Software Verification

Introduction to Abstract Interpretation

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Outline

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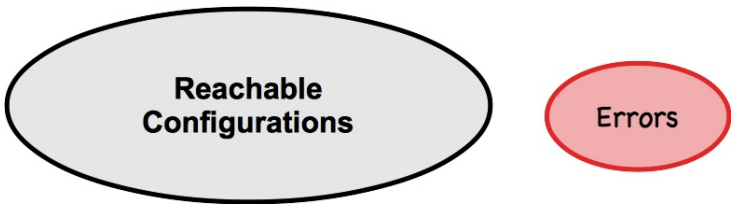
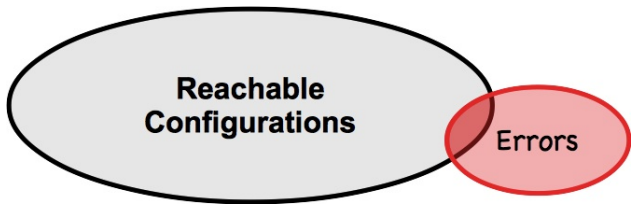
Verification

- We want to answer whether some program behaves correctly. We define "correctness" soon.
- For now, assume that means some erroneous configurations are not reachable
- We say the program is **safe**



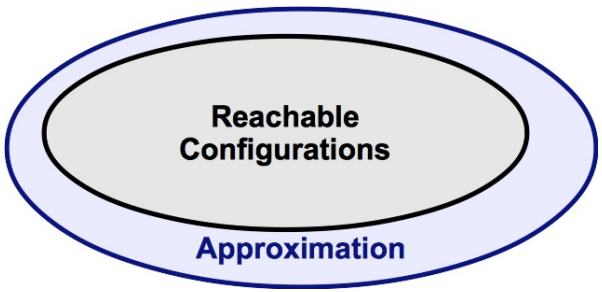
Safe Program



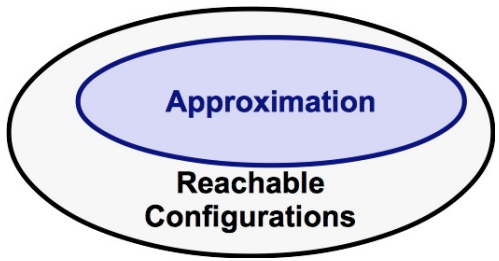
Unsafe Program

Verification problem and approximations

- The idea is then to come up with efficient approximations to give correct answers in as many cases as possible.



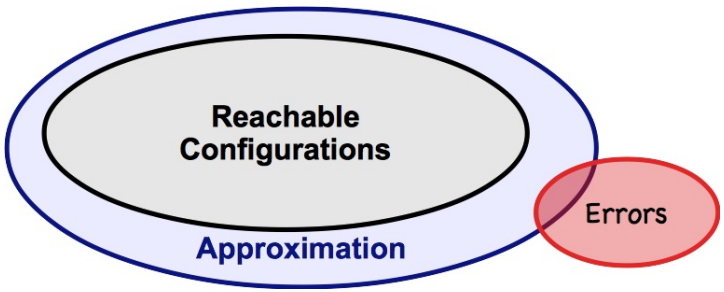
Over-approximation



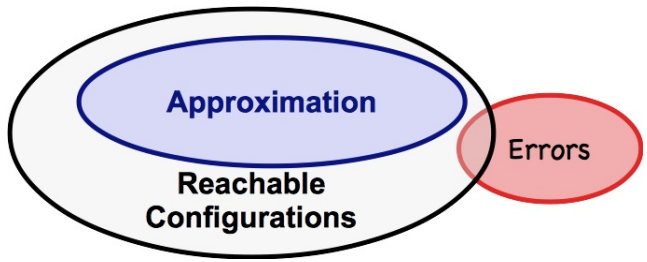
Under-approximation

Program verification and the price of approximations

- A sound analysis cannot give **false negatives**
- A complete analysis cannot give **false positives**



False Positive



False Negative

Approaches

- We discussed:
  - Explicit/symbolic model checking
  - Symbolic Execution relying on SMT solvers
  - Deductive frameworks with Hoare triples
- Today: a systematic and automatic framework for generating over-approximations

Outline

A simple language

Assume a simple language over a finite number of integer variables ranging over $\mathbb{Z}$.

- `exp ::= n | x | exp + exp | exp - exp | *`
- `cond ::= true | false | exp >= exp | cond && cond`
- `cmd ::= x:= exp | if cond goto label | assert cond | end`

```
L1. x:= *
L2. y:= x
L3. x:= x + y
L4. if x >= y goto L6
L5. end
L6. assert y >= 0
L7. end
```

```
L1. x:= 0
L2. y:= 0
L3. x:= x + 1
L4. y:= y + 2
L5. if x < 100 goto L3
L6. assert y = 200
L7. end
```

We want to generate "invariants", e.g., characterize “facts” that hold at each line

Outline

Abstract Interpretation

- In the concrete semantics, one can associate to each label a set of possible mappings from the variables to their values.
- Concrete semantics is precise: it only captures possible states. It is however too inefficient to be feasible in practice (loops, arbitrary inputs, arrays, recursion, concurrent interleavings, etc)
- Instead, efficiently and soundly over-approximate while tracking “facts” of certain “forms”

Abstract Interpretation

Idea:

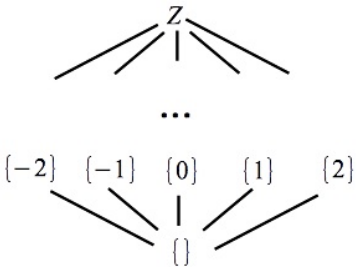
- Assume an "abstract domain" fixing the "form" of the tracked facts (e.g., $a \leq x \leq b$ or $a \leq x - y \leq b$ or ...)
- Define an abstract semantics that over-approximates the concrete semantics (e.g., abstract additions, substractions, comparisons, etc)
- Iterate computation with abstract semantics untill fixpoint. Deduce facts about the original semantics.

A simple abstract domain: the sign example

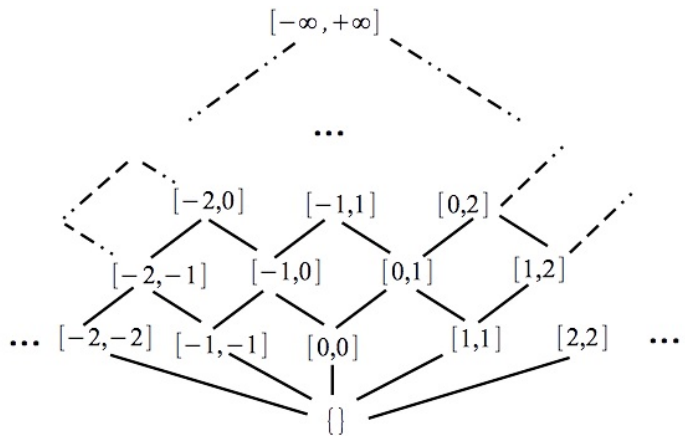
- For an integer variable, the set of concrete values at a location is in $\mathcal{P}(\mathbb{Z})$. Concrete sets of variables' values can be ordered with the subset relation $\sqsubseteq_c$ on $\mathcal{P}(\mathbb{Z})$. We write $S_1 \sqsubseteq_c S_2$ to mean that S_1 is more precise than S_2.
- If you are only interested in the signs of variables' values, you can associate, at each label and to each variable, a subset of $\{-, 0, +\}$.
- We approximate concrete values with an element in $\mathcal{P}(\{-, 0, +\})$. For instance, $\{+\}$ reflects the variable is larger than zero. For A_1, A_2 in $\mathcal{P}(\{-, 0, +\})$, we write $A_1 \sqsubseteq_a A_2$ to mean that A_1 is more precise than A_2.

Lattices

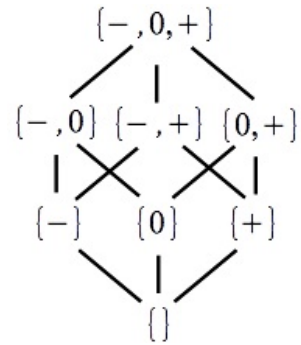
- A lattice is a poset $(Q, \sqsubseteq)$ where each pair p, q in Q has:
  - a greatest lower bound (aka. meet) $p \sqcap q$ wrt. $\sqsubseteq$, and
  - a least upper bound (aka. join) $p \sqcup q$ wrt. $\sqsubseteq$
- If S is a set, then $(\mathcal{P}(S), \subseteq, \bot, \top, \cup, \cap)$ is a complete lattice: i.e, a lattice where each subset has a least upper bound and a greatest lower bound.
- $(\mathcal{P}(\mathbb{Z}), \sqsubseteq_c)$ (resp. $(\mathcal{P}(\{-, 0, +\}), \sqsubseteq_a)$) is a complete lattices where $\top_c = \mathbb{Z}$ and $\bot_c = \{\}$ (resp. $\top_a = \{-, 0, +\}$ and $\bot_a = \{\}$)



subsets



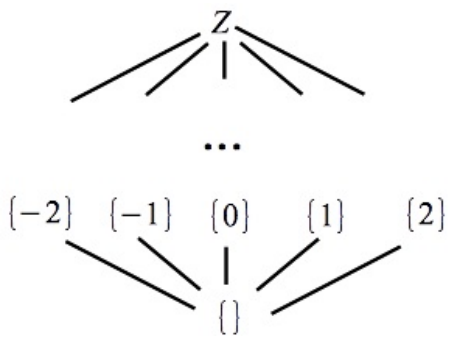
intervals



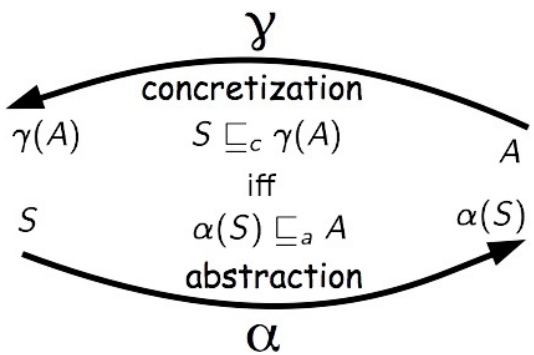
signs

The sign example: abstraction and concretization

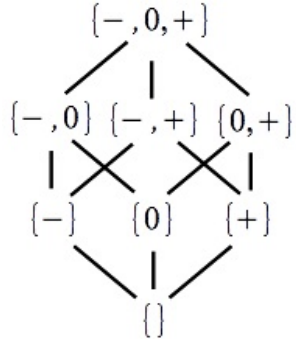
- An abstraction function: $\alpha(\{1, 10, 100, 103, 2021\}) = \{+\}$, $\alpha(\{-10, -3, 10\}) = \{-, +\}$, $\alpha(\{-3, 0\}) = \{-, 0\}$
- A concretization function: $\gamma(\{0\}) = \{0\}$, $\gamma(\{-, 0\}) = \{v | v \leq 0\}$ and $\gamma(\{+\}) = \{v | v > 0\}$



Concrete lattice
$(\mathcal{P}(\mathbb{Z}), \sqsubseteq_c)$



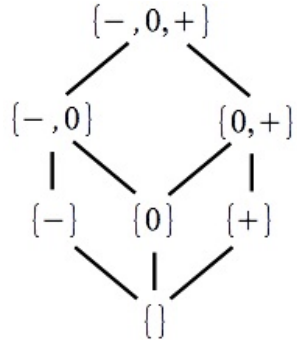
A Galois connection
(α, γ)



Abstract lattice
$(\mathcal{P}(\{-, 0, +\}), \sqsubseteq_a)$

Fixpoint computation: simple example (1/6)

```
L1. x:= *
L2. y:= x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:
//y:
L1. x:= *
//x:
//y:
L2. y:= x
//x:
//y:
L3. if x > 0 goto L5
//x:
//y:
L4. end
//x:
//y:
L5. assert x >= 0
//x:
//y:
L6. end
```

→

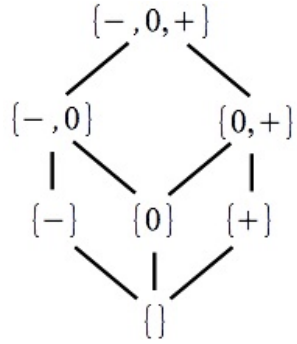
```
//x:⊤
//y:⊤
L1. x:= *
//x:
//y:
L2. y:= x
//x:
//y:
L3. if x > 0 goto L5
//x:
//y:
L4. end
//x:
//y:
L5. assert x >= 0
//x:
//y:
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x:= *
//x:⊥
//y:⊥
L2. y:= x
//x:⊥
//y:⊥
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

Fixpoint computation: simple example (2/6)

```
L1. x:= *
L2. y:= x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:⊤
//y:⊤
L1. x:= *
//x:⊥
//y:⊥
L2. y:= x
//x:⊥
//y:⊥
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

→

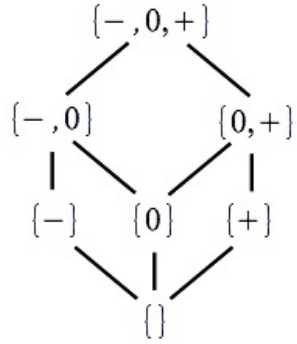
```
//x:⊤⊔⊥
//y:⊤⊔⊥
L1. x:= *
//x:⊥⊔⊤
//y:⊥⊔⊤
L2. y:= x
//x:⊥⊔⊥
//y:⊥⊔⊥
L3. if x > 0 goto L5
//x:⊥⊔(⊥ ⊓ {−,0})
//y:⊥⊔(⊥ ⊓ ⊤)
L4. end
//x:⊥⊔(⊥ ⊓ {+})
//y:⊥⊔(⊥ ⊓ ⊤)
L5. assert x >= 0
//x:⊥⊔(⊥ ⊓ {0,+})
//y:⊥⊔(⊥ ⊓ ⊤)
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x:= *
//x:⊤
//y:⊤
L2. y:= x
//x:⊥
//y:⊥
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

Fixpoint computation: simple example (3/6)

```
L1. x:= *
L2. y:= x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:⊤
//y:⊤
L1. x:= *
//x:⊤
//y:⊤
L2. y:= x
//x:⊥
//y:⊥
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

→

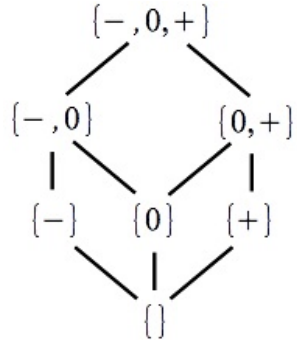
```
//x:⊤⊔⊥
//y:⊤⊔⊥
L1. x:= *
//x:⊤⊔⊤
//y:⊤⊔⊤
L2. y:= x
//x:⊥⊔⊤
//y:⊥⊔⊤
L3. if x > 0 goto L5
//x:⊥⊔(⊥ ⊓ {−,0})
//y:⊥⊔(⊥ ⊓ ⊤)
L4. end
//x:⊥⊔(⊥ ⊓ {+})
//y:⊥⊔(⊥ ⊓ ⊤)
L5. assert x >= 0
//x:⊥⊔(⊥ ⊓ {0,+})
//y:⊥⊔(⊥ ⊓ ⊤)
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x:= *
//x:⊤
//y:⊤
L2. y:= x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

Fixpoint computation: simple example (4/6)

```
L1. x := *
L2. y := x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:⊥
//y:⊥
L4. end
//x:⊥
//y:⊥
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

→

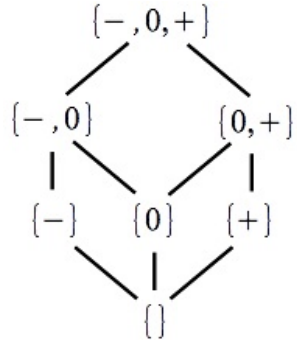
```
//x:⊤⊔⊥
//y:⊤⊔⊥
L1. x := *
//x:⊤⊔⊤
//y:⊤⊔⊤
L2. y := x
//x:⊤⊔⊤
//y:⊤⊔⊤
L3. if x > 0 goto L5
//x:⊥⊔(⊤ ⊓ {−,0})
//y:⊥⊔(⊤ ⊓ ⊤)
L4. end
//x:⊥⊔(⊤ ⊓ {+})
//y:⊥⊔(⊤ ⊓ ⊤)
L5. assert x >= 0
//x:⊥⊔(⊥ ⊓ {0,+})
//y:⊥⊔(⊥ ⊓ ⊤)
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:{−,0}
//y:⊤
L4. end
//x:{+}
//y:⊤
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

Fixpoint computation: simple (5/6)

```
L1. x := *
L2. y := x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:{−,0}
//y:⊤
L4. end
//x:{+}
//y:⊤
L5. assert x >= 0
//x:⊥
//y:⊥
L6. end
```

→

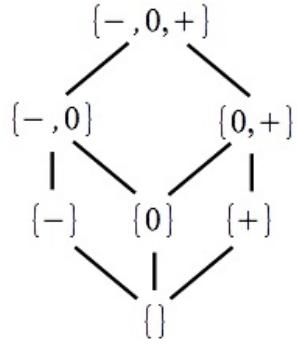
```
//x:⊤⊔⊥
//y:⊤⊔⊥
L1. x := *
//x:⊤⊔⊤
//y:⊤⊔⊤
L2. y := x
//x:⊤⊔⊤
//y:⊤⊔⊤
L3. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
//y:⊤⊔(⊤ ⊓ ⊤)
L4. end
//x:{+}⊔(⊤ ⊓ {+})
//y:⊤⊔(⊤ ⊓ ⊤)
L5. assert x >= 0
//x:⊥⊔({+} ⊓ {0,+})
//y:⊥⊔(⊤ ⊓ ⊤)
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:{−,0}
//y:⊤
L4. end
//x:{+}
//y:⊤
L5. assert x >= 0
//x:{+}
//y:⊤
L6. end
```

Fixpoint computation: simple (6/6)

```
L1. x := *
L2. y := x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```




```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:{−,0}
//y:⊤
L4. end
//x:{+}
//y:⊤
L5. assert x >= 0
//x:{+}
//y:⊤
L6. end
```

→

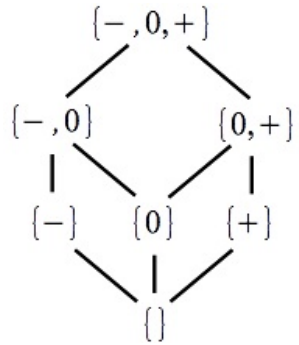
```
//x:⊤⊔⊥
//y:⊤⊔⊥
L1. x := *
//x:⊤⊔⊤
//y:⊤⊔⊤
L2. y := x
//x:⊤⊔⊤
//y:⊤⊔⊤
L3. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
//y:⊤⊔(⊤ ⊓ ⊤)
L4. end
//x:{+}⊔(⊤ ⊓ {+})
//y:⊤⊔(⊤ ⊓ ⊤)
L5. assert x >= 0
//x:{+}⊔({+} ⊓ {0,+})
//y:⊤⊔(⊤ ⊓ ⊤)
L6. end
```

→

```
//x:⊤
//y:⊤
L1. x := *
//x:⊤
//y:⊤
L2. y := x
//x:⊤
//y:⊤
L3. if x > 0 goto L5
//x:{−,0}
//y:⊤
L4. end
//x:{+}
//y:⊤
L5. assert x >= 0
//x:{+}
//y:⊤
L6. end
```

Fixpoint computation: another without $\neq 0$ (1/5)

```
L1. x := *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x := *
//x:⊥
L2. if x > 0 goto L5
//x:⊥
L3. if x < 0 goto L5
//x:⊥
L4. end
//x:⊥
L5. assert x != 0
//x:⊥
L6. end
```

→

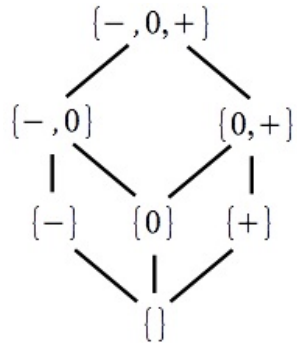
```
//x:⊤,
L1. x := *
//x:⊥⊔⊤
L2. if x > 0 goto L5
//x:⊥⊔(⊥ ⊓ {−,0})
L3. if x < 0 goto L5
//x:⊥⊔(⊥ ⊓ {0,+})
L4. end
//x:⊥⊔(⊥ ⊓ {+}) ⊔ (⊥ ⊓ {−})
L5. assert x != 0
//x:⊥⊔(⊥ ⊓ ⊤)
L6. end
```

→

```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:⊥
L3. if x < 0 goto L5
//x:⊥
L4. end
//x:⊥
L5. assert x != 0
//x:⊥
L6. end
```

Fixpoint computation: another without $\neq 0$ (2/5)

```
L1. x := *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:⊥
L3. if x < 0 goto L5
//x:⊥
L4. end
//x:⊥
L5. assert x != 0
//x:⊥
L6. end
```

→

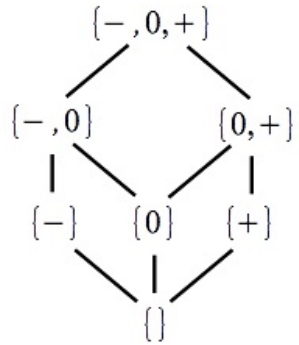
```
//x:⊤,
L1. x := *
//x:⊤⊔⊤
L2. if x > 0 goto L5
//x:⊥⊔(⊤ ⊓ {−,0})
L3. if x < 0 goto L5
//x:⊥⊔(⊥ ⊓ {0,+})
L4. end
//x:⊥⊔(⊤ ⊓ {+}) ⊔ (⊥ ⊓ {−})
L5. assert x != 0
//x:⊥⊔(⊥ ⊓ ⊤)
L6. end
```

→

```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:⊥
L4. end
//x:{+}
L5. assert x != 0
//x:⊥
L6. end
```

Fixpoint computation: another without $\neq 0$ (3/5)

```
L1. x:= *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x:= *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:⊥
L4. end
//x:{+}
L5. assert x != 0
//x:⊥
L6. end
```

→

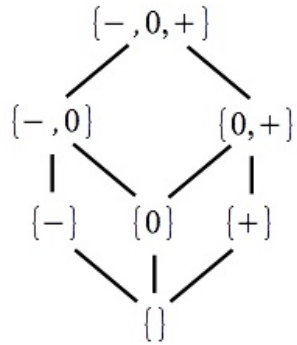
```
//x:⊤,
L1. x:= *
//x:⊤⊔⊤
L2. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
L3. if x < 0 goto L5
//x:⊥⊔({−,0} ⊓ {0,+})
L4. end
//x:{+}⊔(⊤ ⊓ {+}) ⊔ ({−,0} ⊓ {−})
L5. assert x != 0
//x:{+}⊔({+} ⊓ ⊤)
L6. end
```

→

```
//x:⊤,
L1. x:= *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:⊤
L5. assert x != 0
//x:{+}
L6. end
```

Fixpoint computation: another without $\neq 0$ (4/5)

```
L1. x := *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:{+}
L5. assert x != 0
//x:{+}
L6. end
```

→

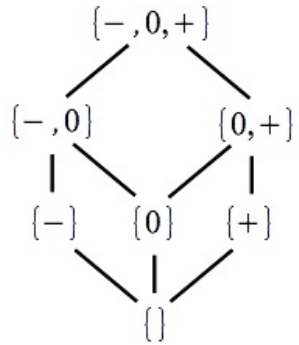
```
//x:⊤,
L1. x := *
//x:⊤⊔⊤
L2. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
L3. if x < 0 goto L5
//x:{0}⊔({−,0} ⊓ {0,+})
L4. end
//x:{+}⊔(⊤ ⊓ {+}) ⊔ ({−,0} ⊓ {−})
L5. assert x != 0
//x:{+}⊔({+} ⊓ ⊤)
L6. end
```

→

```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:⊤
L5. assert x != 0
//x:{+}
L6. end
```

Fixpoint computation: another without $\neq 0$ (5/5)

```
L1. x := *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:⊤
L5. assert x != 0
//x:{+}
L6. end
```

$\longrightarrow$

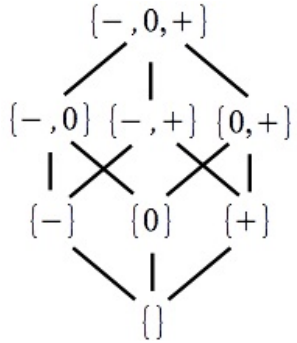
```
//x:⊤,
L1. x := *
//x:⊤⊔⊤
L2. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
L3. if x < 0 goto L5
//x:{0}⊔({−,0} ⊓ {0,+})
L4. end
//x:⊤⊔(⊤ ⊓ {+}) ⊔ ({−,0} ⊓ {−})
L5. assert x != 0
//x:{+}⊔(⊤ ⊓ ⊤)
L6. end
```

$\longrightarrow$

```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:⊤
L5. assert x != 0
//x:⊤
L6. end
```

Fixpoint computation: same with $\neq 0$

```
L1. x := *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```




```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:{−,+}
L5. assert x != 0
//x:{−,+}
L6. end
```

→

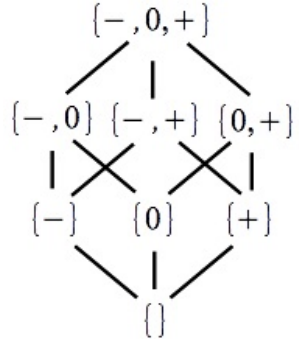
```
//x:⊤,
L1. x := *
//x:⊤⊔⊤
L2. if x > 0 goto L5
//x:{−,0}⊔(⊤ ⊓ {−,0})
L3. if x < 0 goto L5
//x:{0}⊔({−,0} ⊓ {0,+})
L4. end
//x:{−,+}⊔(⊤ ⊓ {+}) ⊔ ({−,0} ⊓ {−})
L5. assert x != 0
//x:{−,+}⊔({−,+} ⊓ {−,+})
L6. end
```

→

```
//x:⊤,
L1. x := *
//x:⊤
L2. if x > 0 goto L5
//x:{−,0}
L3. if x < 0 goto L5
//x:{0}
L4. end
//x:{−,+}
L5. assert x != 0
//x:{−,+}
L6. end
```

Fixpoint computation: signs vs. intervals

```
L1. x:= 0
L2. x:= x + 1
L3. x:= x + 2
L4. assert x >= 2
L5. end
```

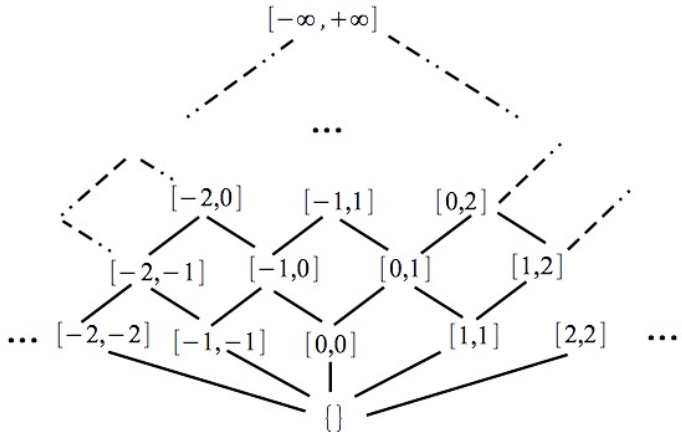



```
//x:⊤
L1. x:= 0
//x:{0}
L2. x:= x + 1
//x:{+}
L3. x:= x + 2
//x:{+}
L4. assert x >= 2
//x:{+}
L5. end
```

```
//x:⊤⊔⊥
L1. x:= 0
//x:{0}⊔{0}
L2. x:= x + 1
//x:{+}⊔{+}
L3. x:= x + 2
//x:{+}⊔{+}
L4. assert x >= 2
//x:{+}⊔{+}
L5. end
```

```
//x:⊤
L1. x:= 0
//x:{0}
L2. x:= x + 1
//x:{+}
L3. x:= x + 2
//x:{+}
L4. assert x >= 2
//x:{+}
L5. end
```

Fixpoint computation: signs vs. intervals



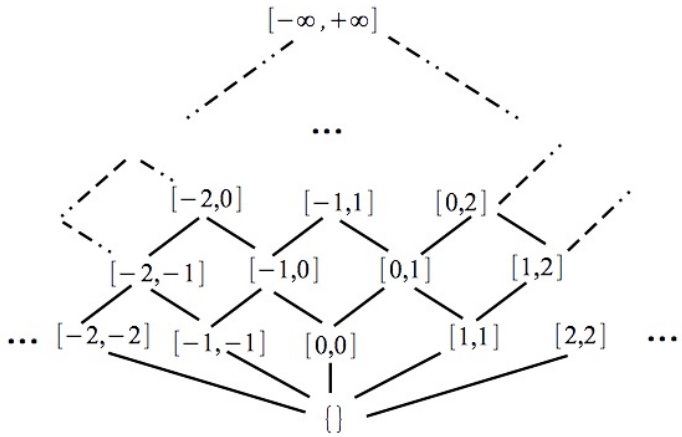
$[a,b] \sqsubseteq [c,d]$ iff $c \leq a$ and $b \leq d$
$[a,b] \sqcup [c,d]$ is $[inf\{a,c\}, sup\{b,d\}]$
$[a,b] \sqcap [c,d]$ is $[sup\{a,c\}, inf\{b,d\}]$

```
//x:⊤
L1. x:= 0
//x:[0,0]
L2. x:= x + 1
//x:[1,1]
L3. x:= x + 2
//x:[3,3]
L4. assert x >= 2
//x:[3,3]
L5. end
```

```
//x:⊤
L1. x:= 0
//x:[0,0]⊔[0,0]
L2. x:= x + 1
//x:[1,1]⊔[1,1]
L3. x:= x + 2
//x:[3,3]⊔[3,3]
L4. assert x >= 2
//x:[3,3]⊔([3,3] ⊓ [2,+oo])
L5. end
```

```
//x:⊤
L1. x:= 0
//x:[0,0]
L2. x:= x + 1
//x:[1,1]
L3. x:= x + 2
//x:[3,3]
L4. assert x >= 2
//x:[3,3]
L5. end
```

Fixpoint computation: loops with intervals



$[a,b] \sqsubseteq [c,d]$ iff $c \leq a$ and $b \leq d$
$[a,b] \sqcup [c,d]$ is $[inf\{a,c\}, sup\{b,d\}]$
$[a,b] \sqcap [c,d]$ is $[sup\{a,c\}, inf\{b,d\}]$

```
//x:⊤
L1. x:= 0
//x:[0,25]
L2. x:= x + 1
//x:[1,26]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

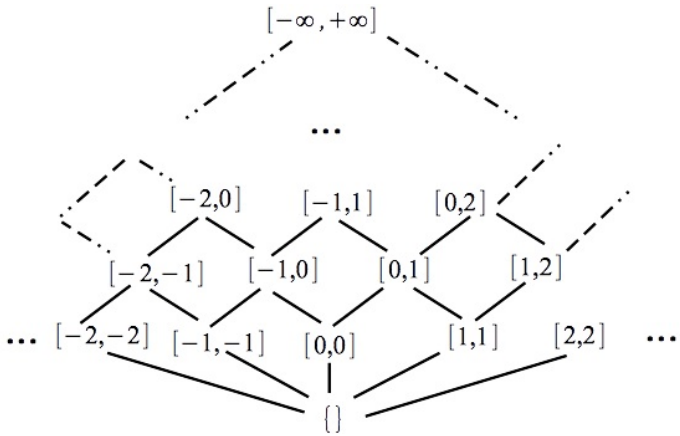
→

```
//x:⊤
L1. x:= 0
//x:[0,25]⊔[0,0]
//   ⊔([1,26] ⊓ [−oo,99])
L2. x:= x + 1
//x:[1,26]⊔[1,26]
L3. if x < 100 goto L2
//x:⊥⊔([1,26] ⊓ [100,+oo])
L4. assert x >= 100
//x:⊥⊔(⊥ ⊓ [100,+oo])
L5. end
```

→

```
//x:⊤
L1. x:= 0
//x:[0,26]
L2. x:= x + 1
//x:[1,26]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

Fixed point computation: widening



$[0,0], [0,1], [0,2], [0,3]$....
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x:⊤
L1. x:= 0
//x:[0,25]

L2. x:= x + 1
//x:[1,26]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:([0,25]⊔[0,0])
// ⊔([1,26]⊓[-oo,99])
L2. x:= x + 1
//x:([1,26]⊔[1,26])
L3. if x < 100 goto L2
//x:⊥⊔([1,26]⊓[100,+oo])
L4. assert x >= 100
//x:⊥⊔(⊥⊓[100,+oo])
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:([0,25]∇[0,0])
// ∇([1,26]⊓[-oo,99])
L2. x:= x + 1
//x:([1,26]∇[1,26])
L3. if x < 100 goto L2
//x:⊥∇([1,26]⊓[100,+oo]
L4. assert x >= 100
//x:⊥∇(⊥⊓[100,+oo])
L5. end
```

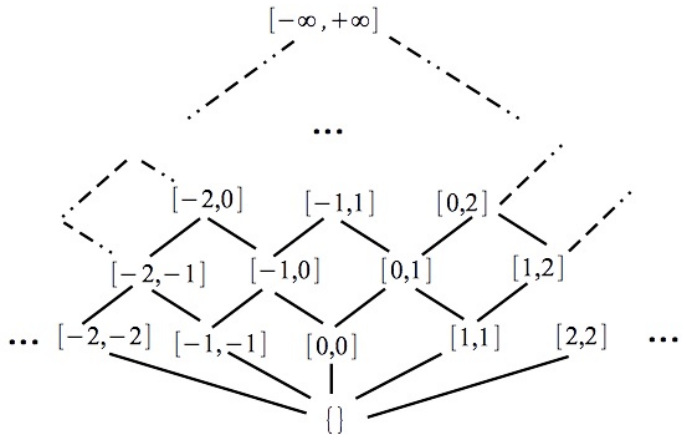
Fixpoint computation: widening

- A widening operator ∇ guarantees termination even in the presence of an infinite-height abstract lattice.
- Conditions on a widening operator ∇
  - for any abstract elements $A^\sharp, B^\sharp$, $(A^\sharp \sqcup_a B^\sharp) \sqsubseteq_a (A^\sharp \nabla B^\sharp)$
  - For any ${C_0}^\sharp \sqcup_a {C_1}^\sharp \sqcup_a \ldots$, let ${D_0}^\sharp = {C_0}^\sharp$ and ${D_{i+1}}^\sharp = {D_i}^\sharp \nabla {C_{i+1}}^\sharp$. The sequence ${D_0}^\sharp \sqcup_a {D_1}^\sharp \sqcup_a \ldots$ stabilizes.
  - Convergence guaranteed when using ∇ instead of $\sqcup_a$.

Widening for the interval domain

- A widening operator ∇ for the interval domain:
  - $[a, b] \nabla \bot = \bot \nabla [a, b] = [a, b]$
  - $[a, b] \nabla [c, d] = [l, r]$ with
    - $l = a$ if $a \leq c$ and $l = -\infty$ otherwise
    - $r = b$ if $b \geq d$ and $r = \infty$ otherwise
- $[3, 10] \nabla [2, 9]$
- ...

Fixpoint computation: widening (1)



$[0,0], [0,1], [0,2], [0,3]....$
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x:⊤
L1. x:= 0
//x:[0,0]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

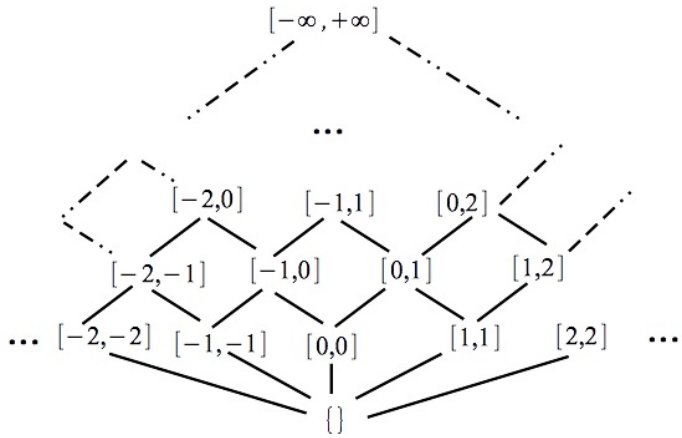
⟶

```
//x:⊤∇⊥
L1. x:= 0
//x:([0,0] ∇ [0,0])
//      ∇([1,1] ⊓ [−oo,99]
L2. x:= x + 1
//x:[1,1] ∇ [1,1]
L3. if x < 100 goto L2
//x:⊥∇([1,1] ⊓ [100,+oo])
L4. assert x >= 100
//x:⊥
L5. end
```

⟶

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x:[1,1]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

Fixpoint computation: widening (2)



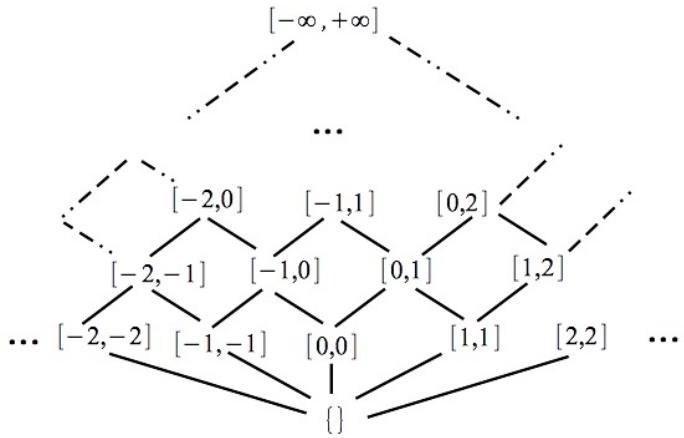
$[0,0], [0,1], [0,2], [0,3]....$
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2 ⟶
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

```
//x:⊤∇⊥
L1. x:= 0
//x:([0,+oo] ∇ [0,0])
//      ∇([1,1] ⊓ [−oo,99]
L2. x:= x + 1
//x:[1,1] ∇ [1,+oo]
L3. if x < 100 goto L2 ⟶
//x:⊥∇([1,1] ⊓ [100,+oo])
L4. assert x >= 100
//x:⊥
L5. end
```

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

Fixpoint computation: widening (3)



$[0,0], [0,1], [0,2], [0,3]....$
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x: [1,+oo]
L3. if x < 100 goto L2
//x:⊥
L4. assert x >= 100
//x:⊥
L5. end
```

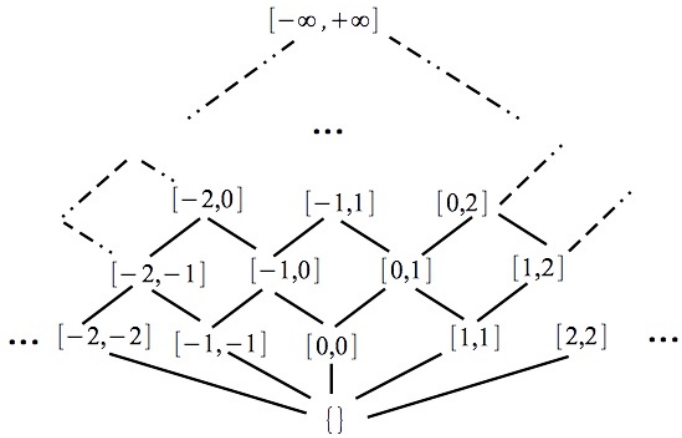
$\longrightarrow$

```
//x:⊤∇⊥
L1. x:= 0
//x:([0,+oo] ∇ [0,0])
//      ∇([1,+oo] ⊓ [−oo,99]
L2. x:= x + 1
//x:[1,+oo] ∇ [1,+oo]
L3. if x < 100 goto L2
//x:⊥∇([1,+oo] ⊓ [100,+oo])
L4. assert x >= 100
//x:⊥∇(⊥ ⊓ [100,+oo])
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x >= 100
//x:⊥
L5. end
```

Fixpoint computation: widening (4)



$[0,0], [0,1], [0,2], [0,3]....$
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:⊥
L5. end
```

⟶

```
//x:⊤
L1. x:= 0
//x:([0,+oo]∇[0,0])
//      ∇[1,+oo]
L2. x:= x + 1
//x:[1,+oo]∇[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]∇([1,+oo]⊓[100,+oo])
L4. assert x <= 100
//x:⊥∇([100,+oo]⊓[−oo,100])
L5. end
```

⟶

```
//x:⊤
L1. x:= 0
//x:[0,oo]
L2. x:= x + 1
//x:[1,oo]
L3. if x < 100 goto L2
//x:[100,oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

Widening can overapproximate too much: narrowing

- After widening, having obtained a too coarse over-approximation of the lfp, use a narrowing operator Δ to regain some precision.
- Conditions on a narrowing operator Δ
  - for any abstract elements $A^\sharp \sqsubseteq_a B^\sharp$, $(A^\sharp \sqsubseteq_a (A^\sharp \Delta B^\sharp) \sqsubseteq_a B^\sharp)$
  - For any ${C_0}^\sharp \sqsupseteq_a {C_1}^\sharp \sqsupseteq_a \ldots$, let ${D_0}^\sharp = {C_0}^\sharp$ and ${D_{i+1}}^\sharp = {D_i}^\sharp \Delta {C_{i+1}}^\sharp$. The sequence ${D_0}^\sharp \sqcup_a {D_1}^\sharp \sqcup_a \ldots$ stabilizes.

Narrowing for the interval domain (1/4)

- A narrowing operator Δ for the interval domain:
  - $[a, b]\Delta\bot = [a, b]$
  - $[a, b]\Delta[c, d] = [l, r]$ with
    - $l = c$ if $a = -oo$ and $l = a$ otherwise
    - $r = d$ if $b = +oo$ and $r = b$ otherwise
- $[0, oo]\Delta[1, 99] = [0, 99]$

```
//x:⊤
L1. x:= 0
//x:[0,+oo]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:([0,+oo]Δ([0, 0] ⊔ [−oo, 99])
L2. x:= x + 1
//x:[1,+oo]Δ[1, +oo]
L3. if x < 100 goto L2
//x:[100, +oo]Δ([1, +oo] ⊓ [100, +oo])
L4. assert x <= 100
//x:[100, 100]Δ([100, +oo] ⊓ [100, +oo])
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

Narrowing for the interval domain (2/4)

- A narrowing operator Δ for the interval domain:
  - $[a, b]\Delta\bot = [a, b]$
  - $[a, b]\Delta[c, d] = [l, r]$ with
    - $l = c$ if $a = -oo$ and $l = a$ otherwise
    - $r = d$ if $b = +oo$ and $r = b$ otherwise
- $[0, oo]\Delta[1, 99] = [0, 99]$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,+oo]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:([0,99]Δ([0,0] ⊔ [−oo,99])
L2. x:= x + 1
//x:[1,+oo]Δ[1,100]
L3. if x < 100 goto L2
//x:[100,+oo]Δ([1,+oo] ⊓ [100,+oo])
L4. assert x <= 100
//x:[100,100]Δ([100,+oo] ⊓ [100,+oo])
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,100]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

Narrowing for the interval domain (3/4)

- A narrowing operator Δ for the interval domain:
  - $[a, b]\Delta\bot = [a, b]$
  - $[a, b]\Delta[c, d] = [l, r]$ with
    - $l = c$ if $a = -oo$ and $l = a$ otherwise
    - $r = d$ if $b = +oo$ and $r = b$ otherwise
- $[0, oo]\Delta[1, 99] = [0, 99]$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,100]
L3. if x < 100 goto L2
//x:[100,+oo]
L4. assert x <= 100
//x:[100,100]
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:([0,99]Δ([0,0] ⊔ [−oo,99])
L2. x:= x + 1
//x:[1,100]Δ[1,100]
L3. if x < 100 goto L2
//x:[100,+oo]Δ([1,100] ⊓ [100,+oo])
L4. assert x <= 100
//x:[100,100]Δ([100,+oo] ⊓ [100,+oo])
L5. end
```

$\longrightarrow$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,100]
L3. if x < 100 goto L2
//x:[100,100]
L4. assert x <= 100
//x:[100,100]
L5. end
```

Narrowing for the interval domain (4/4)

- A narrowing operator Δ for the interval domain:
  - $[a, b]\Delta\bot = [a, b]$
  - $[a, b]\Delta[c, d] = [l, r]$ with
    - $l = c$ if $a = -oo$ and $l = a$ otherwise
    - $r = d$ if $b = +oo$ and $r = b$ otherwise
- $[0, oo]\Delta[1, 99] = [0, 99]$

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,100]
L3. if x < 100 goto L2
//x:[100,100]
L4. assert x <= 100
//x:[100,100]
L5. end
```

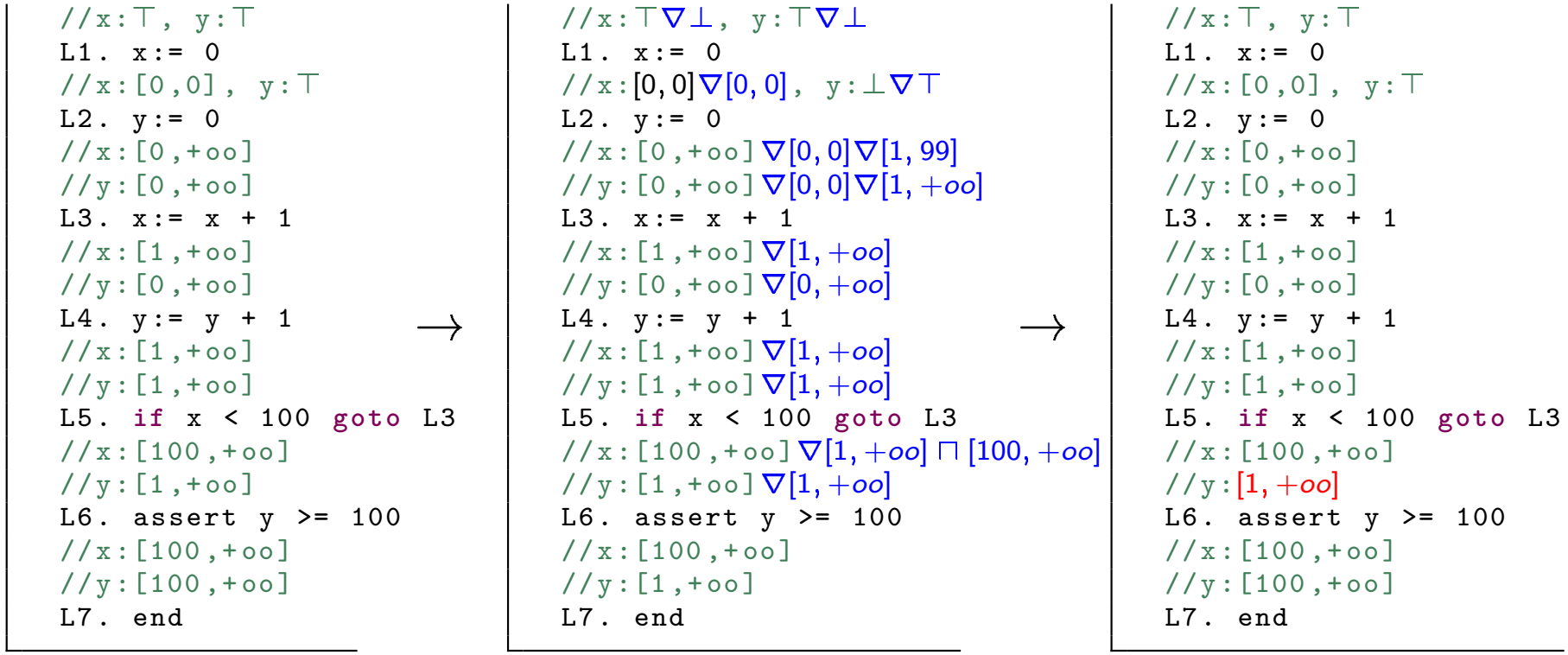
→

```
//x:⊤
L1. x:= 0
//x:([0,99]Δ([0,0] ⊔ [−oo,99])
L2. x:= x + 1
//x:[1,100]Δ[1,100]
L3. if x < 100 goto L2
//x:[100,100]Δ([1,100] ⊓ [100,+oo])
L4. assert x <= 100
//x:[100,100]Δ([100,100] ⊓ [100,+oo])
L5. end
```

→

```
//x:⊤
L1. x:= 0
//x:[0,99]
L2. x:= x + 1
//x:[1,100]
L3. if x < 100 goto L2
//x:[100,100]
L4. assert x <= 100
//x:[100,100]
L5. end
```

Need for relational domains



- Intervals do not capture relations between variables
- DBMs $x - y \leq k$
- Octagons $\pm x \pm y \leq k$
- Polyhedra $a_1 x_1 + \ldots + a_n x_n <= k$
- Shape analysis, arrays, lists, etc

Outline

Further readings


Apron numerical abstract domain library.
`http://apron.cri.ensmp.fr/library/`.
Accessed: 2021-01-14.


General information on the ppl.
`https://www.bugseng.com/products/ppl/documentation/user/ppl-user-1.2-html/index.html`.
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Abstract Interpretation Based Formal Methods and Future Challenges, pages 138–156.
Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.


P. Cousot and R. Cousot.
Systematic design of program analysis frameworks.
POPL '79, 1979.