Software Verification

Satisfiability Modulo Theory and applications

Symbolic representations II

Ahmed Rezine

IDA, Linköpings Universitet

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Outline

SAT solving and verification

Theories and SMTLIB

Symbolic Execution

Further readings

Outline

Basics

Sat solvers usually take propositional formulas written in conjunctive normal (cnf):

- variables: propositional variables, e.g., $x, y, z, \ldots$
- literals: variables or their negations, e.g., $x, \overline{x}, y, \ldots$
- clause: disjunction of literals, e.g., $(x \vee \overline{y})$
- cnf formula: conjunction of clauses, e.g., $(x \vee \overline{y}) \wedge (y \vee z) \wedge (\overline{z}) \wedge (\overline{x})$
- a (partial) variable assignment associates (some of the) variables to Boolean values in $\{0, 1\}$

Basics: a well known NP-problem

Given a propositional formula (usually in cnf), does there exist a satisfying assignment (i.e., a variable assignment for which the formula evaluates to true)?

- $x \mapsto 1, y \mapsto 1, z \mapsto 0$ is a satisfying assignment for $(x \vee \overline{y}) \wedge (y \vee z) \wedge (\overline{z})$. The formula is **satisfiable**
- $(x \vee \overline{y}) \wedge (y \vee z) \wedge (\overline{z}) \wedge (\overline{x})$ has no satisfying assignment, it is **unsatisfiable**
- Success story with applications in hardware/software model checking, planning, combinatorial design, test pattern generation, protocol design, bioinformatics, etc.
- From 100 variables and 200 constraints in early 90s to more than a 1,000,000 variables and 5,000,000 constraints

Unit propagation

- A unit is a clause where there is one unassigned literal while all other literals are assigned 0.
- The only chance for the current assignment to satisfy the formula is to assign 1 to unassigned literals in unit clauses
- $(x) \wedge (\overline{x} \vee y) \wedge (\overline{y} \vee z \vee s) \wedge (z \vee \overline{y} \vee \overline{x}) \wedge \ldots$
  - implied assignment: $x = 1$, antecedent (x)
  - implied assignment: $y = 1$, antecedent $(\overline{x} \vee y)$
  - implied assignment: $z = 1$, antecedent $(z \vee \overline{y} \vee \overline{x})$
  - implied assignment: $s = 1$, antecedent $(\overline{y} \vee z \vee s)$

Davis-Putnam-Logemann-Loveland (DPLL)

- decides satisfiability for sat-cnf problems
- introduced in early 60s
- basis of modern sat solvers
- idea: alternate unit propagation, choosing a value for some variable, and recursively checking the result, if does not give satisfying assignment then backtrack (remove) with opposit value

Davis-Putnam-Logemann-Loveland (DPLL)

Algorithm 1: DPLL-recursive(φ, ρ)

Input: φ: cnf formula, ρ: partial assignment

Output: UNSAT or satisfying assignment

$(\varphi, \rho) := \mathit{UnitPropagate}(\varphi, \rho)$;

if φ *contains an empty clause* **then**

  return *UNSAT*;

if φ *has no clauses left* **then**

  Output ρ;

  return SAT;

$l :=$ a literal not assigned by ρ;

if *(DPLL-recursive*$(\varphi[l], \rho \cup \{l\}))$ **then**

  return *SAT*;

return *DPLL-recursive*$(\varphi[\neg l], \rho \cup \{\neg l\})$;

DPLL, Example

$$\varphi = (x \vee y) \wedge (y \vee z) \wedge (\overline{x} \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee z)$$

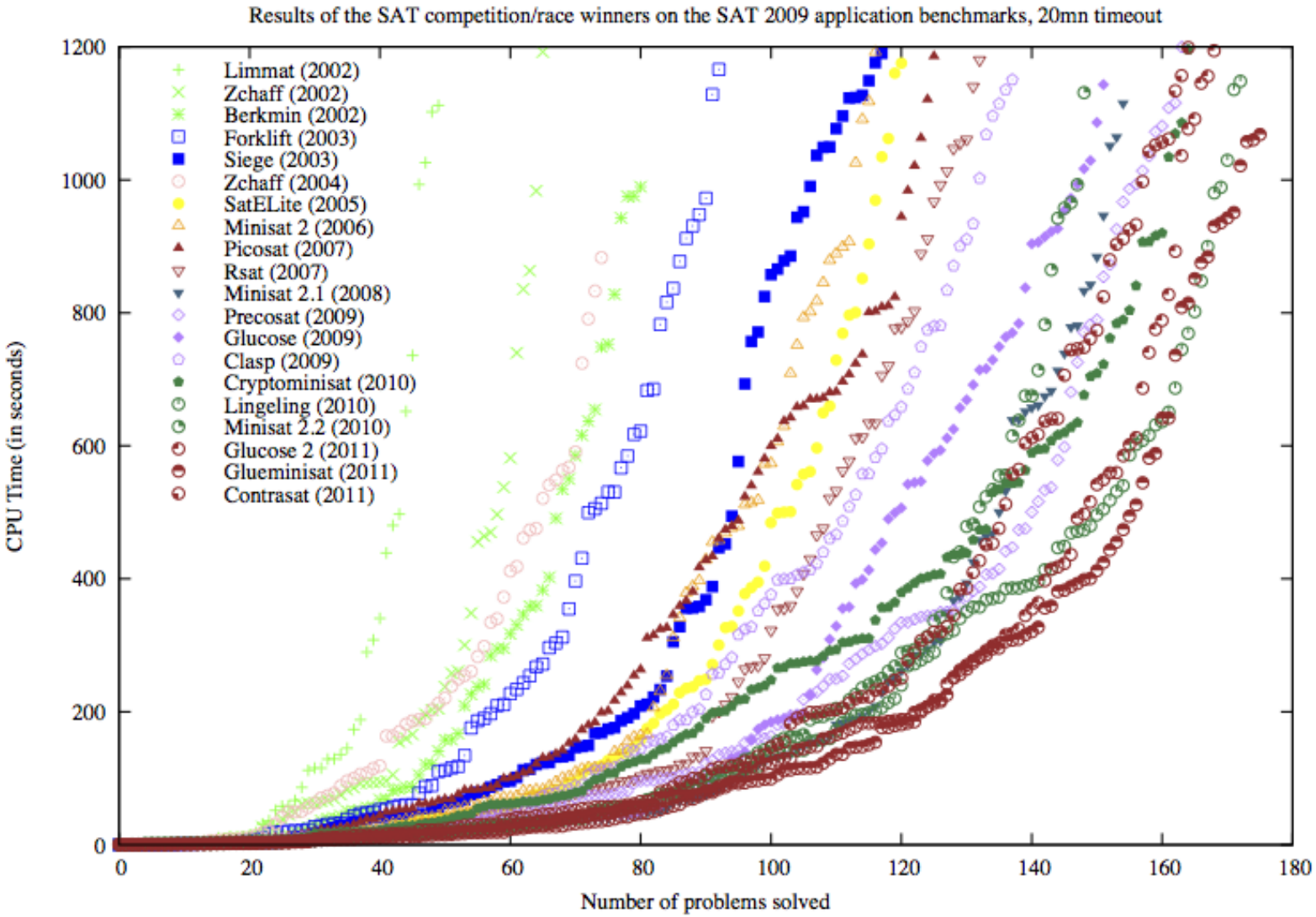
level	decision	formula	unit propagation
0	$x = 1$	$(1 \vee y) \wedge (y \vee z) \wedge (\overline{1} \vee \overline{y} \vee \overline{z}) \wedge (\overline{1} \vee 1)$	$z = 1@0$
1	$y = 1$	$(1 \vee 1) \wedge (1 \vee 1) \wedge (\overline{1} \vee \overline{1} \vee \overline{1}) \wedge (\overline{1} \vee 1)$	
1	$y = 0$	$(1 \vee 0) \wedge (0 \vee 1) \wedge (\overline{1} \vee \overline{0} \vee \overline{1}) \wedge (\overline{1} \vee 1)$	

$$x = 1@0, z = 1@0, y = 0@1$$

DPLL, Example 2

$$\varphi = (a \vee \overline{b} \vee d) \wedge (a \vee \overline{b} \vee e) \wedge (\overline{b} \vee \overline{d} \vee \overline{e}) \wedge (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \overline{d}) \wedge (a \vee b \vee \overline{c} \vee e) \wedge (a \vee b \vee \overline{c} \vee \overline{e})$$

Modern solvers



What made it possible?

- Clause learning with non chronological backtracking
- search restarts
- lazy data-structures
- ...

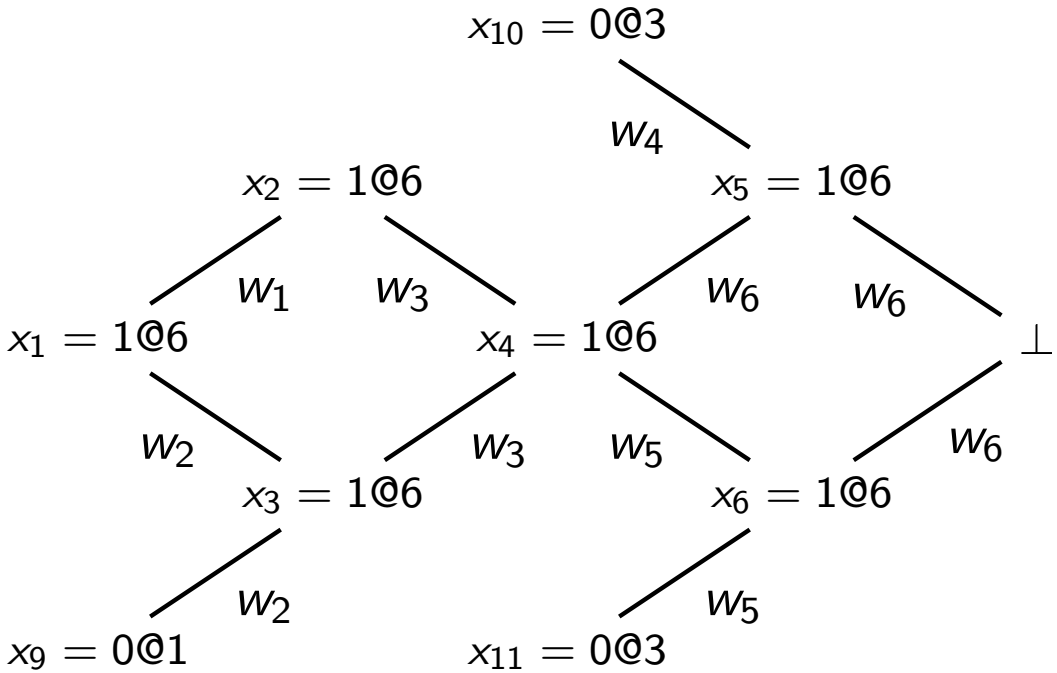
Clause learning

- capture "cause" of encountered conflict as a clause
- the conflict clause is added to the formula
- if deciding $x = 1$ and $y = 0$ led to unsat then remember this by adding $(\overline{x} \vee y)$ to the formula
- learn "reasons" of discovered inconsistencies in order to avoid them in the future

Conflict clauses

- Current assignment: $x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, \ldots$
- Current decision: $x_1 = 1@6$

$$
\begin{aligned}
w_1 &= (\overline{x_1} \vee x_2) \\
w_2 &= (\overline{x_1} \vee x_3 \vee x_9) \\
w_3 &= (\overline{x_2} \vee \overline{x_3} \vee x_4) \\
w_4 &= (\overline{x_4} \vee x_5 \vee x_{10}) \\
w_5 &= (\overline{x_4} \vee x_6 \vee x_{11}) \\
w_6 &= (\overline{x_5} \vee \overline{x_6}) \\
w_7 &= (x_1 \vee x_7 \vee \overline{x_{12}}) \\
w_8 &= (x_1 \vee x_8) \\
w_9 &= (\overline{x_7} \vee \overline{x_8} \vee \overline{x_{13}}) \\
&\ldots.
\end{aligned}
$$



Implication graph

- source nodes of implication graph can be used as a conflict clause
- here, $(\overline{x_1} \vee x_9 \vee x_{10} \vee x_{11})$ can be added as a conflict clause
- better clauses with "unique implication point"
- here, add $(\overline{x_4} \vee x_{10} \vee x_{11})$

SMTs as generalizations of SAT

Originates from automating proof-search for first order logic.

- Variables: $x, y, z, \ldots$
- Constants: $a, b, c, \ldots$
- N-ary functions: $f, g, h, \ldots$
- N-ary predicates: $p, q, r, \ldots$
- Atoms: $\bot, \top, p(t_1, \ldots, t_n)$
- Literals: atoms or their negation
- A FOL formula is a literal, boolean combinations of formulas, or quantified ($\exists$, $\forall$) formulas.

Evaluation of formula φ, with respect to interpretation I over non-empty (possibly infinite) domains for variables and constants gives true or false (resp. $I \models \varphi$ or $I \not\models \varphi$)

Satisfiability and Validity

A formula φ is:

- satisfiable if $I \models \varphi$ for **some** interpretation I
- valid if $I \models \varphi$ for **all** interpretations I

Satisfiability of FOL is undecidable. Instead, target decidable or domain-specific fragments.

Introduction to SMT

Given a quantifier free FOL formula and a combination of theories, is there an interpretation to the free variables that makes the formula true?

$$\varphi \triangleq g(a) = c \wedge (f(g(a)) \neq f(c) \vee g(a) = d) \wedge c \neq d$$

- EUF: Equality over Uninterpreted functions
- Satisfiable?

Introduction to SMT

Given a quantifier free FOL formula and a combination of theories, is there an interpretation to the free variables that makes the formula true?

$$\begin{aligned}\varphi \quad \triangleq \quad & (x_1 \geq 0) \wedge (x_1 < 1) \\ & \wedge((f(x_1) = f(0)) \Rightarrow (rd(wr(a, x_2, x_3), x_2 + x_1) = x_3 + 1)\end{aligned}$$

Introduction to SMT

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$$\begin{aligned}\varphi \quad \triangleq \quad & (x_1 \geq 0) \wedge (x_1 < 1) \\ & \wedge((f(x_1) = f(0)) \Rightarrow (rd(wr(a, x_2, x_3), x_2 + x_1) = x_3 + 1)\end{aligned}$$

- involves arrays with read (rd) and write (wr):
  - $\forall arr\ \forall i\ \forall val.\ (rd(wr(arr, i, val), i) = val)$
  - $\forall arr\ \forall i\ \forall j\ \forall val.\ (i \neq j \Rightarrow rd(wr(arr, i, val), j)) = rd(arr, j))$

Introduction to SMT

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- Linear Integer Arithmetic (LIA)

Introduction to SMT

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- Linear Integer Arithmetic (LIA)
- Equality over Uninterpreted functions (EUF)
- Arrays (A)

Introduction to SMT

Given a quantifier free FOL formula and a combination of theories, is there an interpretation to the free variables that makes the formula true?

$$\begin{aligned}\varphi \quad \triangleq \quad & (x_1 \geq 0) \wedge (x_1 < 1) \\ & \wedge((f(x_1) = f(0)) \Rightarrow (rd(wr(a, x_2, x_3), x_2 + x_1) = x_3 + 1)\end{aligned}$$

- LIA: $x_1 = 0$
- EUF: $f(x_1) = f(0)$
- A: $rd(wr(a, x_2, x_3), x_2) = x_3$
- Bool: $rd(wr(P, x_2, x_3), x_2) = x_3 + 1$
- LIA: $\bot$

Introduction to SMT

- Sometimes more natural to express in logics other than propositional logic
- SMT decide satisfiablity of ground FO formulas wrt. background theory
- Many applications: Model checking, predicate abstraction, symbolic execution, scheduling, test generation, ...

Outline

SMT competition and SMTLIB

- Drive development, since 2005
- 19^{th} instance at `https://smt-comp.github.io/2024/`
- Papers at SAT, CADE, CAV, FMCAD, TACAS, ...
- SMTLIB key initiative to promote common input and output for SMT solvers, benchmarks, tutorials, ...
- at `http://smtlib.cs.uiowa.edu/`

Equality with uninterpreted Functions (EUF)

- Consider $a * (f(b) + f(c)) = d \wedge b * (f(a) + f(c)) \neq d \wedge a = b$
- Formula is unsat, could be abstracted with
- $h(a, g(f(b), f(c))) = d \wedge h(b, g(f(b), f(c))) \neq d \wedge a = b$
- EUF used to abstracted non-supported theories such as non-linear multiplication or ALUs in circuits.

Arithmetic

Several restricted fragments, whether real or integer variables:

- Bounds $x \sim k$ with $\sim \in \{<, \leq, =, \geq, >\}$
- Difference logic $x - y \sim k$ with $\sim \in \{<, \leq, =, \geq, >\}$
- UTVPI $\pm x \pm y \sim k$ with $\sim \in \{<, \leq, =, \geq, >\}$
- Linear Arithmetic $x + 2y - 3z \leq 2$
- Non-linear arithmetic $xy - 4xy^2 + 2z \leq 2$

Arrays

- Special functions *read* and *write*
- Axioms:
  - $\forall a \forall i \forall v (read(write(a, i, v), i) = v)$
  - $\forall a \forall i \forall j \forall v (i \neq j \Rightarrow read(write(a, i, v), j)) = read(a, j))$
- Used for software (arrays) and hardware (memories) verification

Bit vectors

- Operations on vectors of bits
  - String like: concatenation, extraction, ...
  - Logical: bit-wise or, not, and...
  - Arithmetic: add, substract, multiply, ...
- $a[0:1] \neq b[0:1] \wedge (a|b) = c \wedge c[0] = 0 \wedge a[1] + b[1] = 0$

Outline

Testing

- Most common form of software validation
- Explores only one possible execution at a time
- For each new value, run a new test.
- On a 32 bit machine, `if(i==2023) bug()` would require 2^{32} different values to make sure there is no bug.
- The idea in symbolic testing is to associate **symbolic values** to the variables

Symbolic Testing

- Main idea by JC. King in "Symbolic Execution and Program Testing" in the 70s
- Use symbolic values instead of concrete ones
- Along the path, maintain a *Path Constraint* (PC) and a symbolic state (σ)
- PC collects constraints on variables' values along a path,
- σ associates variables to symbolic expressions,
- We get concrete values if PC is satisfiable
- The program can be run on these values
- Negate a condition in the path constraint to get another path

Symbolic Execution: a simple example

- Can we get to the ERROR? explore using SSA forms.
- Useful to check array out of bounds, assertion violations, etc.

```
1 foo(int x,y,z){
2  x = y - z;
3  if(x==z){
4     z = z - 3;
5     if(4*z < x + y){
6        if(25 > x + y) {
7           ...
8        }
9        else{
10          ERROR;
11       }
12    }
13 }
14 ...
```

$PC_1 = true$	
$PC_2 = PC_1$	$x \mapsto x_0, y \mapsto y_0, z \mapsto z_0$
$PC_3 = PC_2 \wedge x_1 = y_0 - z_0$	$x \mapsto (y_0 - z_0), y \mapsto y_0, z \mapsto z_0$
$PC_4 = PC_3 \wedge x_1 = z_0$	$x \mapsto (y_0 - z_0), y \mapsto y_0, z \mapsto z_0$
$PC_5 = PC_4 \wedge z_1 = z_0 - 3$	$x \mapsto (y_0 - z_0), y \mapsto y_0, z \mapsto (z_0 - 3)$
$PC_6 = PC_5 \wedge 4 * z_1 < x_1 + y_0$	$x \mapsto (y_0 - z_0), y \mapsto y_0, z \mapsto (z_0 - 3)$
$PC_{10} = PC_6 \wedge 25 \leq x_1 + y_0$	$x \mapsto (y_0 - z_0), y \mapsto y_0, z \mapsto (z_0 - 3)$

$PC = (x_1 = y_0 - z_0 \wedge x_1 = z_0 \wedge z_1 = z_0 - 3 \wedge 4 * z_1 < x_1 + y_0 \wedge 25 \leq x_1 + y_0)$

Check satisfiability with a solver (e.g., z3, cvc, yices, boolector,stp,...)

Symbolic execution today

- Leverages on the impressive advancements of SMT solvers
- Modern symbolic execution frameworks are not purely symbolic and are often dynamic: Sage, Klee (open source), Pex:
  - They can follow a concrete execution while collecting constraints along the way, or
  - They can treat some of the variables concretely, and some other symbolically
- This allows them to scale, to handle closed code or complex queries

Symbolic execution today

- C (actullay llvm) `http://klee.github.io/`
- Java (more than a symbolic executer)
  `http://babelfish.arc.nasa.gov/trac/jpf`
- C# (actually .net)
  `http://research.microsoft.com/en-us/projects/pex/`
- ...

Outline

Further readings


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