Software Verification

Introduction to Abstract Interpretation

Ahmed Rezine

IDA, Linköpings Universitet

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Outline

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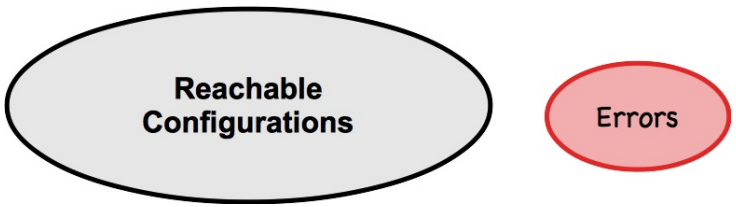
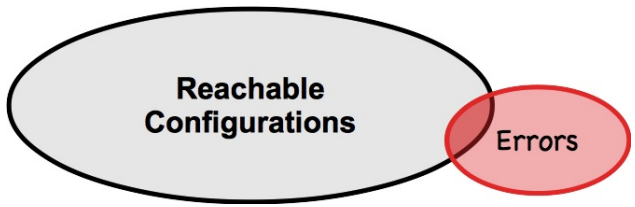
Verification

- We want to answer whether some program behaves correctly. We define "correctness" soon.
- For now, assume that means some erroneous configurations are not reachable
- We say the program is **safe**



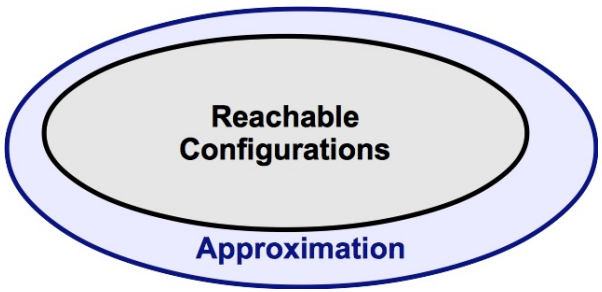
Safe Program



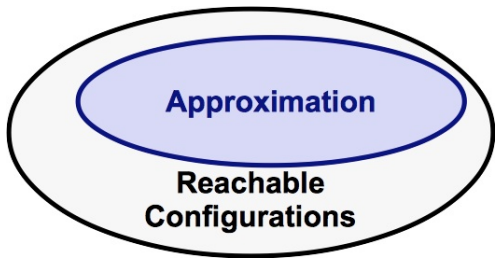
Unsafe Program

Verification problem and approximations

- The idea is then to come up with efficient approximations to give correct answers in as many cases as possible.



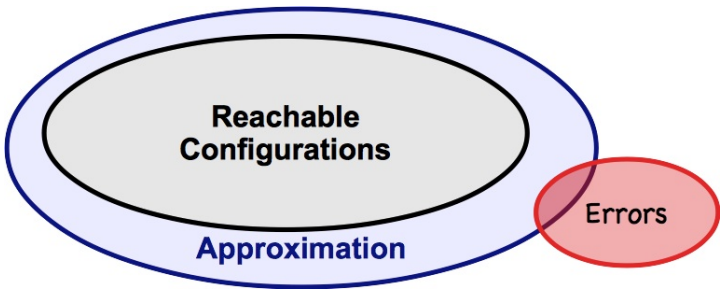
Over-approximation



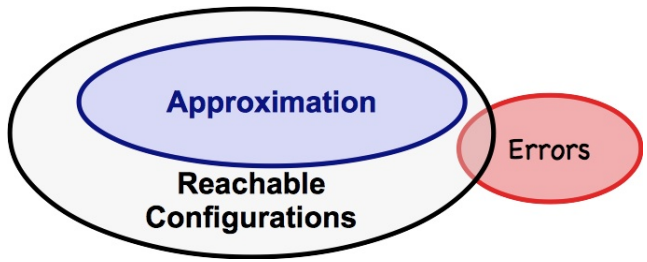
Under-approximation

Program verification and the price of approximations

- A sound analysis cannot give **false negatives**
- A complete analysis cannot give **false positives**



False Positive



False Negative

Approaches

- We discussed:
  - Explicit/symbolic model checking
  - Symbolic Execution relying on SMT solvers
  - Deductive frameworks with Hoare triples
- Today: a systematic and automatic framework for generating over-approximations

Outline

A simple language

Assume a simple language over a finite number of integer variables ranging over $\mathbb{Z}$.

- `exp ::= n | x | exp + exp | exp - exp | *`
- `cond ::= true | false | exp >= exp | cond && cond`
- `cmd ::= x:= exp | if cond goto label | assert cond | end`

```
L1. x:= *
L2. y:= x
L3. x:= x + y
L4. if x >= y goto L6
L5. end
L6. assert y >= 0
L7. end
```

```
L1. x:= 0
L2. y:= 0
L3. x:= x + 1
L4. y:= y + 2
L5. if x < 100 goto L3
L6. assert y = 200
L7. end
```

Generate invariants as in:

`http://pop-art.inrialpes.fr/interproc/interprocweb.cgi`

Outline

Abstract Interpretation

- In the concrete semantics, one can associate to each label a set of possible mappings from the variables to their values.
- Concrete semantics is precise: it only captures possible states. It is however too inefficient to be feasible in practice (loops, arbitrary inputs, arrays, recursion, concurrency, etc)
- Instead, efficiently and soundly over-approximate while tracking properties of certain "forms"

Abstract Interpretation

Idea:

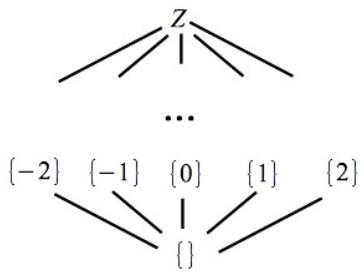
- Assume an "abstract domain" fixing the "form" of the tracked properties (e.g., $a \leq x \leq b$ or $a \leq x - y \leq b$ or ...)
- Define an abstract semantics that over-approximates the concrete semantics
- Iterate computation with abstract semantics untill fixpoint. Deduce facts about the original semantics.

A simple abstract domain: the sign example

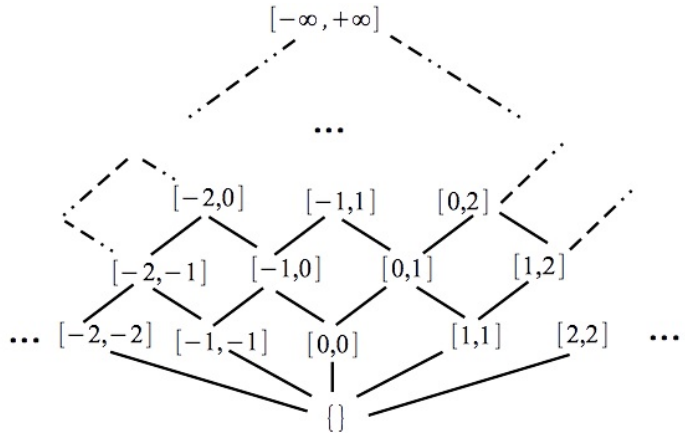
- If you are only interested in the signs of variables' values, you can associate, at each label and to each variable, a subset of $\{neg, zro, pos\}$.
- For an integer variable, the set of concrete values at a location is in $\mathcal{P}(\mathbb{Z})$. Concrete sets of variables' values (if that is what you are interested in) are ordered with the subset relation $\sqsubseteq_c$ on $\mathcal{P}(\mathbb{Z})$. We can associate $\mathbb{Z}$ to each variable in each location, but that is not precise. We write $S_1 \sqsubseteq_c S_2$ to mean that S_1 is more precise than S_2.
- We approximate concrete values with an element in $\mathcal{P}(\{neg, zro, pos\})$. For instance, $\{pos\}$ reflects the variable is larger than zero. For A_1, A_2 in $\mathcal{P}(\{neg, zro, pos\})$, we write $A_1 \sqsubseteq_a A_2$ to mean that A_1 is more precise than A_2.

Lattices

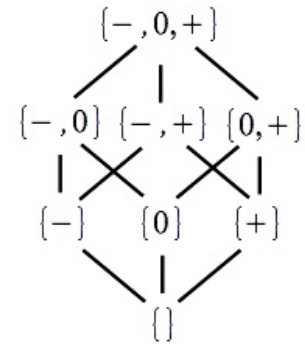
- A lattice is a poset $(Q, \sqsubseteq)$ where each pair p, q in Q has:
  - a greatest lower bound $p \sqcap q$ wrt. $\sqsubseteq$ (aka. meet), and
  - a least upper bound $p \sqcup q$ wrt. $\sqsubseteq$ (aka. join)
- If S is a set, then $(\mathcal{P}(S), \subseteq, \bot, \top, \cup, \cap)$ is a complete lattice: i.e, a lattice where each subset has a least upper bound and a greatest lower bound. $\cap\mathcal{P}(S) = \emptyset = \bot$ is the least element and $\cup\mathcal{P}(S) = S = \top$ is the greatest one.
- $(\mathcal{P}(\mathbb{Z}), \sqsubseteq_c)$ and $(\mathcal{P}(\{-, 0, +\}), \sqsubseteq_a)$ are therefore (complete) lattices



subsets



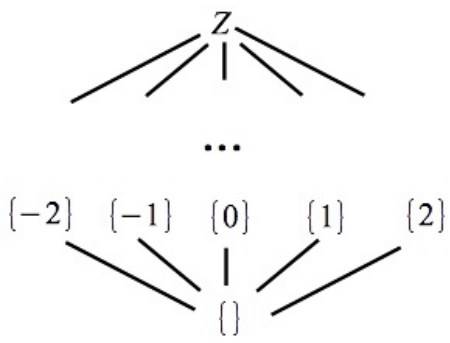
intervals



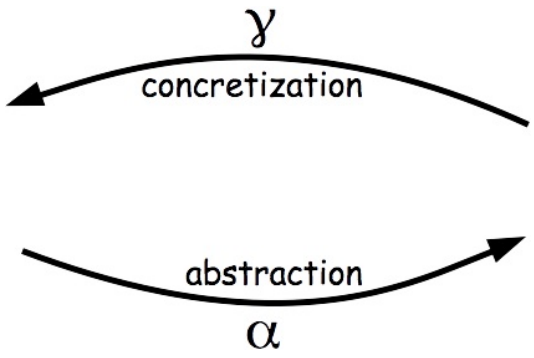
signs

The sign example: abstraction and concretization

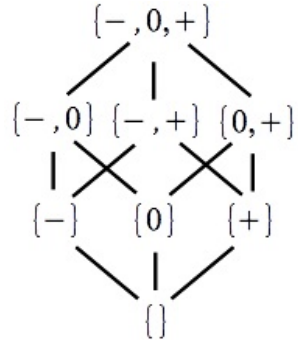
- An abstraction function: $\alpha(\{1, 10, 100, 103, 2021\}) = \{pos\}$, $\alpha(\{-10, -3, 10\}) = \{neg, pos\}$, $\alpha(\{-3, 0\}) = \{neg, zro\}$
- A concretization function: $\gamma(\{zro\}) = \{0\}$, $\gamma(\{neg, zro\}) = \{v | v \leq 0\}$ and $\gamma(\{pos\}) = \{v | v > 0\}$



Concrete lattice
$(\mathcal{P}(\mathbb{Z}), \sqsubseteq_c)$

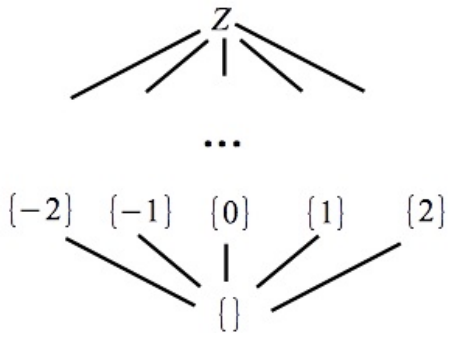


A Galois connection
(α, γ)



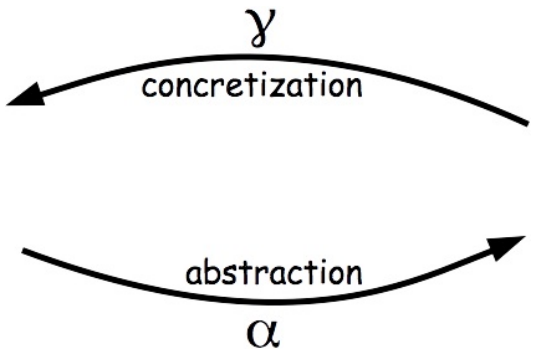
Abstract lattice
$(\mathcal{P}(\{neg, zro, pos\}), \sqsubseteq_a)$

The sign example: Galois connections



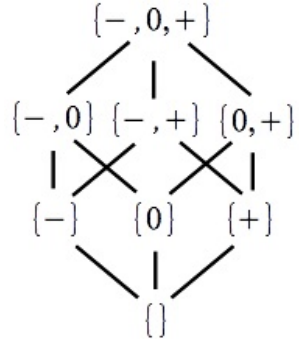
Concrete lattice

$(\mathcal{P}(\mathbb{Z}), \sqsubseteq_c)$



A Galois connection

(α, γ)



Abstract lattice

$(\mathcal{P}(\{neg, zro, pos\}), \sqsubseteq_a)$

- (α, γ) is a Galois connection if, for all $S \in \mathcal{P}(\mathbb{Z})$ and $A \in \mathcal{P}(\{neg, zro, pos\})$, $\alpha(S) \sqsubseteq_a A$ iff $S \sqsubseteq_c \gamma(A)$
  - $\alpha(S)$ is the most precise approximation of S
  - $\gamma(A)$ is least precise element that can be approximated by A
- Prop. (α, γ) is a Galois connection iff α and γ are monotone and $S \sqsubseteq_c \gamma \circ \alpha(S)$ and $\alpha \circ \gamma(A) \sqsubseteq_a A$ for any concrete and abstract elements S, A.
  - approximation/concretization of a more precise is more precise
  - concretizing an approximation is less precise than the original
  - approximating a concretization is precise

Abstract transformers and sound approximations

- You want to interpret the program in the abstract domain
- If A, B are approximated with $A^\sharp, B^\sharp$ (i.e., $\alpha(A) \sqsubseteq_a A^\sharp$ and $\alpha(B) \sqsubseteq_a B^\sharp$), then the abstraction $A^\sharp +^\sharp B^\sharp$ should approximate $A + B$, i.e. $\alpha(A + B) \sqsubseteq_a (A^\sharp +^\sharp B^\sharp)$.

$+^\sharp$	$\{neg\}$	$\{zro\}$	$\{pos\}$
$\{neg\}$	{neg}	{neg}	{neg,zro,pos}
$\{zro\}$	{neg}	{zro}	{pos}
$\{pos\}$	{neg,zro,pos}	{pos}	{pos}

$$A^\sharp +^\sharp B^\sharp = \bigcup_{a^\sharp \in A^\sharp, b^\sharp \in B^\sharp} a^\sharp +^\sharp b^\sharp$$

Abstract transformers and sound approximations

- You want to **soundly** interpret in the abstract domain
- If F is the concrete transformer and $F^\sharp$ is the abstract one, you want that:

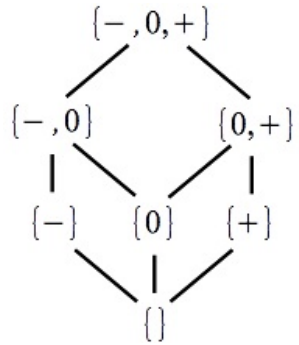
$$\alpha(A) \sqsubseteq_a B^\sharp \implies \alpha(F(A)) \sqsubseteq_a F^\sharp(A^\sharp)$$

.

- If $B^\sharp$ over-approximates A, you want $F^\sharp(B^\sharp)$ to overapproximate $\alpha(F(A))$.

Fixed point computation

```
L1. x:= *
L2. y:= x
L3. if x > 0 goto L5
L4. end
L5. assert x >= 0
L6. end
```



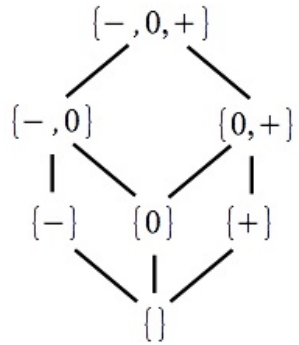

```
//x: ⊤, y: ⊤
L1. x:= *
//x: ⊥, y: ⊥
L2. y:= x
//x: ⊥, y: ⊥
L3. if x > 0 goto L5
//x: ⊥, y: ⊥
L4. end
//x: ⊥, y: ⊥
L5. assert x >= 0
//x: ⊥, y: ⊥
L6. end
```

```
//x: ⊤, y: ⊤
L1. x:= *
//x: ⊤, y: ⊤
L2. y:= x
//x: ⊤, y: ⊤
L3. if x > 0 goto L5
//x: ⊥, y: ⊥
L4. end
//x: ⊥, y: ⊥
L5. assert x >= 0
//x: ⊥, y: ⊥
L6. end
```

```
//x: ⊤, y: ⊤
L1. x:= *
//x: ⊤, y: ⊤
L2. y:= x
//x: ⊤, y: ⊤
L3. if x > 0 goto L5
//x: {neg,zro}, y: ⊤
L4. end
//x: {pos}, y: ⊤
L5. assert x >= 0
//x: {pos}, y: ⊤
L6. end
```

Fixed point computation

```
L1. x:= *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```



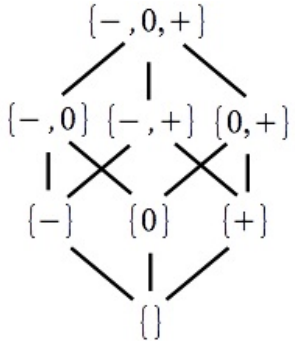

```
//x: ⊤,
L1. x:= *
//x: ⊥
L2. if x > 0 goto L5
//x: ⊥
L3. if x < 0 goto L5
//x: ⊥
L4. end
//x: ⊥
L5. assert x != 0
//x: ⊥
L6. end
```

```
//x: ⊤,
L1. x:= *
//x: ⊤
L2. if x > 0 goto L5
//x: {neg,zro}
L3. if x < 0 goto L5
//x: ⊥
L4. end
//x: {pos}
L5. assert x != 0
//x: {pos}
L6. end
```

```
//x: ⊤,
L1. x:= *
//x: ⊤
L2. if x > 0 goto L5
//x: {neg,zro}
L3. if x < 0 goto L5
//x: {zro}
L4. end
//x: {neg,zro,pos}
L5. assert x != 0
//x: {neg,zro,pos}
L6. end
```

Fixed point computation

```
L1. x:= *
L2. if x > 0 goto L5
L3. if x < 0 goto L5
L4. end
L5. assert x != 0
L6. end
```



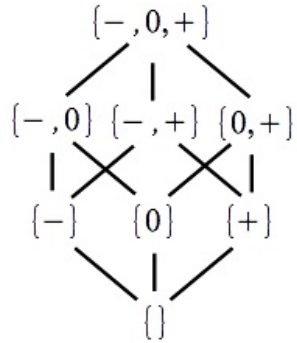

```
//x: ⊤,
L1. x:= *
//x: ⊥
L2. if x > 0 goto L5
//x: ⊥
L3. if x < 0 goto L5
//x: ⊥
L4. end
//x: ⊥
L5. assert x != 0
//x: ⊥
L6. end
```

```
//x: ⊤,
L1. x:= *
//x: ⊤
L2. if x > 0 goto L5
//x: {neg,zro}
L3. if x < 0 goto L5
//x: ⊥
L4. end
//x: {pos}
L5. assert x != 0
//x: {pos}
L6. end
```

```
//x: ⊤,
L1. x:= *
//x: ⊤
L2. if x > 0 goto L5
//x: {neg,zro}
L3. if x < 0 goto L5
//x: {zro}
L4. end
//x: {neg,pos}
L5. assert x != 0
//x: {neg,pos}
L6. end
```

Fixed point computation

```
L1. x:= 0
L2. x:= x + 1
L3. x:= x + 2
L4. assert x >= 2
L5. end
```

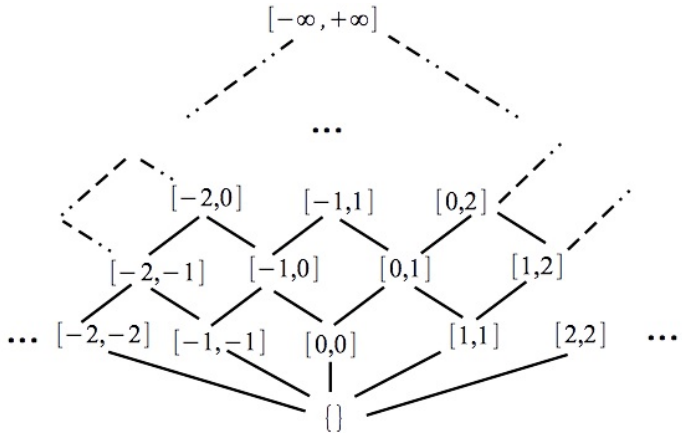



```
//x: ⊤
L1. x:= 0
//x: ⊥
L2. x:= x + 1
//x: ⊥
L3. x:= x + 2
//x: ⊥
L4. assert x >= 2
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: {zro}
L2. x:= x + 1
//x: {pos}
L3. x:= x + 2
//x: {pos}
L4. assert x >= 2
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: {zro}
L2. x:= x + 1
//x: {pos}
L3. x:= x + 2
//x: {pos}
L4. assert x >= 2
//x: {pos}
L5. end
```

Fixed point computation



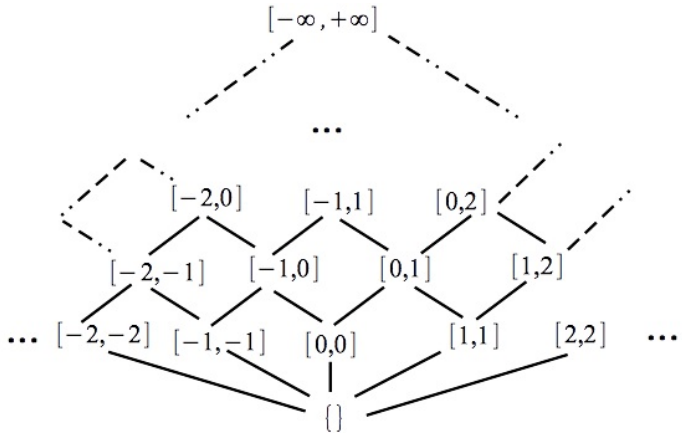
$[a,b] \sqsubseteq [c,d]$ iff $c \leq a$ and $b \leq d$
$[a,b] \sqcup [c,d]$ is $[inf\{a,c\}, sup\{b,d\}]$
$[a,b] \sqcap [c,d]$ is $[sup\{a,c\}, inf\{b,d\}]$

```
//x: ⊤
L1. x:= 0
//x: ⊥
L2. x:= x + 1
//x: ⊥
L3. x:= x + 2
//x: ⊥
L4. assert x >= 2
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. x:= x + 2
//x: [3,3]
L4. assert x >= 2
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. x:= x + 2
//x: [3,3]
L4. assert x >= 2
//x: [3,3]
L5. end
```

Fixed point computation



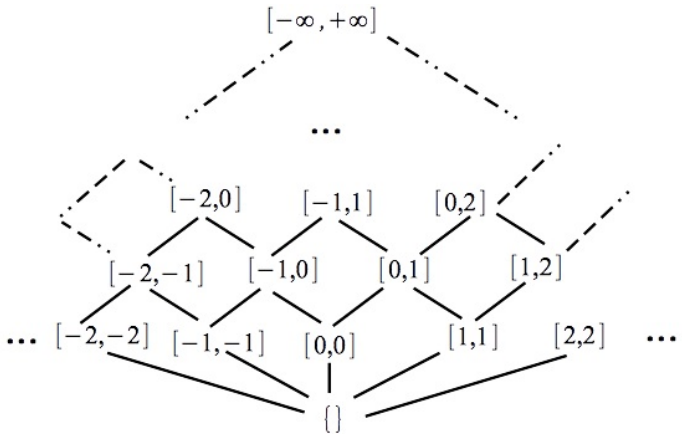
$[a,b] \sqsubseteq [c,d]$ iff $c \leq a$ and $b \leq d$
$[a,b] \sqcup [c,d]$ is $[inf\{a,c\}, sup\{b,d\}]$
$[a,b] \sqcap [c,d]$ is $[sup\{a,c\}, inf\{b,d\}]$

```
//x: ⊤
L1. x:= 0
//x: ⊥
L2. x:= x + 1
//x: ⊥
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,25]
L2. x:= x + 1
//x: [1,26]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

Fixed point computation: widening



$[0,0], [0,1], [0,2], [0,3]$....
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x: ⊤
L1. x:= 0
//x: ⊥
L2. x:= x + 1
//x: ⊥
L3. if x < 100 goto L2
//x: ⊥
L4. assert x <= 101
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,25]
L2. x:= x + 1
//x: [1,26]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

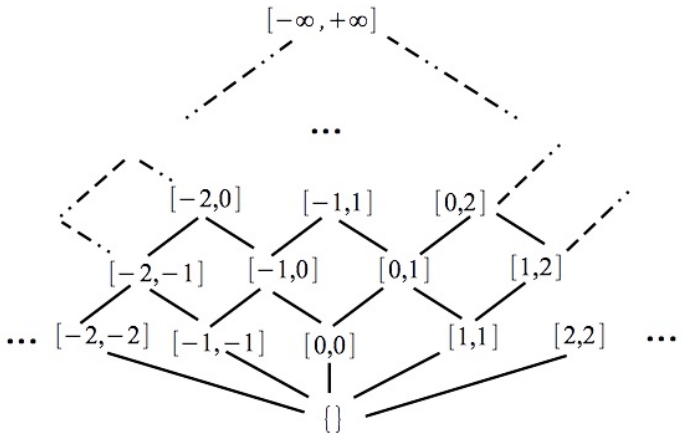
Fixed point computation: widening

- A widening operator ∇ guarantees termination even in the presence of an infinite-height abstract lattice.
- Conditions on a widening operator ∇
  - for any abstract elements $A^\sharp, B^\sharp$, $(A^\sharp \sqcup_a B^\sharp) \sqsubseteq_a (A^\sharp \nabla B^\sharp)$
  - For any ${C_0}^\sharp \sqcup_a {C_1}^\sharp \sqcup_a \ldots$, let ${D_0}^\sharp = {C_0}^\sharp$ and ${D_{i+1}}^\sharp = {D_i}^\sharp \nabla {C_{i+1}}^\sharp$. The sequence ${D_0}^\sharp \sqcup_a {D_1}^\sharp \sqcup_a \ldots$ stabilizes.
  - Convergence guaranteed when using ∇ instead of $\sqcup_a$.

Widening for the interval domain

- A widening operator ∇ for the interval domain:
  - $[a, b] \nabla \bot = \bot \nabla [a, b] = [a, b]$
  - $[a, b] \nabla [c, d] = [l, r]$ with
    - $l = a$ if $a \leq c$ and $l = -\infty$ otherwise
    - $r = b$ if $b \geq d$ and $r = \infty$ otherwise
- $[3, 10] \nabla [2, 9]$
- ...

Fixed point computation: widening



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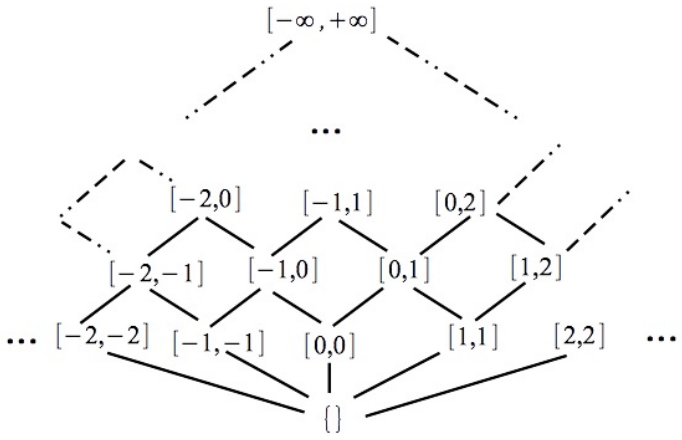
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0] ∇ [1,1]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x >= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,oo]
L2. x:= x + 1
//x: [1,oo]
L3. if x < 100 goto L2
//x: [100,oo]
L4. assert x >= 100
//x: [100,oo]
L5. end
```

Fixed point computation: widening



$[0,0], [0,1], [0,2], [0,3]$....
would take long time to converge.
For this use some widening operator ∇.
Intuitively, an acceleration that ensures termination

```
//x: ⊤
L1. x:= 0
//x: [0,0]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x <= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,0] ∇ [1,1]
L2. x:= x + 1
//x: [1,1]
L3. if x < 100 goto L2
//x: ⊥
L4. assert x <= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,oo]
L2. x:= x + 1
//x: [1,oo]
L3. if x < 100 goto L2
//x: [100,oo]
L4. assert x <= 100
//x: [100,oo]
L5. end
```

Widening can overapproximate too much: narrowing

- After widening, having obtained a too coarse over-approximation of the lfp, use a narrowing operator Δ to regain some precision.
- Conditions on a narrowing operator Δ
  - for any abstract elements $A^\sharp \sqsubseteq_a B^\sharp$, $(A^\sharp \sqsubseteq_a (A^\sharp \Delta B^\sharp) \sqsubseteq_a B^\sharp)$
  - For any ${C_0}^\sharp \sqsupseteq_a {C_1}^\sharp \sqsupseteq_a \ldots$, let ${D_0}^\sharp = {C_0}^\sharp$ and ${D_{i+1}}^\sharp = {D_i}^\sharp \Delta {C_{i+1}}^\sharp$. The sequence ${D_0}^\sharp \sqcup_a {D_1}^\sharp \sqcup_a \ldots$ stabilizes.

Narrowing for the interval domain

- A narrowing operator Δ for the interval domain:
  - $[a, b]\Delta\bot = \bot\Delta[a, b] = [a, b]$
  - $[a, b]\Delta[c, d] = [l, r]$ with
    - $l = c$ if $a = -oo$ and $l = a$ otherwise
    - $r = d$ if $b \geq oo$ and $r = b$ otherwise
- $[0, oo]\Delta[1, 99] = [0, 99]$

```
//x: ⊤
L1. x:= 0
//x: ⊥
L2. x:= x + 1
//x: ⊥
L3. if x < 100 goto L2
//x: ⊥
L4. assert x <= 100
//x: ⊥
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,oo]
L2. x:= x + 1
//x: [1,oo]
L3. if x < 100 goto L2
//x: [100,oo]
L4. assert x <= 100
//x: [100,oo]
L5. end
```

```
//x: ⊤
L1. x:= 0
//x: [0,99]
L2. x:= x + 1
//x: [1,100]
L3. if x < 100 goto L2
//x: [100,100]
L4. assert x <= 100
//x: [100,100]
L5. end
```

Need for relational domains

```
//x: ⊤, y: ⊤
L1. x:= 0
//x: ⊤, y: ⊤
L2. y:= 0
//x: ⊤, y: ⊤
L3. x:= x + 1
//x: ⊤, y: ⊤
L4. y:= y + 1
//x: ⊤, y: ⊤
L5. if x < 100 goto L3
//x: ⊤, y: ⊤
L6. assert y <= 100
//x: ⊤, y: ⊤
L7. end
```

```
//x: ⊤, y: ⊤
L1. x:= 0
//x: [0,0], y: ⊤
L2. y:= 0
//x: [0,oo], y: [0,oo]
L3. x:= x + 1
//x: [1,oo], y: [0,oo]
L4. y:= y + 1
//x: [1,oo], y: [1,oo]
L5. if x < 100 goto L3
//x: [100,oo], y: [100,oo]
L6. assert y <= 100
//x: [100,oo], y: [100,oo]
L7. end
```

```
//x: ⊤, y: ⊤
L1. x:= 0
//x: [0,0], y: ⊤
L2. y:= 0
//x: [0,99], y: [0,oo]
L3. x:= x + 1
//x: [1,100], y: [0,oo]
L4. y:= y + 1
//x: [1,100], y: [1,oo]
L5. if x < 100 goto L3
//x: [100,100], y: [100,oo]
L6. assert y <= 100
//x: [100,100], y: [100,oo]
L7. end
```

- Interval domain does not capture relations between values of variables
- DBMs $x - y \leq k$
- Octagons $\pm x \pm y \leq k$
- Polyhedra $a_1 x_1 + ... + a_n x_n <= k$
- Shape analysis, arrays, lists, etc

Outline

Further readings


Apron numerical abstract domain library.
`http://apron.cri.ensmp.fr/library/`.
Accessed: 2021-01-14.


General information on the ppl.
`https://www.bugseng.com/products/ppl/documentation/user/ppl-user-1.2-html/index.html`.
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Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.


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POPL '79, 1979.