the proof of Theorem 3.4. The proofs of membership in P can be found in part A and the NP-hardness results in part B.

A Polynomial-Time Problems

Proving that $\mathcal{V}$-SAT($\mathcal{V}^{23}$) $\in$ P is straightforward.

Proposition A.1 Deciding satisfiability of a set of PA formulae is a problem that can be solved in polynomial-time.

Proof: See Vilain *et al.* [19]. $\square$

Lemma A.2 $\mathcal{V}$-SAT($\mathcal{V}^{23}$) is in P.

Proof: Follows immediately from the definition of $\mathcal{V}^{23}$ and the previous proposition. $\square$

Before we can show that the other algebras in Table 3 are tractable, we need an auxiliary definition.

Definition A.3 Let $S \subseteq \mathbb{R}$ be finite and denote the absolute value of x with $\mathrm{abs}(x)$. The *minimal distance in* S, MD(S), is defined as

$$\min\{\mathrm{abs}(x \Leftrightarrow y) \mid x, y \in S \text{ and } x \neq y\}.$$

Observe that $|S| \geq 2$ in order to make MD(S) defined. For all such S, MD(S) > 0. The definition of minimal distance can be extended to PA- and $\mathcal{V}$-interpretations in the obvious way.

Lemma A.4 $\mathcal{V}$-SAT($\mathcal{V}^{20}_{\mathtt{s}}$) is in P.

Proof: Recall that $\mathcal{V}^{20}_{\mathtt{s}}$ consists of relations r satisfying either $r \subseteq \{\mathtt{b}, \mathtt{s}\}$ or $\{\mathtt{a}\} \subseteq r$.

Define the function $g : \mathcal{V}^{20}_{\mathtt{s}} \rightarrow 2^{\{<,=,>\}}$ such that

$$< \in g(r) \text{ iff } \mathtt{b} \in r;$$

$$= \in g(r) \text{ iff } \mathtt{s} \in r;$$

$$> \in g(r) \text{ iff } \mathtt{a} \in r;$$

Clearly, g is a total function. Given an arbitrary instance Θ of $\mathcal{V}$-SAT($\mathcal{V}_{\mathtt{s}}^{20}$), construct a set Θ' of PA formulae as follows:

$$\Theta' = \{p'g(r)I' \mid prI \in \Theta\}.$$

It can be decided whether Θ' has a PA-model or not in polynomial time. We show that Θ' has a PA-model iff Θ has a $\mathcal{V}$-model.

only-if: Assume that Θ has a $\mathcal{V}$-model $\Im$. Define a PA-interpretation $\Im'$ of Θ' such that

$$\Im'(p') = \Im(p);$$

$$\Im'(I') = \Im(I^-).$$

Arbitrarily choose a formula $p'r'I'$ in Θ'. Now,

if $\Im(p)(\mathtt{b})\Im(I)$, then $\Im'(p') < \Im'(I')$ and $< \in r'$ by the definition of g;

if $\Im(p)(\mathtt{s})\Im(I)$, then $\Im'(p') = \Im'(I')$ and $= \in r'$ by the definition of g;

if $\Im(p)(\mathtt{d})\Im(I)$, then $\Im'(p') > \Im'(I')$. However, if $\mathtt{d} \in r$, then $\mathtt{a} \in r$ by the definition of $\mathcal{V}_{\mathtt{s}}^{20}$ and $> \in r'$ by the definition of g;

if $\Im(p)(\mathtt{f})\Im(I)$, then $\Im'(p') > \Im'(I')$. However, if $\mathtt{f} \in r$, then $\mathtt{a} \in r$ by the definition of $\mathcal{V}_{\mathtt{s}}^{20}$ and $> \in r'$ by the definition of g;

if $\Im(p)(\mathtt{a})\Im(I)$, then $\Im'(p') > \Im'(I')$ and $> \in r'$ by the definition of g.

if: Assume that $\Im'$ is a PA-model of Θ'. Define the PA-interpretation $\Im$ of Θ such that

$$\Im(p) = \Im'(p');$$

$$\Im(I^-) = \Im'(I');$$

$$\Im(I^+) = \Im'(I') + \epsilon;$$

where $\epsilon = \mathrm{MD}(\Im')/2$. Arbitrarily choose a formula prI in Θ. The following facts hold:

if $\Im'(p') < \Im'(I')$, then $\mathtt{b} \in r$ (by the definition of g) and $p(\mathtt{b})I$ under $\Im$;

if $\Im'(p') = \Im'(I')$, then $\mathtt{s} \in r$ (by the definition of g) and $p(\mathtt{s})I$ under $\Im$;

if $\Im'(p') > \Im'(I')$, then $\Im(p) > \Im(I^+)$ by the choice of ϵ. The definition of g gives that $\mathtt{a} \in r$ and $p(\mathtt{a})I$ under $\Im$. $\square$

Corollary A.5 $\mathcal{V}$-SAT($\mathcal{V}^{20}_{\mathtt{f}}$) is in P.

Proof: It can easily be verified that $\mathcal{V}^{20}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{20}_{\mathtt{s}})$. □

Lemma A.6 $\mathcal{V}$-SAT($\mathcal{V}^{17}_{\mathtt{s}}$) is in P.

Proof: Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($\mathcal{V}^{17}_{\mathtt{s}}$). If a formula of the form $p \bot I$ is in Θ then Θ is not satisfiable. Otherwise, consider the following $\mathcal{V}$-interpretation: $\Im(p) = 0$ for every point p and $\Im(I^-) = 0$ and $\Im(I^+) = 1$ for every interval I. Let pRI be an arbitrary formula in Θ. By the definition of $\mathcal{V}^{17}_{\mathtt{s}}$, $\mathtt{s} \in R$. Obviously, $\Im$ satisfies pRI. Since it is a polynomial-time problem to check whether $p \bot I \in \Theta$ or not, the lemma follows. □

Corollary A.7 $\mathcal{V}$-SAT($\mathcal{V}^{17}_{\mathtt{f}}$) is in P.

Proof: $\mathcal{V}^{17}_{\mathtt{s}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{17}_{\mathtt{f}})$. □

B NP-Hardness Results

This section provides NP-hardness proofs for the subclasses of $\mathcal{V}$ presented in Table 4. The reductions are mostly made from different subalgebras of Allen's interval algebra. Consequently, we begin this section by recapitulating some results concerning this algebra. To make the proofs of NP-hardness less cumbersome, we will employ a technique which we refer to as *model transformations*; the definitions and results needed are collected in Section B.2.

B.1 Allen's Algebra

Allen's interval algebra [1] is based on the notion of *relations between pairs of intervals*. An interval X is represented as an ordered pair $\langle X^-, X^+ \rangle$ of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*. Their exact definitions can be found in Table 5. Such disjunctions are represented as sets of basic relations. The algebra is provided with the operations of *converse, intersection and composition* on intervals. The definitions of these operations can be found in [1]. By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. The decision problem we will consider is the problem of *satisfiability* ($\mathcal{A}$-SAT) of a set of interval variables with relations between them, *i.e.*, deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals hold. Such an assignment is said to be a $\mathcal{A}$-*model* for the interval variables and relations. For $\mathcal{A}$, we have the following result.

Basic relation	Symbol	Example	Endpoint relations
X before Y	$\prec$	xxx	$X^- < Y^-$, $X^- < Y^+$,
Y after X	$\succ$	yyy	$X^+ < Y^-$, $X^+ < Y^+$
X meets Y	m	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y met-by X	$\mathsf{m}^{\smile}$	yyyy	$X^+ = Y^-$, $X^+ < Y^+$
X overlaps Y	o	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y overlapped-by X	$\mathsf{o}^{\smile}$	yyyy	$X^+ > Y^-$, $X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y includes X	$\mathsf{d}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$, $X^- < Y^+$,
Y started by X	$\mathsf{s}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X finishes Y	f	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y finished-by X	$\mathsf{f}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^- < Y^+$
X equals Y	$\equiv$	xxxx	$X^- = Y^-$, $X^- < Y^+$,
		yyyy	$X^+ > Y^-$, $X^+ = Y^+$

Table 5: The thirteen basic relations of the $\mathcal{A}$-algebra. The endpoint relations $X^- < X^+$ and $Y^- < Y^+$ that are valid for all relations have been omitted.

Theorem B.1 Define the sets $\mathcal{N}_3$ and $\mathcal{F}$ as follows:

$$\mathcal{N}_3 = \{(\prec \succ), (\mathsf{o}\ \mathsf{o}^{\smile})\}$$

and

$$\mathcal{F} = \{(\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{f}^{\smile})\}.$$

$\mathcal{A}$-SAT($\mathcal{N}_3$) and $\mathcal{A}$-SAT($\mathcal{F}$) are NP-hard problems.

Proof: The result for $\mathcal{N}_3$ can be found in [4].

To prove that $\mathcal{A}$-SAT($\mathcal{F}$) is NP-hard, we will use a closure operation for Allen's algebra which was defined by Nebel and Bürckert [15]. Assume $\mathcal{S} \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{S})$ the $\mathcal{A}$-*closure* of $\mathcal{S}$ under converse, intersection and composition, *i.e.*, the least subset of $\mathcal{A}$ containing $\mathcal{S}$ closed under the three operations. Nebel and Bürckert [15] have shown that $\mathcal{A}$-SAT($\mathcal{S}$) is NP-hard if and only if $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is. Hence, to show NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$), we can study $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{F})$) instead of $\mathcal{A}$-SAT($\mathcal{F}$).

It can be verified that $\mathcal{A} = \mathcal{C}_{\mathcal{A}}(\mathcal{F})$ and NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$) follows. □

In the previous lemma, $\mathcal{C}_{\mathcal{A}}$ was computed by the utility `aclose` [14].

B.2 Model Transformations

One of our main vehicles for showing NP-hardness of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem.

Definition B.2 A *model transformation* is a mapping on $\mathcal{V}$-interpretations.

This definition is very general. To make it applicable in practice, we need a way of describing model transformations.

Definition B.3 Let T be a model transformation. A function $f_T : \mathbf{B} \rightarrow 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{V}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and pbJ under $\Im$, then $pf_T(b)J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$.

Lemma B.4 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{V}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{V}$ be such that $\mathcal{V}$-SAT($\mathcal{R}$) is NP-hard and $r_k \subseteq r'_k$, $1 \leq k \leq n$. If there exists a model transformation T with a description f_T such that $f_T(r'_k) \subseteq r_k$ for every $1 \leq k \leq n$, then $\mathcal{V}$-SAT($\mathcal{R}'$) is NP-hard.

Proof: Arbitrarily choose an instance Θ of $\mathcal{V}$-SAT($\mathcal{R}$) and let $\Theta' = \{pr'I \mid prI \in \Theta\}$. Obviously, this is a polynomial-time transformation and Θ' is an instance of $\mathcal{V}$-SAT($\mathcal{R}'$). We show that Θ is satisfiable iff Θ' is satisfiable.
only-if: Let $\Im$ be a model of Θ. Recall that $r_k \subseteq r'_k$, $1 \leq k \leq n$. Hence, $\Im$ is a model of Θ' since every formula $pr'I \in \Theta'$ is weaker than the corresponding formula $prI \in \Theta$.
if: Let $\Im'$ be a model of Θ'. We show that $T(\Im')$ is a model of Θ. Arbitrarily choose a formula prI in Θ. By the construction of Θ', there exists a formula $pr'I \in \Theta'$. Thus, we have $pr'I$ under $\Im'$ which implies $pf_T(r')I$ under $T(\Im')$ since f_T is a description of T. Furthermore, $f_T(r') \subseteq r$ so prI holds under $T(\Im')$. Since prI was arbitrarily chosen, $T(\Im')$ is a model of Θ. □

Definition B.5 Assume that $\Im$ is an arbitrary $\mathcal{V}$-interpretation and let $\epsilon = \mathrm{MD}(\Im)/2$. Define the model transformation $T_{o,o'}$ for $o, o' \in \{+, \Leftrightarrow, 0\}$ as

1. for every point p, let $T_{o,o'}(\Im)(p) = \Im(p)$;
2. for every interval I, let $T_{o,o'}(\Im)(I^-) = \Im(I^-)\ o\ \epsilon$ if $o \in \{+, \Leftrightarrow\}$ and $T_{o,o'}(\Im)(I^-) = \Im(I^-)$ otherwise;
3. for every interval I, let $T_{o,o'}(\Im)(I^+) = \Im(I^+)\ o'\ \epsilon$ if $o' \in \{+, \Leftrightarrow\}$ and $T_{o,o'}(\Im)(I^+) = \Im(I^+)$ otherwise.

Descriptions of the model transformations in the previous definition can be found in Table 6.

	b	s	d	f	a
$T_{-,-}$	b	d	d	a	a
$T_{-,0}$	b	d	d	f	a
$T_{-,+}$	b	d	d	d	a
$T_{0,-}$	b	s	d	a	a
$T_{0,0}$	b	s	d	f	a
$T_{0,+}$	b	s	d	d	a
$T_{+,-}$	b	b	d	a	a
$T_{+,0}$	b	b	d	f	a
$T_{+,+}$	b	b	d	d	a

Table 6: Descriptions of some model transformations.

B.3 NP-Hard Subclasses of $\mathcal{V}$

Lemma B.6 $\mathcal{V}$-SAT(A_0) is NP-hard.

Proof: Reduction from $\mathcal{A}$-SAT($\mathcal{N}_3$) which is NP-hard by Lemma B.1. Let I and J be two intervals. We will show how to express $I(\mathtt{o}\ \mathtt{o}^{\smile})J$ and $I(\prec\ \succ)J$ in the point-interval algebra by only using the point-interval relations in A_0, *i.e.*, ($\mathtt{d}$) and ($\mathtt{b}\ \mathtt{a}$). By doing so, we have shown NP-hardness of $\mathcal{V}$-SAT(A_0).

Assume we want to relate the intervals I and J with the relation ($\mathtt{o}\ \mathtt{o}^{\smile}$). Observe that $I(\mathtt{o}\ \mathtt{o}^{\smile})J$ holds iff there exists three real numbers a, b, c with the following properties:

$I^- < a < I^+$ but $a < J^-$ or $a > J^+$

$I^- < b < I^+$ and $J^- < b < J^+$

$J^- < c < J^+$ but $c < I^-$ or $c > I^+$

Obviously, we can relate I and J with ($\mathtt{o}\ \mathtt{o}^{\smile}$) by introducing three fresh points a, b, c and the following six point-interval formulae:

$a(\mathtt{d})I \quad a(\mathtt{b}\ \mathtt{a})J$
$b(\mathtt{d})I \quad b(\mathtt{d})J$
$c(\mathtt{d})J \quad c(\mathtt{b}\ \mathtt{a})I.$

In order to relate two intervals I, J with the relation ($\prec\ \succ$), we introduce two fresh intervals K, L and relate them with ($\mathtt{o}\ \mathtt{o}^{\smile}$) by using three fresh points a, b and c as above. Observe that either $\Im(a) < \Im(b) < \Im(c)$ or $\Im(c) < \Im(b) < \Im(a)$ in any model $\Im$.

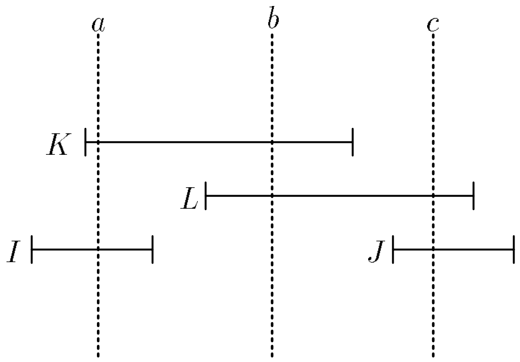

Figure 1: The construction of $I(\prec\ \succ)J$ in the proof of Lemma B.6. The figure shows the situation when $\Im(I^+) < \Im(b) < \Im(J^-)$; the other case is analogous.

Now, relate a, b, c to I, J as follows:

$$\begin{array}{ll} a(\texttt{d})I & a(\texttt{b a})J \\ b(\texttt{b a})I & b(\texttt{b a})J \\ c(\texttt{b a})I & c(\texttt{d})J. \end{array}$$

Let $\Im$ be a model satisfying these restrictions. It follows that either $\Im(I^+) < \Im(b) < \Im(J^-)$ or $\Im(J^+) < \Im(b) < \Im(I^-)$ and, consequently, $I(\prec\ \succ)J$. An explanation of this construction can be found in Figure 1. □

Lemma B.7 $\mathcal{V}$-SAT(A_i), $1 \leq i \leq 8$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(A_0). Use the following model transformations

$$\begin{array}{ll} i = 1 : T_{+,0} & i = 2 : T_{0,-} \\ i = 3 : T_{-,0} & i = 4 : T_{-,-} \\ i = 5 : T_{0,+} & i = 6 : T_{+,+} \\ i = 7 : T_{-,+} & i = 8 : T_{+,-} \end{array}$$

and apply Lemma B.4. □

Lemma B.8 $\mathcal{V}$-SAT(B_0) is NP-hard.

Proof sketch: The proof of Lemma B.6 goes through if the relation (`b a`) is replaced by (`b f`). □

Lemma B.9 $\mathcal{V}$-SAT(B_i), $1 \leq i \leq 2$, is NP-hard.

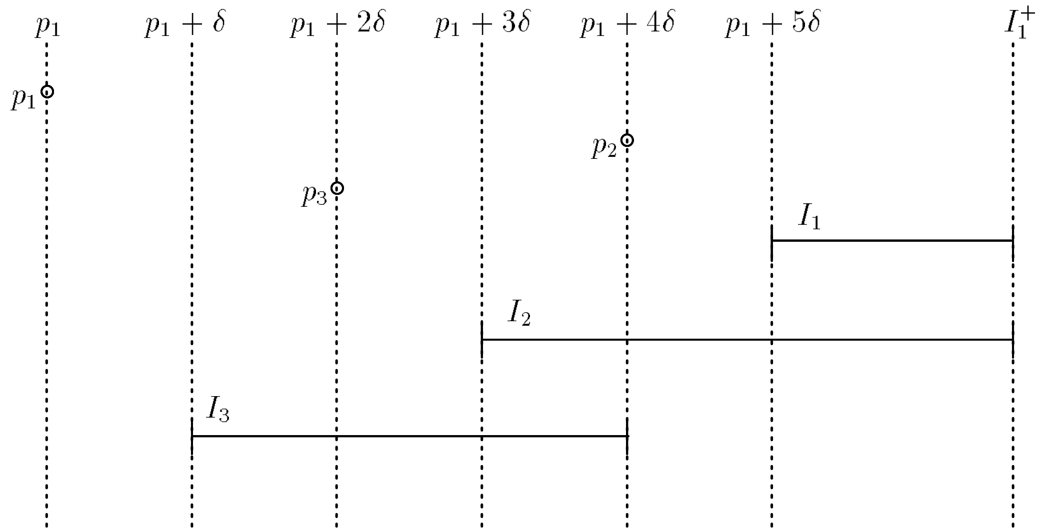

Figure 2: The construction used in Lemma B.10.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_0). Use the following model transformations

$$i = 1 : T_{+,0} \qquad i = 2 : T_{-,0}$$

and apply Lemma B.4. $\square$

Lemma B.10 $\mathcal{V}$-SAT(B_3) is NP-hard.

Proof: By Lemma B.8, $\mathcal{V}$-SAT(B_0) is NP-hard. Let $E = \{(\texttt{b}), (\texttt{d a}), (\texttt{b f})\}$. Then $(\texttt{b f}) \otimes (\texttt{d a}) \otimes (\texttt{b}) = (\texttt{b s d})$ and $(\texttt{b s d}) \cap (\texttt{d a}) = (\texttt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem. We show how to construct an instance Θ' of the $\mathcal{V}$-SAT(B_3) problem that is satisfiable iff Θ is satisfiable.

We begin by showing how to relate a point p_1 and an interval I_1 such as $p_1(\texttt{b})I_1$ by only using the relations in B_3. We introduce two fresh points p_2 and p_3 together with two fresh intervals I_2 and I_3. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\texttt{b f})I_1 & (2)\ p_1(\texttt{b f})I_2 & (3)\ p_1(\texttt{b f})I_3 \\ (4)\ p_2(\texttt{b f})I_1 & (5)\ p_2(\texttt{d a})I_2 & (6)\ p_2(\texttt{b f})I_3 \\ (7)\ p_3(\texttt{b f})I_1 & (8)\ p_3(\texttt{b f})I_2 & (9)\ p_3(\texttt{d a})I_3. \end{array}$$

We denote this set of formulae with Ω. Assume that $\Im$ is a $\mathcal{V}$-model of Ω. For the sake of brevity, we identify the points and intervals with their values when interpreted by $\Im$. Hence, instead of writing $\Im(p_1) < \Im(I_1^-)$, we simply write $p_1 < I_1^-$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\delta = (I_1^- \Leftrightarrow p_1)/5$. Consider the following assignment of values:

$$\begin{array}{lll} I_3^- = p_1 + \delta & p_3 = p_1 + 2\delta & I_2^- = p_1 + 3\delta \\ p_2 = p_1 + 4\delta & I_3^+ = p_1 + 4\delta & I_2^+ = I_1^+. \end{array}$$

This assignment is depicted in Figure 2. Obviously, the assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. By formula (4), we can see that $p_2 < I_1^-$ or $p_2 = I_1^+$. By assumption, $p_1 = I_1^+$. Hence, either $p_2 < I_1^-$ or $p_2 = p_1$. If $p_2 = p_1$, then formula (2) is equivalent to $p_2(\texttt{b f})I_2$ which clearly contradicts formula (5). Thus, $p_2 < I_1^-$ and $p_2(\texttt{b})I_1$. By analogous reasoning one can see that $p_3 < I_1^-$ and $p_3(\texttt{b})I_1$.

Next, observe that formulae (2) and (3) imply $p_1 \le I_2^+$ and $p_1 \le I_3^+$. Furthermore, $p_2 < I_1^-$ and $p_3 < I_1^-$ which implies $p_2 < I_1^+$ and $p_3 < I_1^+$. By our initial assumption $p_1 = I_1^+$ we get

A: $p_2 < I_1^+ = p_1 \le I_3^+$

B: $p_3 < I_1^+ = p_1 \le I_2^+$

Consequently, $p_2 < I_3^+$ and $p_3 < I_2^+$. Observe that $p_2(\texttt{b f})I_3$ and $p_3(\texttt{b f})I_2$ by formulae (6) and (8), respectively. Hence, $p_2 < I_3^-$ and $p_3 < I_2^-$.

We have to study four cases, depending on how the $\mathcal{V}$-model assigns values to the variables in formula (5) and (9).

1. $I_2^- < p_2 < I_2^+$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < p_2$ which leads to a contradiction.

2. $I_2^+ < p_2$ and $I_3^+ < p_3$. Then $p_2 < I_3^- < I_3^+ < p_3 < I_2^- < I_2^+ < p_2$ which is a contradiction.

3. $I_2^+ < p_2$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < I_2^+ < p_2$. Contradiction.

4. $I_2^- < p_2 < I_2^+$ and $I_3^+ < p_3$. This case is analogous to the previous case.

Consequently, every $\mathcal{V}$-model of Ω satisfies $p_1 < I_1^-$.

We have thus shown how to express the relation ($\texttt{b}$) by only using ($\texttt{d a}$) and ($\texttt{b f}$). Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_3) problem. NP-hardness of $\mathcal{V}$-SAT(B_3) follows immediately.

□

Lemma B.11 $\mathcal{V}$-SAT(B_i), $4 \leq i \leq 5$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_3). Use the following model transformations

$$i = 4 : T_{-,0} \qquad i = 5 : T_{+,0}$$

and apply Lemma B.4. □

Lemma B.12 $\mathcal{V}$-SAT(B_6) is NP-hard.

Proof: Let $E = \{(\texttt{b}), (\texttt{b f}), (\texttt{s a})\}$. Then $(\texttt{b f}) \otimes (\texttt{s a}) \otimes (\texttt{b}) = (\texttt{b s d})$, $(\texttt{s a}) \otimes (\texttt{b}) \otimes (\texttt{s a}) = (\texttt{d f a})$ and $(\texttt{b s d}) \cap (\texttt{d f a}) = (\texttt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT(E). We show how to construct an instance Θ' of $\mathcal{V}$-SAT(B_6) that is satisfiable iff Θ is satisfiable.

We begin showing how to relate a point p_1 and an interval I_1 such as $p_1(\texttt{b})I_1$ by only using the relations in B_6. We introduce two fresh points p_2 and p_3 together with a fresh interval I_2. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\texttt{b f})I_1 & (2)\ p_1(\texttt{b f})I_2 & (3)\ p_2(\texttt{b f})I_1 \\ (4)\ p_2(\texttt{s a})I_2 & (5)\ p_3(\texttt{s a})I_1 & (6)\ p_3(\texttt{b f})I_2 \end{array}$$

We denote this set of formulae with Ω. Let $\Im$ be a $\mathcal{V}$-model of Ω. As in the proof of Lemma B.10, we identify the points and intervals with their value when interpreted by $\Im$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\Delta = (I_1^+ \Leftrightarrow I_1^-)/3$. Consider the following assignment of values:

$$p_3 = I_1^-$$

$$p_2 = I_1^+$$

$$I_2^- = I_1^+$$

$$I_2^+ = I_2^- + 1$$

It is not hard to see that this assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. We consider two cases:

Case 1: $p_1(\texttt{f})I_2$. Then $p_1 = I_1^+ = I_2^+$. Hence, if $p_2(\texttt{f})I_1$, then $p_2(\texttt{f})I_2$ which contradicts formula (4). Consequently, $p_2(\texttt{b})I_1$ and, by formula (4), $p_2(\texttt{s})I_2$. Obviously, $I_2^- < I_1^- < I_2^+$.

By formula (5), $p_3(\mathtt{s\ a})I_1$. Assume $p_3(\mathtt{s})I_1$. Then $p_3(\mathtt{d})I_2$ which contradicts formula (6). Assume to the contrary that $p_3(\mathtt{a})I_1$. But this implies $p_3(\mathtt{a})I_2$ which also contradicts formula (6).

Case 2: $p_1(\mathtt{b})I_2$. Since $p_1(\mathtt{f})I_1$, I_1 and I_2 are disjoint intervals and I_1 is strictly before I_2. By formula (3), $p_2(\mathtt{b\ f})I_1$. If $p_2(\mathtt{b})I_1$, then $p_2(\mathtt{b})I_2$ which contradicts formula (4). If $p_2(\mathtt{f})I_1$, then $p_2(\mathtt{b})I_2$ which, again, contradicts formula (4).

We have thus shown how to express the relation ($\mathtt{b}$) by only using ($\mathtt{b\ f}$) and ($\mathtt{s\ a}$). Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_6) problem. NP-hardness of $\mathcal{V}$-SAT(B_6) follows immediately. □

Lemma B.13 $\mathcal{V}$-SAT(C_0) is NP-hard.

Proof: Reduction from $\mathcal{A}$-SAT($\mathcal{F}$) which is NP-hard by Lemma B.1. Let Θ be an instance of $\mathcal{A}$-SAT($\mathcal{F}$). We construct a set Θ' as follows.

For each formula of the type $I(\prec\ \mathtt{d}^{\smile}\ \mathtt{o\ m\ m}^{\smile}\ \mathtt{f}^{\smile})J$ in Θ, introduce a new point $p_{I,J}$ and let $p_{I,J}(\mathtt{s})I$ and $p_{I,J}(\mathtt{b\ f})J$ in Θ';

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_0) problem. It is a routine verification to show that Θ is satisfiable iff Θ' is satisfiable. □

Theorem B.14 $\mathcal{V}$-SAT(D_0) is NP-hard.

Proof: Proof by reduction from Graph 3-Colourability, which is NP-complete [7]. Let $G = \langle V, E\rangle$ be an undirected graph. Construct a corresponding set of point-interval formulae as follows (the construction is illustrated graphically in Figure 3 for the connected two-vertex graph $\langle\{v_1, v_2\}, \{\{v_1, v_2\}\}\rangle$).

First, we construct a *paint-box* defining our three "colours". It consists of a point p and two intervals I_1 and I_2, plus the two relations $p(\mathtt{s\ f})I_1$ and $p(\mathtt{s\ f})I_2$. Because of these constraints, the two intervals must have at least one end-point in common, so there is a total of two or three interval endpoints, which constitute our available colours. Hence, each model has a palette with two or three colours. We may, thus, occasionally get models corresponding to 2-colourings of G, but this is unimportant since every 2-colouring is a 3-colouring. Of course, the actual denotations of these colours differ between models, but this is also unimportant since they denote two or three different colours in each and every model.

Next, for each vertex $v_i \in V$ we introduce a *selector* consisting of two intervals J_i^{sel}, J_i^{sep} and three points q_i^{sub1}, q_i^{sub2} and q_i^{sel}. The interval J_i^{sep} acts

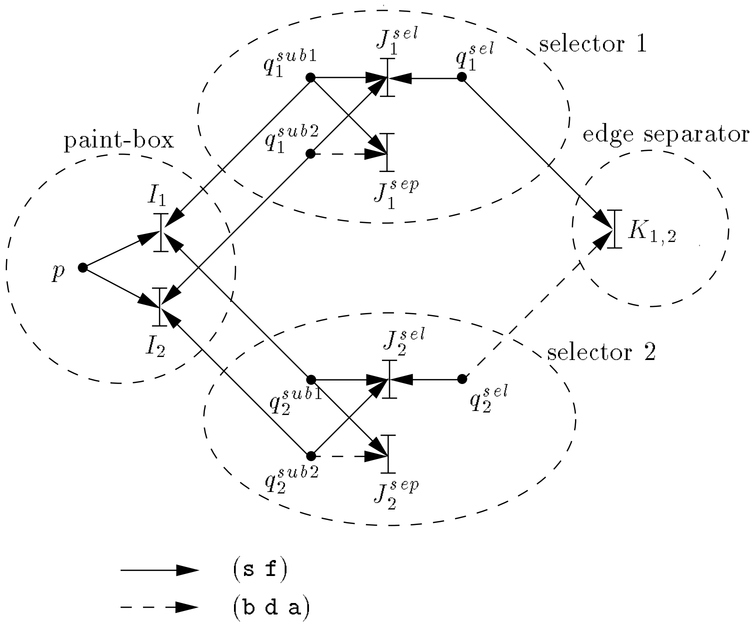

Figure 3: An example of the construction in the proof of Theorem B.14 for a connected two-vertex graph.

as a *separator* for the points q_i^{sub1} and q_i^{sub2}. This means that by introducing the relations $q_i^{sub1}(\mathtt{s\ f})J_i^{sep}$ and $q_i^{sub2}(\mathtt{b\ d\ a})J_i^{sep}$ these two points are forced not to coincide. Further, both these points are related to the interval J_i^{sel} via the relations $q_i^{sub1}(\mathtt{s\ f})J_i^{sel}$ and $q_i^{sub2}(\mathtt{s\ f})J_i^{sel}$, which forces each endpoint of this interval to coincide with either of the two points. Then the point q_i^{sel} is related to interval J_i^{sel} as $q_i^{sel}(\mathtt{s\ f})J_i^{sel}$, which forces it to coincide with either of the two points q_i^{sub1} and q_i^{sub2}. Finally, this whole gadget is connected to the paint-box via the relations $q_i^{sub1}(\mathtt{s\ f})I_1$ and $q_i^{sub2}(\mathtt{s\ f})I_2$. Hence, these two points select a subpalette of exactly two of the available colours, and the point q_i^{sel} then further selects one of these two colours as the colour of vertex v_i.

Finally, for each edge $\{v_i, v_j\} \in E$ we introduce an interval $K_{i,j}$ acting as a separator for the points q_i^{sel} and q_j^{sel}, in the same way as the separator interval within the selectors. More precisely, the relations $q_i^{sel}(\mathtt{s\ f})K_{i,j}$ and $q_j^{sel}(\mathtt{b\ d\ a})K_{i,j}$ are introduced. Obviously, these two formulae can be simultaneously satisfied iff vertices v_i and v_j have different colours, since the two points q_i^0 and q_j^0 must not coincide.

It is obvious that G is 3-colourable iff the set of point-interval formulae just constructed is $\mathcal{V}$-satisfiable, so NP-hardness of the algebra follows. $\square$

Corollary B.15 $\mathcal{V}$-SAT(D_i), $1 \leq i \leq 4$, is NP-hard.

Proof: Use the same construction as in the proof of Theorem B.14. We note that relations of the type (`b d a`) are used only in separators, forcing two points apart. Consider a separator interval I for two points p and q which are related as p(`s f`)I and $q\ r\ I$, where $r =$ (`b d a`) in the original construction. We now consider how r can be constrained. First suppose that $p = q$. In this case at most one of the two relations must be satisfied, which holds as long as r does not contain either s or f. Conversely, suppose $p \neq q$. In this case, there must exist a choice of endpoints for I such that both formulae are satisfied. It is obviously a sufficient criterion that r contains either `d` or both `a` and `b` (either of the latter alone is not sufficient, since other constraints in the model may dictate whether $p < q$ or $q < p$). It follows that r can be chosen as either of the relations (`b a`), (`b d`), (`d a`) and (`d`), which proves the corollary. □