TDDD95 Algorithmic Problem Solving
Le 7 – Graphs III

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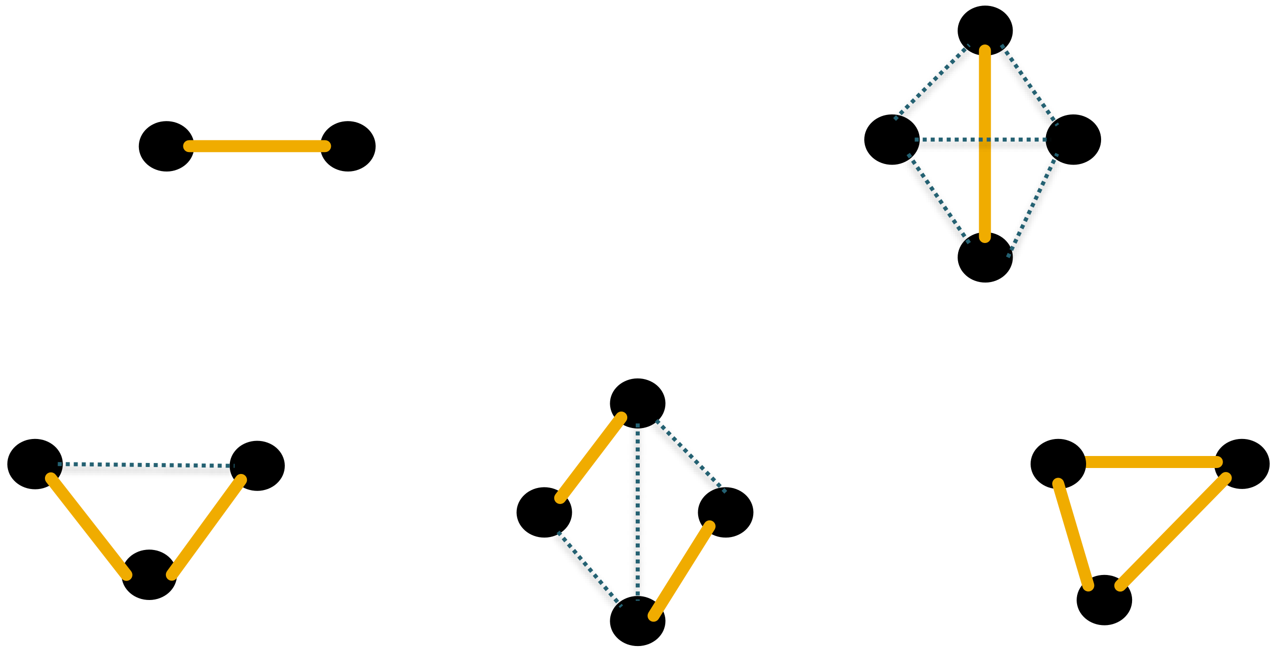
Outline

- Matching Problems
  - Graph Matching
  - Maximum Cardinality Matching
  - Maximum Cardinality Bipartite Matching
  - Maximum Weighted Matching
  - Maximum Weighted Bipartite Matching
  - Augmenting Paths Algorithm
- Covering Problems
  - Maximum Independent Set
  - Minimum Vertex Cover
  - Euler Path and Hoerholzer's Algorithm (lab 2.9)

Graph Matching

- A **<u>matching</u>** (marriage) in a graph G (life) is a subset of edges (relationships) in G <u>without common</u> vertices (no affairs!).



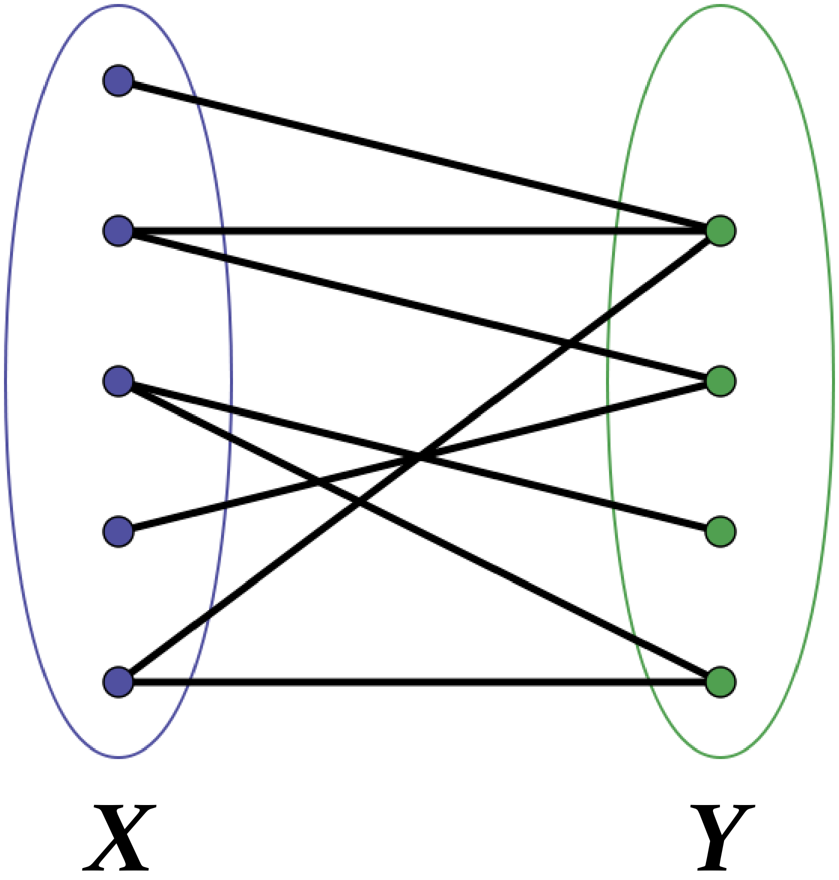
Are any of these graphs matchings?

Cardinality Matching

- **Maximum cardinality matching** (MCM) is the problem of finding the **size** (cardinality) of the largest possible matching in a graph.
- Not to be mixed up with **maximal matching**. A maximal matching is a matching for which we cannot add any more edges (is not necessarily MCM).

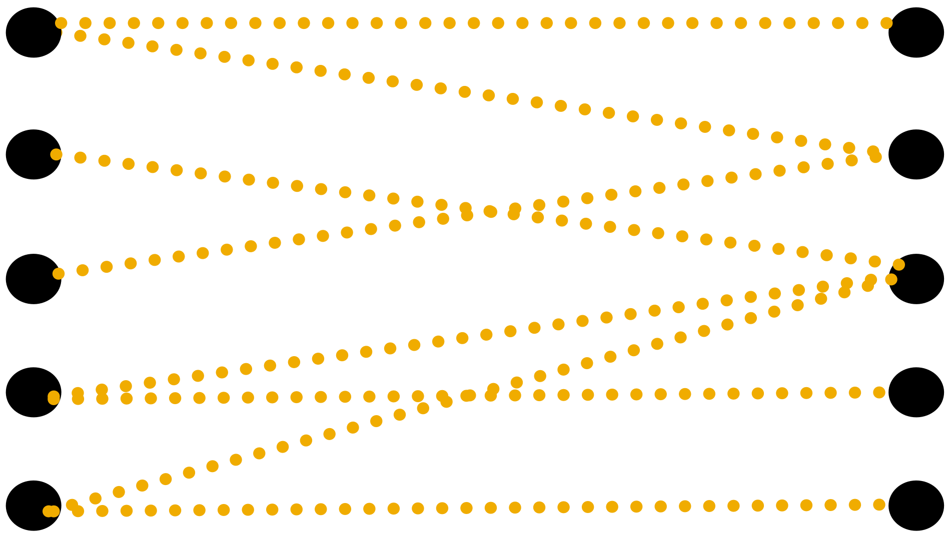
Bipartite Graph

- A **bipartite graph** is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y.



Maximum Cardinality Bipartite Matching

- **<u>Maximum cardinality bipartite matching</u>** (MCBM) is the problem to find the **<u>size</u>** (cardinality) of the largest possible matching in a bipartite graph.



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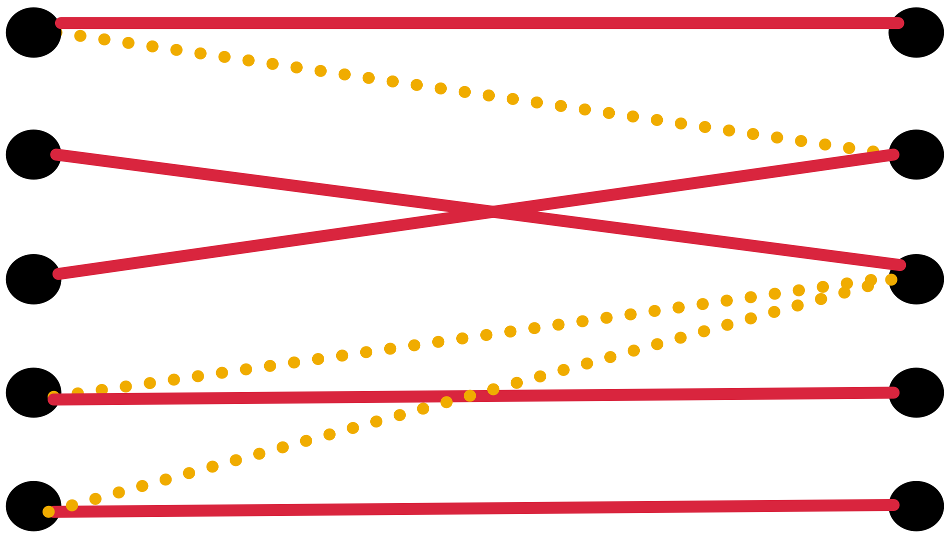
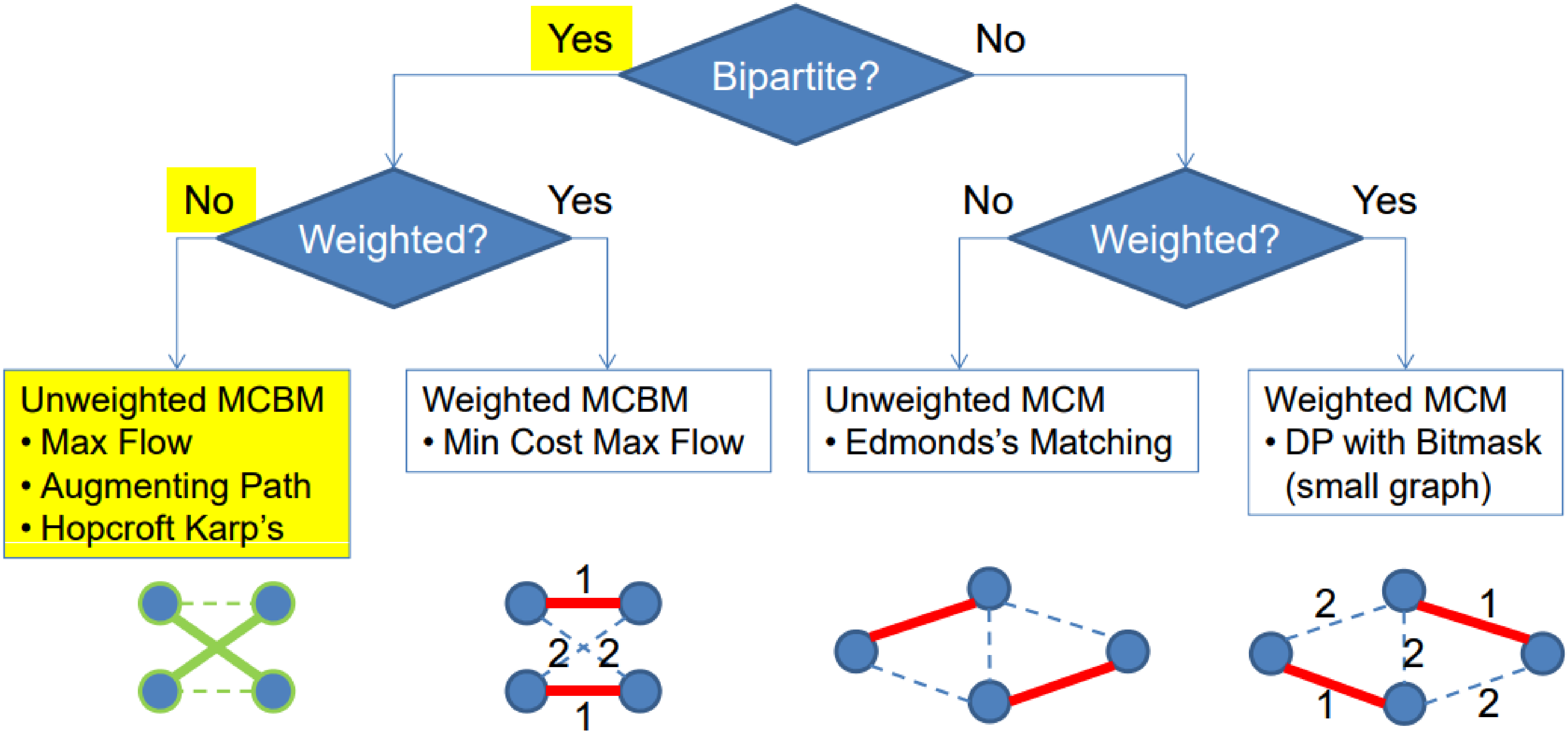


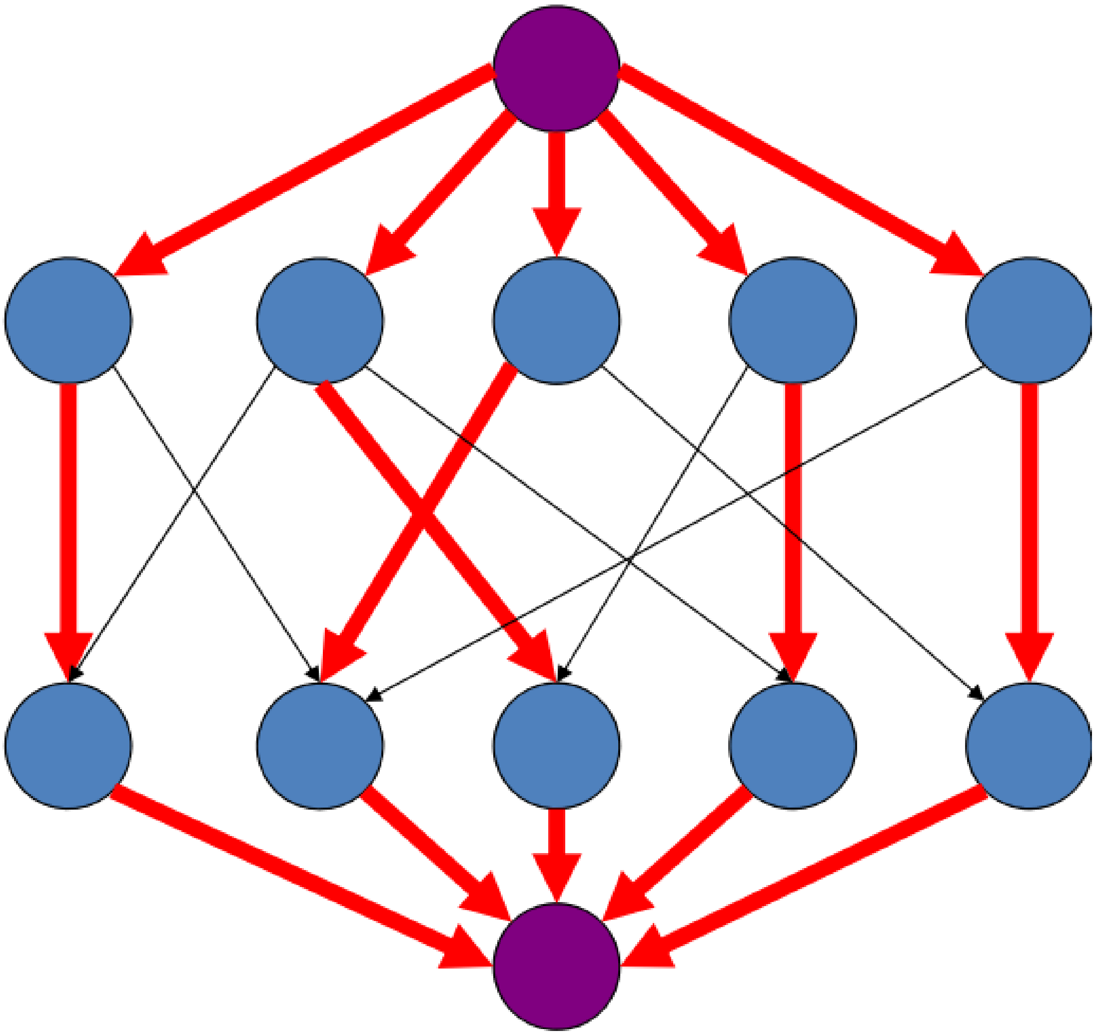
Weighted Maximum Cardinality Matching

- ***Weighted*** MCM involve finding the maximum/minimum MCM among all possible MCMs in a graph with weighted edges.

Graph Matching Solutions



A Max Flow Solution for MCBM



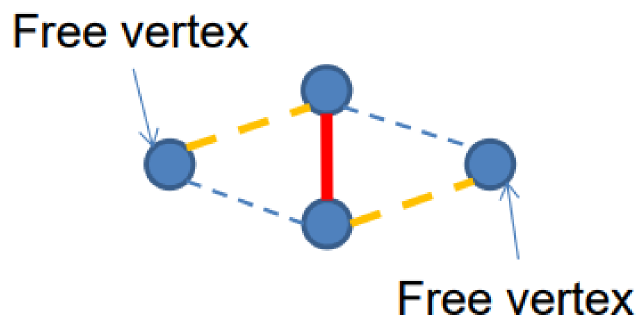
All edges have capacity = 1

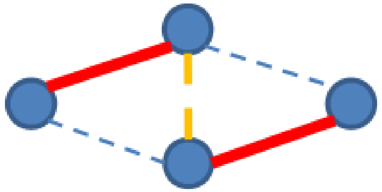
Augmenting Path

- A path $P = v_1, v_2, \ldots, v_k$ is ***alternating*** if the edges v_{i-1},v_i and v_i,v_{i+1} alternate between ***matched*** and ***unmatched***.
- P is **augmenting** if it is alternating and v_1 and v_k are unmatched.

Augmenting Path

- In this graph,
  the path colored
  orange(unmatched)-
  red(matched)-orange
  is an augmenting path
- We can flip the edge
  status to **red-orange-red**
  and the number of
  edges in the matching set
  increases by 1





The Augmenting Path Algorithm

- Lemma (Claude Berge 1957):

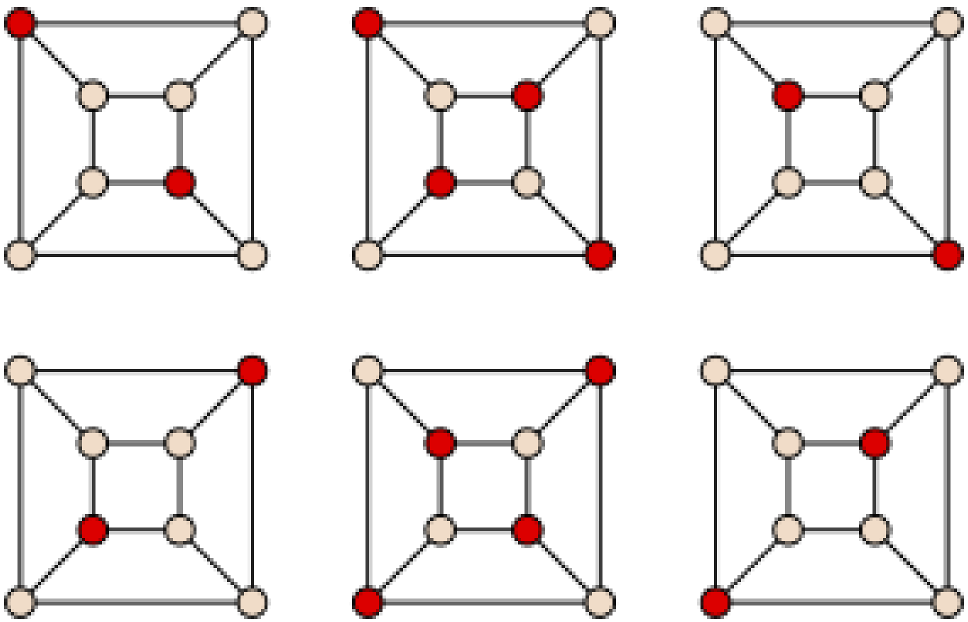
A matching M in G is maximum
iff there is no more augmenting path in G

- Augmenting Path Algorithm is a simple $O(V*(V+E)) = O(V^2 + VE) \sim= O(VE)$ implementation of that lemma

Recall Edmond-Karp $O(VE^2)$.

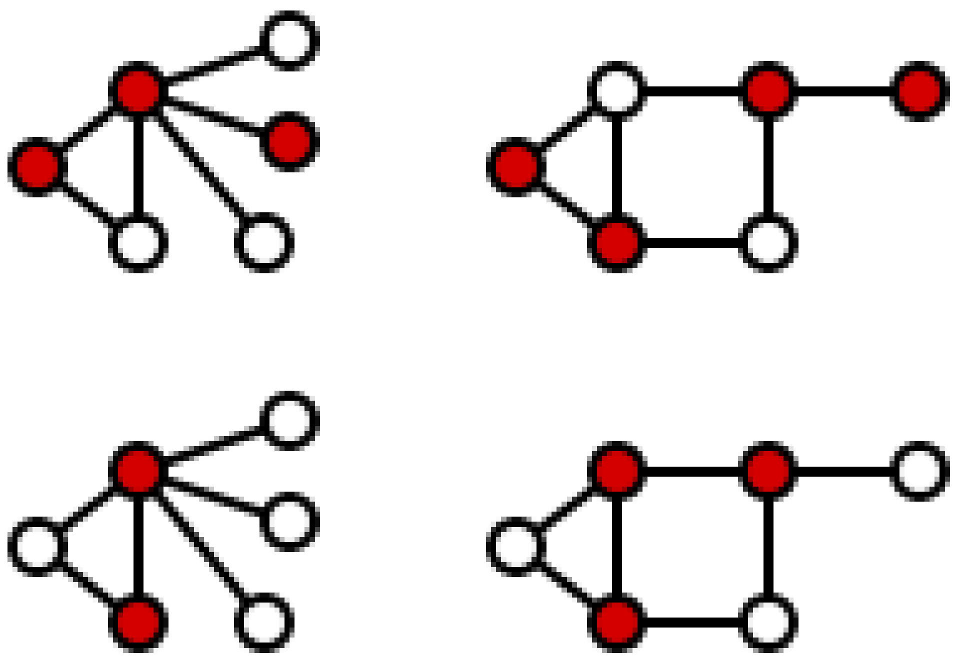
Independent Set

- An ***independent set*** (IS) is a set of vertices in a graph for which no two vertices are adjacent.
- A ***maximal independent set*** (MIS) is such a set that we cannot add additional vertices to.
- A ***maximum independent set*** is a maximum MIS.



Vertex Cover

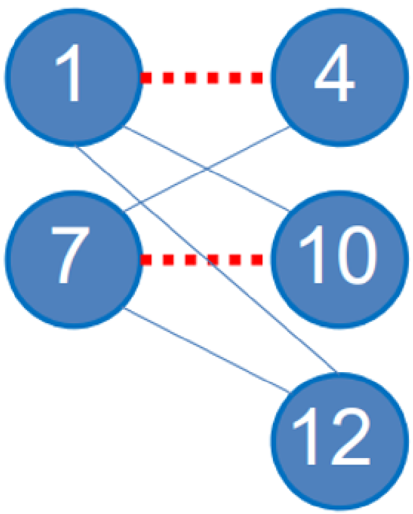
- A ***vertex cover*** in a graph is a set of vertices that includes at least one endpoint of every edge.
- A ***minimum vertex cover*** is a vertex cover of smallest possible size.



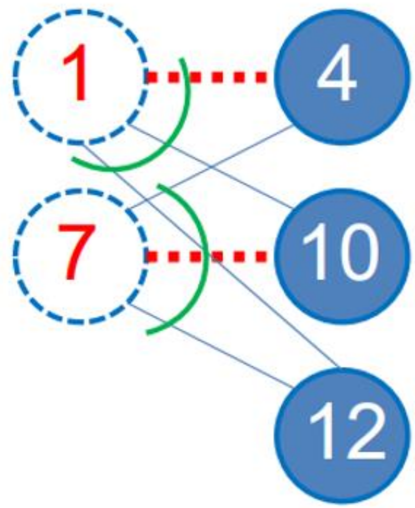
König's Theorem

- ***König's theorem:*** in any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.
  (can be derived from the max-flow min-cut theorem)
- In a bipartite graph, the complement of a maximum independent set is a minimum vertex cover.

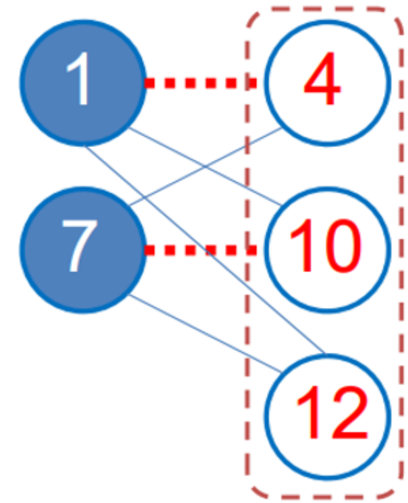
Applications of König's Theorem



Maximum Cardinality Bipartite Matching



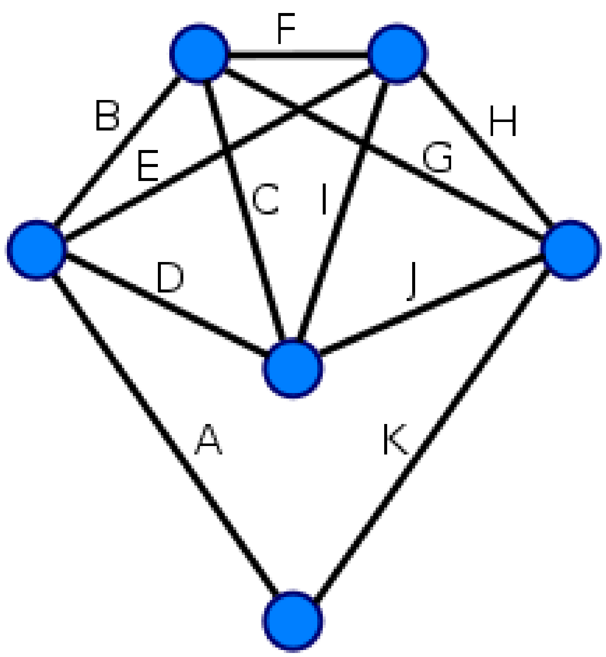
Minimum Vertex Cover
(König's Theorem)

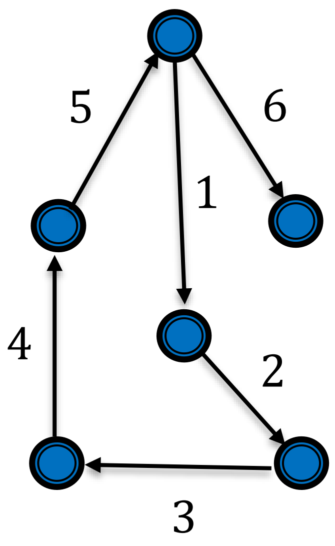


Maximum Independent Set

Eulerian Path

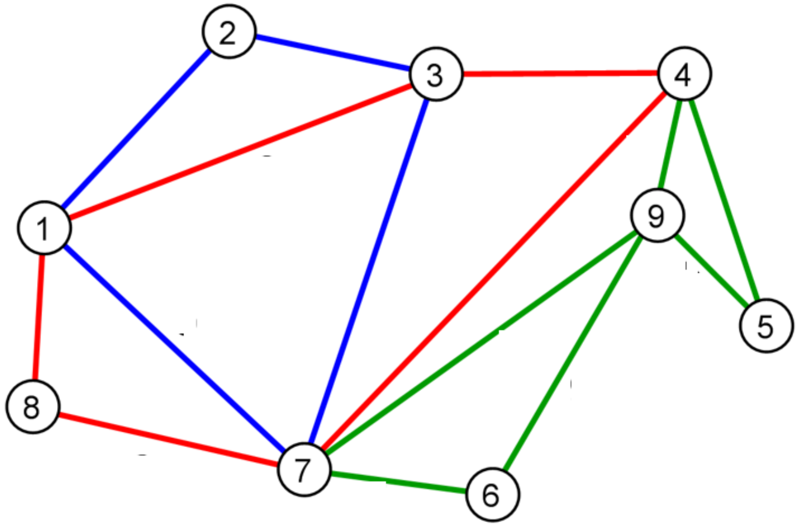
- A **<u>Eulerian path</u>** is a path in a graph that visits every edge exactly once.





Eulerian Path

- **<u>Hierholzer's algorithm</u>**
  - Choose any starting vertex *v*, and follow a trail of edges from that vertex until returning to *v*. The tour formed in this way is a closed tour but may not cover all the vertices and edges of the initial graph.
  - As long as there exists a vertex *v* that belongs to the current tour but that has adjacent edges not part of the tour, start another trail from *v*, following unused edges until returning to *v*, and join the tour formed in this way to the previous tour.
- Time complexity $\boldsymbol{O}(\boldsymbol{E})$.



Source: https://www.geeksforgeeks.org/hierholzers-algorithm-directed-graph/

Summary

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