Equilibrium in a P2P-system

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Abstract

We consider a P2P-system with two kinds of users: "seeders" and "leechers". Seeders provide data to others in a random manner with a known distribution. Leechers maximize amount of downloaded data by trying to guess seeders' behavior. An equilibrium condition is derived for this system.

1 Introduction

P2P-networks are wide-spread in the Internet. A user of a P2P network provides data and consumes it simultaneously. However, it is possible that a user only gets data but gives nothing in exchange for it. We call such users "leechers".

Obviously, the network of this type still exists if there are users which provide more data than they consume. We will call such parties "seeders". Considering the relationship between leechers and seeders unfair, we want to encourage all participants in the system to provide some data. We offer another way of interaction where a user must offer data with the aim to obtain data from other users.

2 Problem statement

The network brings together several parties which are either seeders or leechers. Players (leechers) are seeking data provided by the seeders. The amount of data available to other network users we call a contribution. For simplicity assume that a valid contribution belongs to $[0, 1]$. The game starts when all parties make contributions to the network; the contribution of a seeder is a random value from the interval $[0, 1]$ with some known distribution function. The player whose contribution was closest to the value, but not less, gets this value minus the amount proportional to its contribution. In this case, we say that the player predicts the seeder's contribution. Assume also that if the player did not predict the contribution of any seeder, she loses the value proportional to its contribution. In real networks such losses can be considered the cost of the presence in the system or the value of traffic.

Each player tries to maximize her own expected payoff.

Related work describes a competitive prediction number game proposed by Sakaguchi and Szajowsky [1]. The problem was solved under asymmetrical conditions by Belkovskii and Garnaev [2]. The case of an arbitrary number of players has been considered by Sakaguchi and Mazalov [3]. Particulars of a P2P-system in which players choose their type of behavior were considered by Feldman, Papadimitriou, Chuang and Stoica [4].

3 A case of two leechers and one seeder

For simplicity we assume that a seeder's contribution is uniformly distributed in $[0, 1]$. Let a be the seeder's contribution, x_1 be the contribution of player I, and x_2 be the contribution of player II. Then if $a \leq x_1 < x_2$ or $x_2 < a \leq x_1$ then player I payoff is $a - cx_1$; otherwise player I loses cx_1. Similarly if $a \leq x_2 < x_1$ or $x_1 < a \leq x_2$ player II payoff is $a - cx_2$; otherwise player II loses cx_2.

Denote $H_1(x_1, x_2)$ player I payoff if players make contributions x_1 and

x_2. Then

$$H_1(x_1,x_2)=\begin{cases}\displaystyle\int_0^{x_1} a da - cx_1 = \frac{x_1^2}{2} - cx_1, & \text{for } x_1 < x_2\\ \displaystyle\int_{x_2}^{x_1} a da - cx_1 = \frac{x_1^2 - x_2^2}{2} - cx_1, & \text{for } x_1 > x_2\end{cases}$$

Let player II use the mixed strategy with probability density function $g(x_2)$ in $[b,1]$ in such a way that $H_1(x_1, g(x_2)) = v$, where v is a game value. In these conditions, player I payoff is:

for $x_1 < b$:

$$H_1(x_1, g(x_2)) = \frac{x_1^2}{2} - cx_1;$$

for $x_1 \in [b,1]$:

$$\begin{aligned} H_1(x_1, g(x_2)) &= \int_b^{x_1} \Big(\frac{x_1^2 - x_2^2}{2} - cx_1\Big) g(x_2) dx_2 \\ &+ \int_{x_1}^{1} \Big(\frac{x_1^2}{2} - cx_1\Big) g(x_2) dx_2 \end{aligned} \tag{1}$$

Simplifying (1) we obtain

$$H_1(x_1, g(x_2)) = \frac{x_1^2}{2} - cx_1 - \frac{1}{2}\int_b^{x_1} x_2^2 g(x_2) dx_2$$

Differentiating expression $H_1(x_1, g(x_2)) = v$ in x_1 we get

$$x_1 - c - \frac{x_1^2}{2} g(x_1) = 0.$$

Finally we find the probability density function

$$g(x_1) = \frac{2}{x_1} - \frac{2c}{x_1^2}.$$

Distribution function $G(x) = \int g(x)dx$ has the form:

$$G(x) = 2\ln x + \frac{2c}{x} + const,$$

and from property $G(1) = 1$ it follows that an optimal strategy for player II is:

$$G(x) = 2\ln x + \frac{2c}{x}(1-x) + 1 \tag{2}$$

in $[b, 1]$ and the value of b is found in $G(b) = 0$ or

$$2\ln b + \frac{2c}{b}(1-b) + 1 = 0. \tag{3}$$

Player I payoff is

$$H_1(x_1, g(x_2)) = \frac{b^2}{2} - cb.$$

The strategy (2) is optimal if

$$H_1(x_1, g(x_2)) = x_1^2/2 - cx_1 < b^2/2 - cb \quad \forall x_1 < b. \tag{4}$$

Function $x^2/2 - cx$ has roots $x = 0$ and $x = 2c$ so if $b > 2c$ the inequality (4) is satisfied. Using $b = 2c$ in (3) we get condition for c:

$$\ln(2c) + 1 - c = 0 \Longrightarrow c^* \approx 0.23196. \tag{5}$$

Since the game is symmetric we have proved

Theorem 3.1 *Let $c \in [0, c^*]$ where c^* be a root of (5). Then the optimal strategies of players coincide and have the form*

$$G(x) = \begin{cases} 0, & \text{for } x < b \\ 2\ln x + \frac{2c}{x}(1-x) + 1, & \text{for } x \in [b, 1] \end{cases},$$

where b satisfies

$$2\ln b + \frac{2c}{b}(1-b) + 1 = 0$$

The player's payoff is $b^2/2 - cb$.

4 A case of two leechers and n seeders

Let us define a game with two players and an arbitrary number of seeders as follows. As before players (leechers) make contributions in a network (spending c for a unit of contribution) and seeders make their contribution in a random manner. After that players get data as a payoff from predicted seeders.

Assuming that seeders' contributions are independent identically distributed (iid) values uniformly distributed in $[0,1]$ we obtain player I payoff in a game with n seeders:

$$H_1(x_1,x_2) = \begin{cases} n\int_0^{x_1} ada - cx_1 = n\frac{x_1^2}{2} - cx_1, & \text{for } x_1 < x_2 \\ n\int_{x_2}^{x_1} ada - cx_1 = n\frac{x_1^2 - x_2^2}{2} - cx_1, & \text{for } x_1 > x_2 \end{cases}$$

Further we will find a solution similarly with the case of one seeder.

Let player II use the mixed strategy with probability density function $g(x_2)$ in $[b,1]$ in such a way that $H_1(x_1, g(x_2)) = v$, where v is a game value. In these conditions player I payoff is:

for $x_1 < b$:

$$H_1(x_1, g(x_2)) = n\frac{x_1^2}{2} - cx_1;$$

for $x_1 \in [b,1]$:

$$\begin{aligned} H_1(x_1, g(x_2)) &= \int_b^{x_1} \left(n\frac{x_1^2 - x_2^2}{2} - cx_1\right) g(x_2)dx_2 \\ &+ \int_{x_1}^{1} \left(n\frac{x_1^2}{2} - cx_1\right) g(x_2)dx_2 \end{aligned} \tag{6}$$

Simplifying (6) we obtain

$$H_1(x_1, g(x_2)) = n\frac{x_1^2}{2} - cx_1 - n\frac{1}{2}\int_b^{x_1} x_2^2 g(x_2)dx_2$$

Differentiating expression $H_1(x_1, g(x_2)) = v$ in x_1 we get

$$nx_1 - c - n\frac{x_1}{2}g(x_1) = 0.$$

Finally we find the probability density function

$$g(x_1) = \frac{2}{x_1} - \frac{2c}{nx_1^2}.$$

Distribution function $G(x) = \int g(x)dx$ has the form:

$$G(x) = 2\ln x + \frac{2c}{nx} + const,$$

and from property $G(1) = 1$ it follows that an optimal strategy for player II is:

$$G(x) = 2\ln x + \frac{2c(1-x)}{nx} + 1$$

in $[b, 1]$ wherein b is found in $G(b) = 0$ or

$$2\ln b + \frac{2c(1-b)}{nb} + 1 = 0. \tag{7}$$

Player I payoff is

$$H_1(x_1, g(x_2)) = n\frac{b^2}{2} - cb.$$

For optimality it is necessary that $nx_1^2/2 - cx_1 < nb^2/2 - cb \quad \forall x_1 < b$ or equivalent $b > 2c/n$. Using $b = 2c/n$ in (7) we get the condition for c:

$$\ln 2c/n + 1 - \frac{c}{n} = 0. \tag{8}$$

Since the game is symmetric we have proved

Theorem 4.1 *Let $c \in [0, c^*]$ where c^* is a root of (8). Then the optimal strategies of players coincide and have the form*

$$G(x) = \begin{cases} 0, & for\ x < b \\ 2\ln x + \dfrac{2c(1-x)}{nx} + 1, & for\ x \in [b, 1] \end{cases},$$

where b satisfies

$$2\ln b + \frac{2c(1-b)}{nb} + 1 = 0.$$

The player's payoff is $nb^2/2 - cb$.

5 A case of m leechers and n seeders

Similarly to a case of n seeders, we change the definition of the game for an arbitrary number of players. There are m players in a game. As before, players (leechers) make contributions in a network spending c for unity of contribution and seeders make their contributions in a random manner. After that players get data as a payoff from predicted seeders.

To solve the problem we use the technique offered by Mazalov and Sakaguchi [3].

Let each player use the mixed strategy with probability density function $g(x)$ in $[b,1]$. Let $G_n(x) = I(x \geq b)\int_b^x g(t)dt$. Denote $\overline{G}(x) = 1 - G(x)$. Then the expected payoff to player I if he contributes x and the other players employ their strategies $g(x)$ is

$$H_1(x, g, \ldots, g) = n\frac{x^2}{2} - cx, \quad \text{for } x < b.$$

for $x \in [b, 1]$:

$$\begin{aligned} H_1(x, g, \ldots, g) &= -cx + n\frac{x^2}{2}(\overline{G}(x))^{m-1} \\ &+n\sum_{k=1}^{m-1}\binom{m-1}{k}k(\overline{G}(x))^{m-1-k} \\ &\times \int_b^x g(t)(G(t))^{k-1}\frac{(x^2-t^2)}{2}dt, \end{aligned} \tag{9}$$

since k players may contribute less than x and the other $m-1-k$ contribute greater than x.

Since we have, by integration by parts,

$$\int_b^x g(t)(G(t))^{k-1}\frac{(x^2-t^2)}{2}dt = \frac{1}{k}\int_b^x t(G(t))^k dt,$$

we can rewrite (9) as

$$\begin{aligned} H_1(x, g, \ldots, g) &= -cx + n\frac{x^2}{2}(\overline{G}(x))^{m-1} \\ &+n\sum_{k=1}^{m-1}\binom{m-1}{k}(\overline{G}(x))^{m-1-k}\int_b^x t(G(t))^k dt, \end{aligned} \tag{10}$$

Since players use optimal strategies for a game value $v = H_1(x, g, \ldots, g)$ for $x \in [b, 1]$ and differentiating (10) by x we obtain

$$\begin{aligned}\tfrac{\partial H_1(x,g,\ldots,g)}{\partial x} &= -c + n\Big[x(\overline{G}(x))^{m-1} - (m-1)\frac{x^2}{2}(\overline{G}(x))^{m-2}g(x)\\ &+ \sum_{k=1}^{m-1}\binom{m-1}{k}\Big[(m-1-k)(-g(x))(\overline{G}(x))^{m-2-k}\\ &\times \int_b^x t(G(t))^k dt\\ &+ x(\overline{G}(x))^{m-1-k}(G(t))^k\Big]\Big] = 0.\end{aligned}$$

After dividing both sides by $(\overline{G}(x))^{m-1}$ and simplifying we finally get

$$\begin{aligned} nx\Big[1 + \sum_{k=1}^{m-1}\binom{m-1}{k}\left(\frac{G(x)}{\overline{G}(x)}\right)^k\Big] - \frac{c}{(\overline{G}(x))^{m-1}} = \\ n\frac{g(x)}{\overline{G}(x)}\Big[(m-1)\frac{x^2}{2} \\ + \sum_{k=1}^{m-1}\binom{m-1}{k}(m-1-k)\int_b^x t\left(\frac{G(t)}{\overline{G}(x)}\right)^k dt\Big]. \end{aligned} \tag{11}$$

It is easy to see that

$$1 + \sum_{k=1}^{m-1}\binom{m-1}{k}\left(\frac{G(x)}{\overline{G}(x)}\right)^k = \frac{1}{(\overline{G}(x))^{m-1}},$$

and

$$\begin{aligned}(m-1)\frac{x^2}{2} + \sum_{k=1}^{m-1}\binom{m-1}{k}(m-1-k)\int_b^x t\left(\frac{G(t)}{\overline{G}(x)}\right)^k dt = \\ \frac{b^2}{2} + \int_b^x t\left(1 + \frac{G(t)}{\overline{G}(x)}\right)^{m-2} dt.\end{aligned}$$

Taking it into account we can present (11) as

$$\frac{nx - c}{(\overline{G}(x))^{m-1}} = n(m-1)\frac{g(x)}{\overline{G}(x)}\Big[\frac{b^2}{2} + \int_b^x t\left(1 + \frac{G(t)}{\overline{G}(x)}\right)^{m-2} dt\Big],$$

or more suitably

$$\frac{nx-c}{g(x)n(m-1)} = \frac{b^2}{2}(\overline{G}(x))^{m-2} + \int_b^x t(\overline{G}(x)+G(t))^{m-2}dt. \tag{12}$$

Expression (12) matters for $b < x < 1$ and $\forall m \geq 2$.

Consider the sequence of functions

$$s_k(x) = \frac{2}{x^2}\left[\frac{b^2}{2}(\overline{G}(x))^k + \int_b^x t(\overline{G}(x)+G(t))^k dt\right], \quad \forall k = 1,2,\ldots,m-2, \tag{13}$$

which clearly satisfies

$$1 \equiv s_0(x) \geq s_1(x) \geq s_2(x) \geq \ldots \geq s_{m-2}(x) \geq 0, \quad \forall x \in [b,1].$$

Multiplying $x^2/2$ on both sides of (13) and differentiating we get a recurrent differential equation

$$xs_k(x) + \frac{x^2}{2}s_k'(x) = x - \frac{x^2}{2}kg(x)s_{k-1}(x),$$

or equivalently

$$\frac{2}{x}(1-s_k(x)) - s_k'(x) = kg(x)s_{k-1}(x), \quad \forall k = 1,2,\ldots,m-2 \tag{14}$$

with boundary conditions

$$s_k(b) = 1, \quad \forall k = 1,2,\ldots,m-2.$$

By (12)-(13) we see

$$s_{m-2}(x) = \frac{2(nx-c)}{nx^2(m-1)g(x)}. \tag{15}$$

From above we obtain $g(x)$

$$g(x) = \frac{2(nx-c)}{nx^2(m-1)s_{m-2}(x)} \geq \frac{2(nx-c)}{nx^2(m-1)}.$$

We get b from

$$\int_b^1 g(x)dx = 1. \tag{16}$$

For optimality it is necessary that $nx_1^2/2 - cx_1 < nb^2/2 - cb \quad \forall x_1 < b$ or equivalent $b > 2c/n$. Using $b = 2c/n$ in (16) we get the condition for c:

$$\int_{2c/n}^{1} g(x)dx = 1. \tag{17}$$

Hence we have proved

Theorem 5.1 *Let $c \in [0, c^*]$ where c^* is a root of (17). Also let $\{s_1, \ldots, s_{m-2}\}$ be a solution of the system of differential equations (14) and*

$$g(x) = \frac{2(nx - c)}{nx^2(m-1)s_{m-2}(x)}.$$

Let us choose b from condition $\int_b^1 g(x)dx = 1$. Then $g(x)$ is an optimal strategy.

The player's payoff is $nb^2/2 - cb$.

The system (14) together with (15) can be used to find the solution of the problem with a following algorithm. We fix some value b and consider the system of differential equations (14) in the interval $[b, 1]$. When we found the solution with boundary conditions $s_k(b) = 1, \quad \forall k = 1, 2, \ldots, m-2$, we calculate the density function $g(x) = 2(nx - c)/(nx^2(m-1)s_{m-2}(x)), \quad x \in [b, 1]$. Then we can determine b from the condition $\int_b^1 g(x)dx = 1$.

6 Conclusion

We have considered a P2P-system in which players have to predict the behavior of seeders who make contributions in a random manner. Thus, we have a competitive prediction number game with payment for observation and the predicted value as a payoff.

We have solved the problem for an arbitrary number of players and seeders in case a seeder uses uniform distribution in $[0, 1]$. The theorem 5.1 presents a solution in mixed strategies in $[b, 1]$ with player's payoff of $nb^2/2 - cb$ where c is a cost of observation and n is a number of seeders.

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