Applying a Reputation Metric in a Two-Player Resource Sharing Game

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Abstract. In this paper we introduce and analyze a reputation scheme for common resource sharing problem from game theory perspective. The paper tries to answer the question: If each player is allocated resources proportional to its reputation, will the player's strategy be good for the system? The problem is formulated for a general form of reputation metric and analyzed for more specific, but still representative form of the reputation metric.

Keywords: game theory, reputation system, p2p.

1. Introduction

Many popular Internet applications including data backup, media streaming and software updates are built on peer-to-peer architecture (Lua et al., 2005). Such systems require proper motivation to users to operate (Akella et al., 2002), (Mazalov et al.,2007), (Zhang et al., 2005). It means that effectiveness of these systems is based on free will of individuals to contribute to the systems (Qiu et al.,2003), (Shenker, 1995). A problem that limits the usage for such system is called free-riding (Feldman et al., 2004); it is selfish behavior of individuals who do not contribute to the system while trying to use the system. To prevent such behavior of individuals, reputation systems were proposed. In reputation systems a reputation metric is associated with each individual. While individuals contribute to the system they increase their metric, and decrease during selfish use of the system. While the reputation metric provide some incentives and were used in many systems, the study of metric itself was not extensive.

The incentive problem for distributed systems often can be formulated in terms of distributed file (data) sharing system. In our model we will state two ways of individual behavior. The first one is 'to contribute', and the second one is 'to consume'. The reputation metric is in the form of accumulation of all previous contribution and consumption. We assume that every player of the system is trying to individually maximize its profit. Hence we are trying to find a Nash equilibrium in such a system, and study the equilibrium dependence on the reputation metric.

The paper is organized as follows. In Section 2, we state a formal description of the resource sharing problem. In Section 3, the necessary background on the maximum principle and the optimal control is given. In Section 4, we analyze the problem for a special case of the reputation function $f(x) = \frac{\exp x}{1+\exp x}$. In Section 5, the optimal trajectory is defined. Section 6 concludes the paper.

2. Formulation

Let us have N players. Player i at time moment t contributes $c_i(t)$ to the system ($0 \leq c_i(t) \leq 1$), and requests $r_i(t)$ from the system ($0 \leq r_i(t) \leq 1$). Reputation value (or we will call it history) $x_i(t)$ starting from initial value (0) changes linearly $\dot{x_i}(t) = c_i(t) - r_i(t)$. The profit of player i is calculated over time t (assuming that every player started to work at time 0 and stopped at time T) is $\int_0^T \left[\frac{f(x_i(t))r_i(t)}{\sum_{j=0}^N f(x_j(t))r_j(t)+\epsilon} - lc_i(t)\right] dt$, where f is the reputation metric function. As we can see, every player receives not the amount of the requested data $r_i(t)$, but the proportion of it, weighted by the reputation function. l is the price of contribution (in different systems it has different values). We want to study dependence of equilibrium upon the function f. For simplicity we will reformulate this game as a two-player game:

$$\left.\begin{aligned}
&\int_0^T \left[\frac{f(x_1)r_1(t)}{f(x_1)r_1(t) + f(x_2)r_2(t) + \epsilon} - lc_1(t)\right] dt \to \max \\
&\int_0^T \left[\frac{f(x_2)r_2(t)}{f(x_1)r_1(t) + f(x_2)r_2(t) + \epsilon} - lc_2(t)\right] dt \to \max \\
&\dot{x_1}(t) = c_1(t) - r_1(t) \\
&\dot{x_2}(t) = c_2(t) - r_2(t) \\
&0 \leq c_1(t), c_2(t), r_1(t), r_2(t) \leq 1 \\
&x_1(0) = x_2(0) = 0
\end{aligned}\right\} \tag{1}$$

From the system (1) we can see that the accumulated history $x(t)$ changes depending on the behavior in interval $(-\infty, \infty)$. Function f should weight the history, and it clearly maps from interval $(-\infty, \infty)$ onto $[0, 1]$ (there is no restriction that the upper bound equals to 1, but for simplicity we will bound it from above by 1). Proportion under the integral sign means that the resource first and second players want to gain is not sufficient for both of them ($r_1(t)+r_2(t) > 1$). Hence they receive amount proportional to their behavior.

While the study of such system is complicated, in this paper we attempt to study a special case of the equation where $f(x) = \frac{\exp x}{1+\exp x}$. This view of the function is quite natural, it maps all reputations from domain $(-\infty, \infty)$ to domain $(0, 1)$. While reputation x can be negative, the value of the function just compresses the x closer to 0. In such form we may say that it is some kind of goodness probability of a player (probability metric). By changing the variable we can formulate the task by the following system:

$$\left.\begin{aligned}
&\int_0^T \left[\frac{k_1 r_1(t)}{k_1 r_1(t) + k_2 r_2(t) + \epsilon} - l c_1(t)\right] dt \to \max \\
&\int_0^T \left[\frac{k_2 r_2(t)}{k_1 r_1(t) + k_2 r_2(t) + \epsilon} - l c_2(t)\right] dt \to \max \\
&\dot{k}_1(t) = k_1(1-k_1)(c_1(t) - r_1(t)) \\
&\dot{k}_2(t) = k_2(1-k_2)(c_2(t) - r_2(t)) \\
&0 \le c_1(t), c_2(t), r_1(t), r_2(t) \le 1 \\
&k_1(0) = k_2(0) = 0.5
\end{aligned}\right\} \quad (2)$$

Now history x_1, x_2 itself takes values in interval $(0,1)$, hence those can be used as reputation functions. The rule $\dot{k}_i(t) = k_i(1-k_i)(c_i(t) - r_i(t))$ with initial value $k_i(0) = 0.5$ restricts function $k_i(t)$ to always stay on the $(0,1)$ interval. This paper is devoted to the study of this special case of a 2-player game. The study of the special case helps in understanding the behavior and study of more general system (1).

3. Background. Maximum principle of Pontryagin.

To analyze the stated problem we will formulate the *maximum principle* of Pontryagin (Pontryagin et al., 1962) first and then give some required background. We will use notation and Theorem 5.4 from (Basar and Olsder, 1999). Let $x(t)$ be the trajectory, $u(t)$ – control and f – law of the dynamic system, which connects trajectory, control and time. Let also T be the final time, $L(u)$ the cost function and γ some mapping from time space onto space of admissible control space (S). Then the problem in general can be written as the following system:

$$\left.\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)), x(0) = x_0, t \ge 0, \\
u(t) &= \gamma(t) \in S, \\
L(u) &= \textstyle\int_0^T g(t, x(t), u(t))dt + q(T, x(T)), \\
T &= \min_{t \ge 0}\{t : l(t, x(t)) = 0\},
\end{aligned}\right\} \quad (3)$$

where condition $l(t, x(t)) = 0$ defines achievement of terminal state. For such a system we can form the following system of equations, which are called *canonical* *equations*:

$$\left.\begin{array}{rcl} \dot{x}^*(t) & = & \left(\frac{\partial H}{\partial p}\right)' = f(t, x^*, u^*), x(t_0) = x_0; \\ \dot{\lambda}'(t) & = & -\frac{\partial H(t,\lambda,x^*,u^*)}{\partial x}, \\ \lambda'(T) & = & \frac{\partial q(T,x^*)}{\partial x} \quad \text{along } l(T,x) = 0; \\ H(t,\lambda,x,u) & \triangleq & g(t,x,u) + \lambda' f(t,x,u), \\ u^*(t) & = & \arg\min_{u\in S} H(t,\lambda,x^*,u), \end{array}\right\} \tag{4}$$

where λ is a co-state variable, function $H(t,p,x,u)$ is called the Hamiltonian function and operation $(\dots)'$ is a transposition operation. Now we can state the maximum principle of Pontryagin in the following theorem:

Theorem 1. (Theorem 5.4 from (Basar and Olsder, 1999)) *Consider the optimal control problem defined by (3) and under the open loop information structure. If the functions f,g,q and l are continuously differentiable in x and continuous in t and u, then relations (4) provide a set of necessary conditions for the optimal control and the corresponding optimal trajectory.*

For a more formal description of all game-theoretic notations and theorem, see (Basar and Olsder, 1999), and for the proof of the theorem above, see (Pontryagin et al., 1962).

4. Analysis

In this section, we will give the analysis for the system (2) using the maximum principle, described in the previous section. First of all we construct the Hamiltonians; for the system (2) they have the following form:

$$H_1 = \left(lc_1 - \frac{k_1 r_1}{k_1 r_1 + k_2 r_2 + \epsilon}\right) + \lambda_{11} k_1 (1-k_1)(c_1 - r_1) + \lambda_{12} k_2 (1-k_2)(c_2 - r_2) \tag{5}$$

$$H_2 = \left(lc_2 - \frac{k_2 r_2}{k_1 r_1 + k_2 r_2 + \epsilon}\right) + \lambda_{21} k_1 (1-k_1)(c_1 - r_1) + \lambda_{22} k_2 (1-k_2)(c_2 - r_2) \tag{6}$$

Note, in this section we will be using i, j as indexes. Whenever it is not mentioned explicitly we assume that following conditions hold $i, j \in \{1, 2\}$ and $j \neq i$, i.e., i is the index of first player and j is the index of the second player.

Theorem 2. *For the problem stated above the co-state variables have the following form*

$$\lambda_{11}(t) = \frac{-1}{k_1(1-k_1)} \int_t^T \frac{(k_2r_2+\epsilon)k_1r_1(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$
$$\lambda_{12}(t) = \frac{1}{k_2(1-k_2)} \int_t^T \frac{k_1r_1k_2r_2(1-k_2)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$
$$\lambda_{22}(t) = \frac{-1}{k_2(1-k_2)} \int_t^T \frac{(k_1r_1+\epsilon)k_2r_2(1-k_2)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$
$$\lambda_{21}(t) = \frac{1}{k_1(1-k_1)} \int_t^T \frac{k_1r_1k_2r_2(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$

Proof. Knowing the Hamiltonians we can get the co-state variables using following equation:

$$\dot{\lambda}_{ij}(t) = -\frac{\partial H_i}{\partial k_j}, \text{ where } i,j \in \{1,2\}. \tag{7}$$

Thus, for λ_{ii}, $i \in \{1,2\}$ we have (we also use $j \in \{1,2\}$, $j \neq i$)

$$\dot{\lambda}_{ii}(t) = \frac{r_i(k_jr_j+\epsilon)}{(k_1r_1+k_2r_2+\epsilon)^2} - \lambda_{ii}(t)(1-2k_i)(c_i-r_i) \tag{8}$$

Multiplying both parts of equation above by $e^{\int_0^t (1-2k_i)(c_i-r_i)}dt$ we get

$$\frac{d}{dt}\left(\lambda_{ii}(t)e^{\int_0^t(1-2k_i)(c_i-r_i)dt}\right) = e^{\int_0^t(1-2k_i)(c_i-r_i)dt}\frac{r_i(k_jr_j+\epsilon)}{(k_1r_1+k_2r_2+\epsilon)^2}. \tag{9}$$

Now, we can simplify $e^{\int_0^t(1-2k_i)(c_i-r_i)}dt$, from (2) we have that $dk_i(t) = k_i(1-k_i)(c_i(t)-r_i(t))dt$, thus

$$\begin{aligned}\int_0^t (1-2k_i)(c_i-r_i)dt &= \int_0^t \frac{(1-2k_i(t))}{k_i(t)(1-k_i(t))}dk_i(t)\\ &= \int_0^t \frac{1}{k_i(t)(1-k_i(t))}d[k_i(t)(1-k_i(t))]\\ &= \ln\left(k_i(t)(1-k_i(t))\right) - \ln\left(k_i(0)(1-k_i(0))\right). \end{aligned} \tag{10}$$

Hence, we have

$$e^{\int_0^t(1-2k_i)(c_i-r_i)dt} = \frac{k_i(1-k_i)}{k_0(1-k_0)} \tag{11}$$

And, finally

$$\lambda_{ii}(t)k_i(1-k_i) = c + \int_0^t \frac{(k_2r_2+\epsilon)k_1r_1(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt, \tag{12}$$

where c is an arbitrary constant defined by boundary condition $\lambda_{ii}(T) = 0$. Now, putting $\lambda_{ii}(T)k_i(T)(1-k_i(T)) = 0$, as $k_i(t) \neq 0$ for all t, we can find the constant c

$$c = -\int_0^T \frac{(k_2r_2+\epsilon)k_1r_1(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt. \tag{13}$$

Thus, we conclude the derivation of λ_{ii}

$$\lambda_{ii}(t) = \frac{-1}{k_i(1-k_i)}\int_t^T \frac{(k_jr_j+\epsilon)k_ir_1(i-k_i)}{(k_ir_i+k_jr_j+\epsilon)^2}dt. \tag{14}$$

By analogy, we find that

$$\lambda_{ij}(t) = \frac{1}{k_j(1-k_j)}\int_t^T \frac{k_ir_ik_jr_j(1-k_j)}{(k_ir_i+k_jr_j+\epsilon)^2}dt. \tag{15}$$

■

Lemma 1. *$\lambda_{11}(t) \leq 0$, $\lambda_{22}(t) \leq 0$, $\lambda_{12}(t) \geq 0$, $\lambda_{21}(t) \geq 0$ for any $t \in [0,T]$.*

Proof. We know that $k_i(t) \in (0,1)$ for all i and t; also $r_i(t) \geq 0$ for all t, thus functions under all integrals are non-negative and, hence, the value of integral itself is non-negative, and function $k_i(1-k_i)$ outside the integral is always positive, thus, the sign before integral defines the non-negativity or non-positivity of corresponding co-state variable.

■

Now for simplicity we define new functions:

$$L_{11}(t) = -\lambda_{11}k_1(1-k_1) \text{ i.e. } L_{11}(t) = \int_t^T \frac{(k_2r_2+\epsilon)k_1r_1(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$

$$L_{12}(t) = \lambda_{12}k_2(1-k_2) \quad \text{i.e. } L_{12}(t) = \int_t^T \frac{k_1r_1k_2r_2(1-k_2)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$

$$L_{21}(t) = \lambda_{21}k_1(1-k_1) \quad \text{i.e. } L_{21}(t) = \int_t^T \frac{k_1r_1k_2r_2(1-k_1)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$

$$L_{22}(t) = -\lambda_{22}k_2(1-k_2) \text{ i.e. } L_{22}(t) = \int_t^T \frac{(k_1r_1+\epsilon)k_2r_2(1-k_2)}{(k_1r_1+k_2r_2+\epsilon)^2}dt.$$

Lemma 2. *$L_{ij}(t) \geq 0$ for $\forall t \in [0,T]$ and $L_{ij}(t)$ is decreasing in time t and at the final moment of time $L_{ij}(T) = 0$ for all $i,j \in \{1,2\}$.*

Proof. From Lemma 1 we know that $\lambda_{ii} \leq 0$ and $\lambda_{ij} \geq 0$ for $i,j \in \{1,2\}$ and $i \neq j$. Now the sign of L_{ii} is opposite to λ_{ii} and sign of L_{ij} is the same with λ_{ij}. For decreasing it is enough to notice that function the function under integral is positive and for some fixed (optimal) control function the trajectory is also fixed (as the solution is in open loop form), from this we immediately have that with increase of time the value of integral decreases or remains the same.

■

The idea of maximum principle for finding optimal controls can be formulated in the following way:

$$H_1(c^*, r^*, k^*, \lambda^*) = \min_{c_1, r_1} H_1(c_1, r_1, c_2^*, r_2^*, k^*, \lambda^*). \tag{16}$$

$$H_2(c^*, r^*, k^*, \lambda^*) = \min_{c_2, r_2} H_2(c_2, r_2, c_1^*, r_1^*, k^*, \lambda^*). \tag{17}$$

It means that player's optimal controls minimize the corresponding Hamiltonian. In our case the Hamiltonians have the following forms

$$H_1 = (l + \lambda_{11} k_1 (1 - k_1)) c_1 - k_1 \left(\frac{1}{k_1 r_1 + k_2 r_2 + \epsilon} + \lambda_{11}(1 - k_1) \right) r_1 + \lambda_{12} k_2 (1 - k_2)(c_2 - r_2). \tag{18}$$

$$H_2 = (l + \lambda_{22} k_2 (1 - k_2)) c_2 - k_2 \left(\frac{1}{k_1 r_1 + k_2 r_2 + \epsilon} + \lambda_{22}(1 - k_2) \right) r_2 + \lambda_{21} k_1 (1 - k_1)(c_1 - r_1). \tag{19}$$

Now, we define new functions that we will be needed for solution

$$a_i(t) = \frac{k_i}{k_j r_j + \epsilon} \tag{20}$$

and

$$b_i(t) = \frac{k_i (k_j r_j + \epsilon)}{(k_i + k_j r_j + \epsilon)^2}. \tag{21}$$

Lemma 3. *The following inequality always holds $a(t) \geq b(t)$.*

Proof. Putting the function for $a(t)$ (20) and function for $b(t)$ (21) together, we can see that the inequality always holds for positive k_i,r_i $i \in \{1, 2\}$.

$$\frac{k_i}{k_j r_j + \epsilon} \geq \frac{k_i (k_j r_j + \epsilon)}{(k_i + k_j r_j + \epsilon)^2}, \tag{22}$$

which corresponds to inequality

$$\frac{1}{(k_j r_j + \epsilon)^2} \geq \frac{1}{(k_i + k_j r_j + \epsilon)^2}. \tag{23}$$

∎

Let $d_i(t) = -\frac{1}{a} + \sqrt{\frac{1}{a L_{ii}(t)}}$, then the following theorem holds

Theorem 3. *Optimal control for player i has the following form:*

$$c_i = \begin{cases} 0, & \text{if } L_{ii}(t) \le l, \\ 1, & \text{if } L_{ii}(t) > l. \end{cases} \tag{24}$$

$$r_i = \begin{cases} 0, & \text{if } L_{ii}(t) > a_i(t), \\ 1, & \text{if } L_{ii}(t) \le b_i(t), \\ d_i(t), & \text{otherwise }. \end{cases} \tag{25}$$

Proof. From (18) and (19) we have that c_i independently minimizes corresponding Hamiltonian from r_i, thus, $c_i = 0$, when $l + \lambda_{11} k_1(1 - k_1) \ge 0$, and $c_i = 1$, otherwise. From the other hand, $l + \lambda_{11} k_1(1 - k_1) \ge 0$ is equivalent to $l - L_{ii}(t) \ge 0$, from which we get the fist system (24).

In order to find r_i, which minimizes corresponding Hamiltonian, we need to find r_i, which maximizes the following function

$$\left(\frac{k_i}{k_i r_i + k_j r_j + \epsilon} + \lambda_{ii} k_i (1 - k_i) \right) r_i \tag{26}$$

Let's find the derivative by r_i of this function

$$\frac{k_i(r_j k_j + \epsilon)}{(k_i r_i + k_j r_j + \epsilon)^2} - L_{ii}(t) = 0 \tag{27}$$

In turn, its solution is equivalent to the solution of the following quadratic equation

$$k_i(r_j k_j + \epsilon) - L_{ii}(t)(k_i r_i + k_j r_j + \epsilon)^2 = 0, \tag{28}$$

It has two solutions of the form $-\frac{1}{a_i} \pm \sqrt{\frac{1}{L_{ii} a_i}}$, the negative one is always outside admissible set of r_i. Additionally, it can be easily checked that function (26) decreases until the point $-\frac{1}{a_i} - \sqrt{\frac{1}{L_{ii} a_i}}$ after that it increases until new point $d_i(t) = -\frac{1}{a_i} + \sqrt{\frac{1}{L_{ii} a_i}}$ and after that point it decreases.

From this we can conclude that if point $d_i(t) < 0$ then $r_i(t) = 0$ maximizes the function (26), if pint $d_i(t) > 1$ then $r_i(t) = 1$ maximizes function (26), otherwise $r_i(t) = d_i(t)$ maximizes that function. The first condition $d_i(t) < 0$ is equivalent to $L_{ii}(t) > a_i(t)$, the second $d_i(t) > 1$ is equivalent to $L_{ii}(t) \le b_i(t)$. This concludes the proof.

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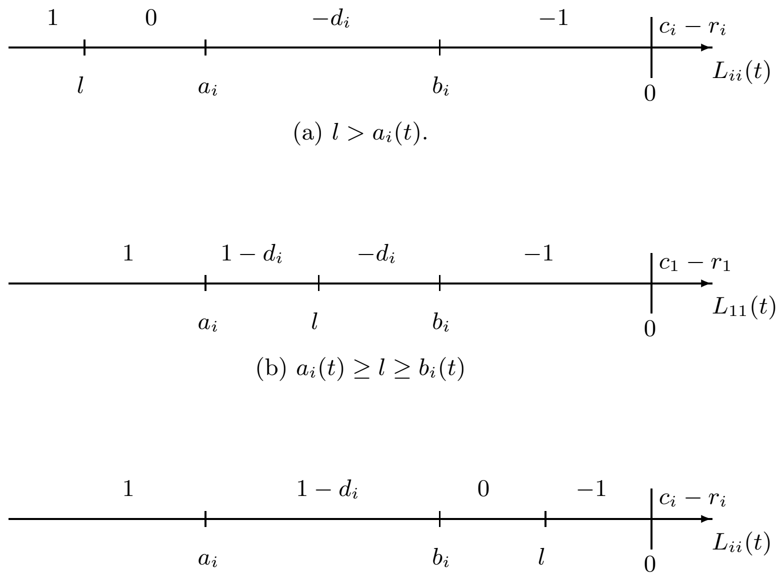
Note, that the optimal control is written in closed loop form, which depends on the remaining path. It means that the current decision is based on the decisions we will make from the current point path until the final moment (T). It is a natural way of dynamic programing to find the optimal solution, when we know values at the end of the optimal trajectory and find them in backward direction until the initial position ($t = 0$).

5. Optimal trajectory

Based on Theorem 3 we can construct the optimal trajectory, knowing optimal control on it. See three possible cases in Fig. 1. Now, we are going to give an



(a) $l > a_i(t)$.

(b) $a_i(t) \geq l \geq b_i(t)$

(c) $l < b_i$

Fig. 1: Three possible cases.

iterative scheme to construct the optimal trajectory based on the knowledge of optimal controls. First of all, assume that in the final point T we are in the position $k_i(T)$ (we actually have $k_i(0)$ as the initial position, but we will use it in the end of computation to find the $k_i(T)$ point, for now assume that is given instead of $x_i(0)$). From Lemma 2 we know that at final point $L_{ii}(T) = 0$. In Figure 1 it corresponds to selfish behavior $c_i - r_i = -1$, i.e., when we request a lot but contribute nothing. Based on that we construct a backward path, $k_i(t)$ based on the dynamic system (2) and the final point $k_i(T)$. Function $b_i(t)$ also depends on the value k_i, hence it has also some dynamics in time. When the function $L_{ii}(T)$, which depends on the remaining part starts to be greater than $b_i(t)$ or l we will need to switch to another corresponding optimal control, and hence to the new function of dynamics for k_i. Notice, that by Lemma 2 the function $L_{ii}(T)$ can intersect the value l only once (though, it can remain on it for a long time), while $L_{ii}(T)$ can many times intersect $a_i(t)$ and $b_i(t)$ thus changing the optimal control value on the new part of trajectory.

6. Conclusion

In this paper we have formulated a common resource sharing problem using game theory in a new form, where we introduced dynamically changing ranks, based on which every player obtains resources. The ranks are more closely relevant to the form of interaction history of a player. Although, it is natural to assume that the ranking itself restricts players from selfish behavior, the study of this is rare and not easily always solvable. In this work we formulated the reputation based resource sharing problem in general form, for a wide set of ranks. We analyzed a more specified form of the reputation recomputation, which does follow the behavior and idea of general reputation form. Finally, we give solution in a decision form based on three cases. We determine the optimal controls and the optimal trajectory.

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