Algorithmic Problem Solving

Exercise 06 + Graph III

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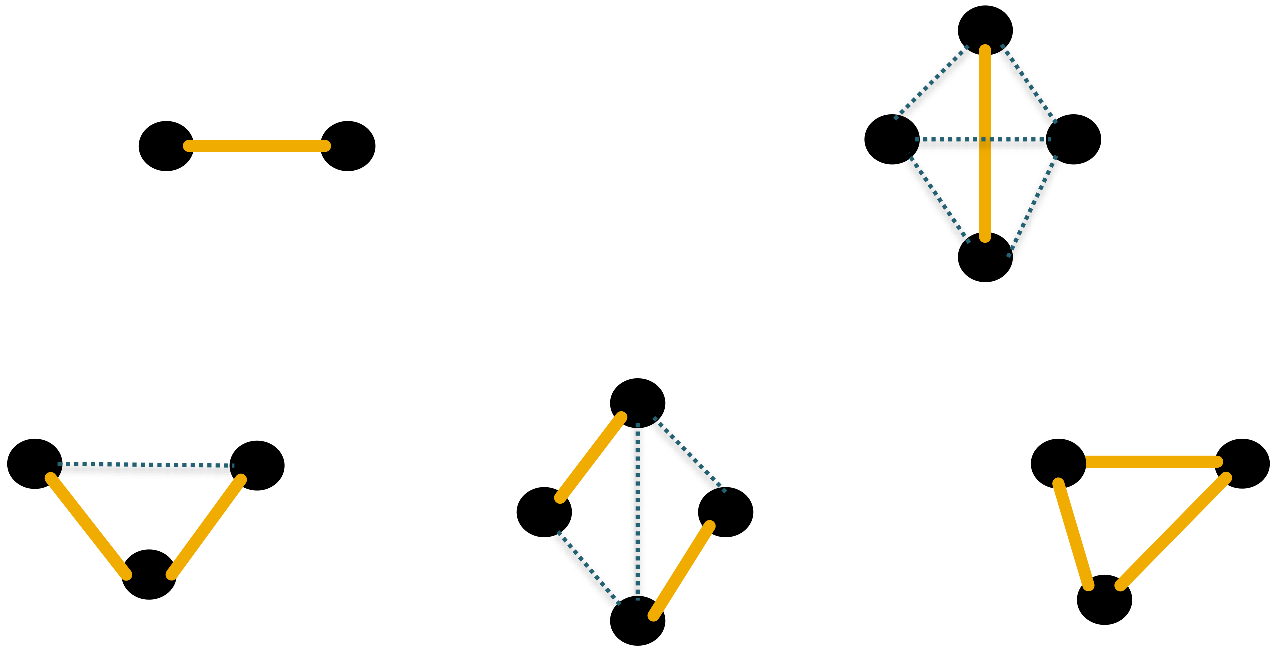
Outline

- ~~**This Week's Problems**~~
  ~~*(Island Hopping, George, Full Tank?, Councilling)*~~
- Matching Problems
  - Graph Matching
  - Maximum Cardinality Matching
  - Maximum Cardinality Bipartite Matching
  - Maximum Weighted Matching
  - Maximum Weighted Bipartite Matching
  - Augmenting Paths Algorithm
- Covering Problems
  - Maximum Independent Set
  - Minimum Vertex Cover
  - Euler Path and Hoerholzer's Algorithm (lab 2.9)

Graph Matching

- A **matching** (marriage) in a graph G (life) is a subset of edges (relationships) in G without common vertices (no affairs!).



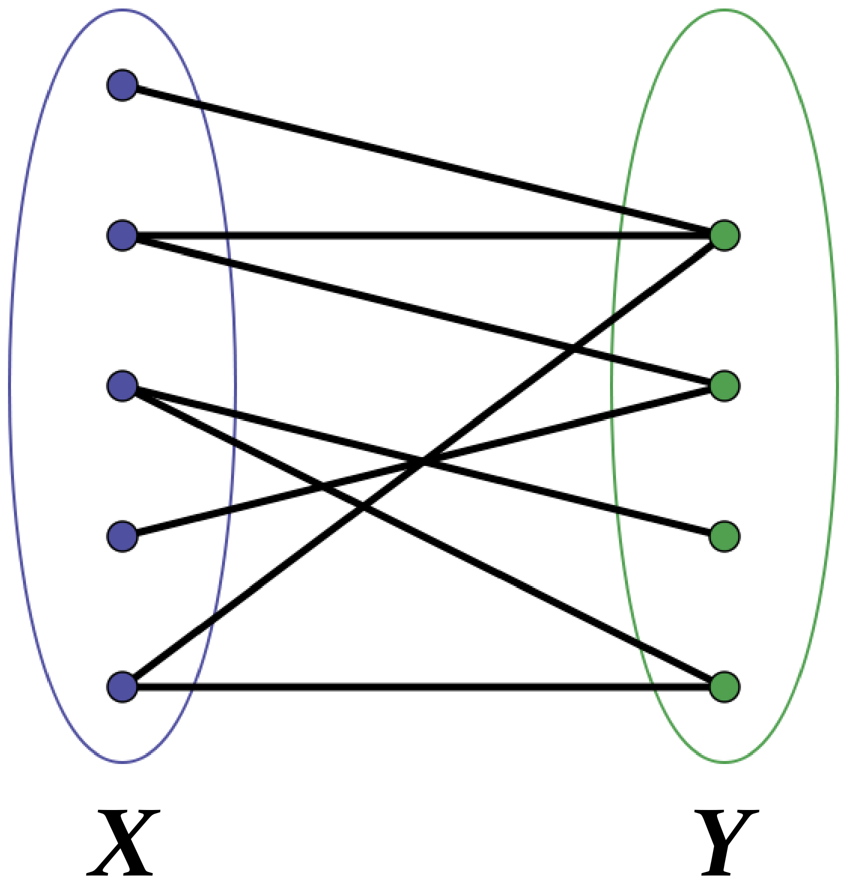
Are any of these graphs matchings?

Cardinality Matching

- **Maximum cardinality matching** (MCM) is the problem of finding the **size** (cardinality) of the largest possible matching in a graph.
- Not to be mixed up with **maximal matching**. A maximal matching is a matching for which we cannot add any more edges (is not necessarily MCM).

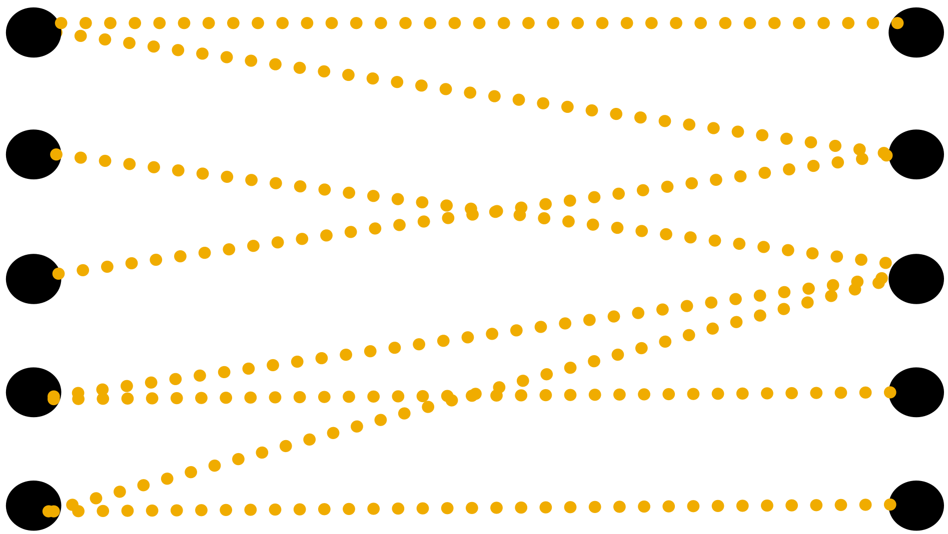
Bipartite Graph

- A **bipartite graph** is a graph whose vertices can be divided into two disjoint sets ***X*** and ***Y*** such that every edge connects a vertex in ***X*** to one in ***Y***.



Maximum Cardinality Bipartite Matching

- **Maximum cardinality bipartite matching** (MCBM) is the problem to find the **size** (cardinality) of the largest possible matching in a bipartite graph.



Maximum Cardinality Bipartite Matching

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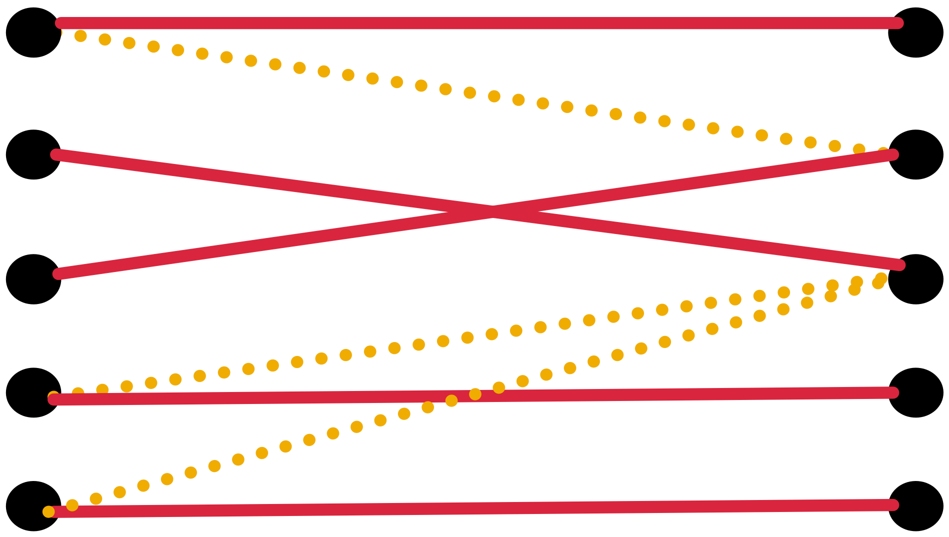
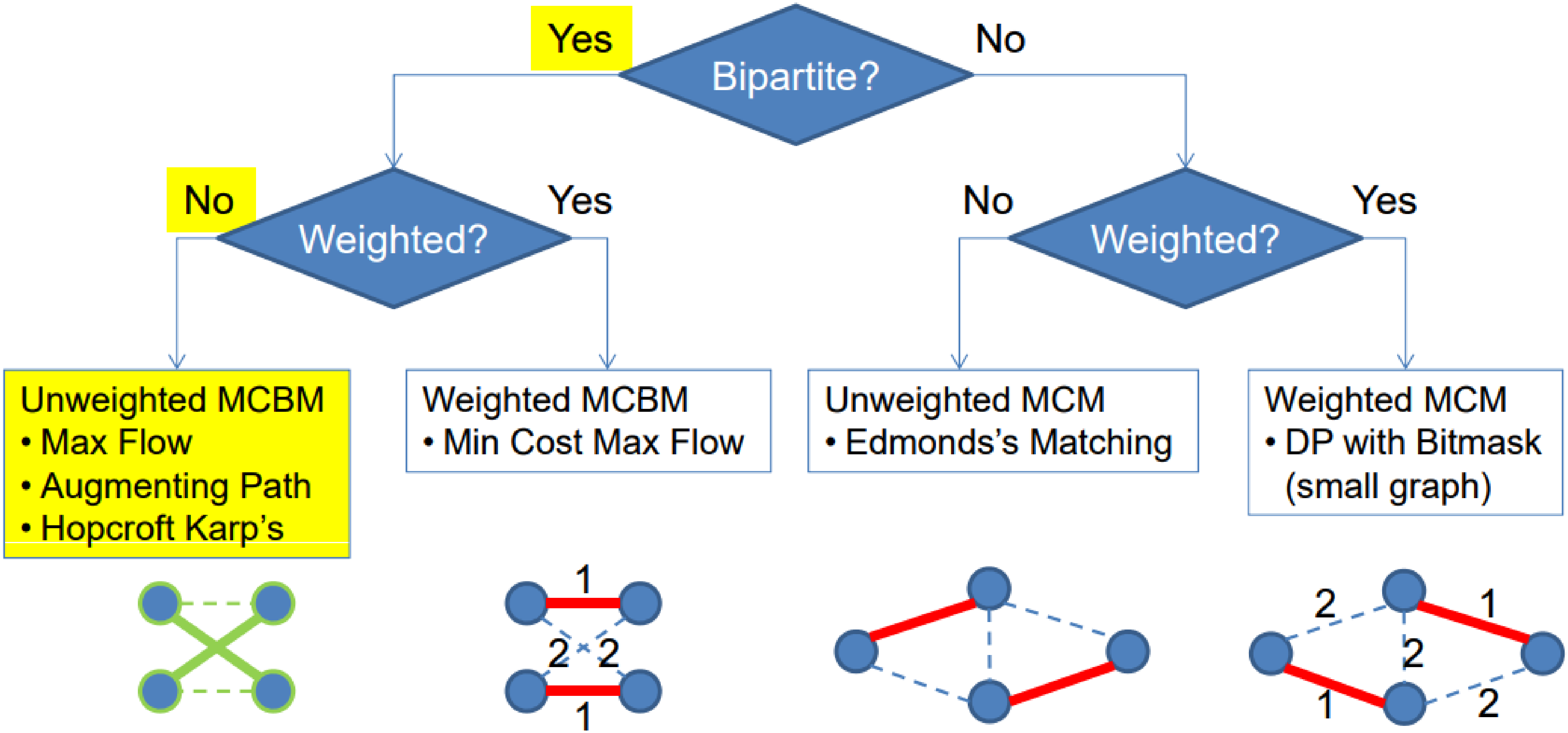


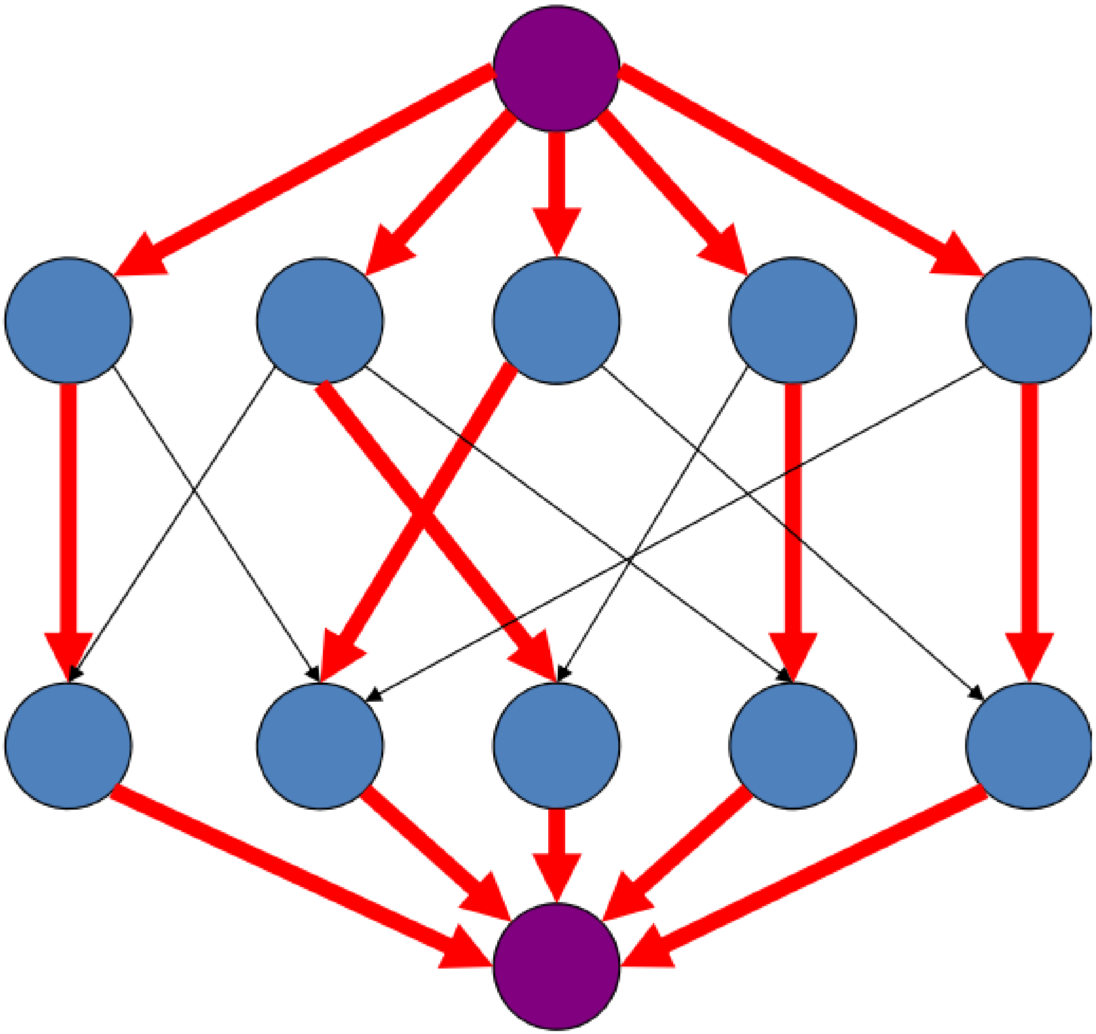
Weighted Maximum Cardinality Matching

- ***Weighted*** MCM involve finding the maximum/minimum MCM among all possible MCMs in a graph with weighted edges.

Graph Matching Solutions



A Max Flow Solution for MCBM



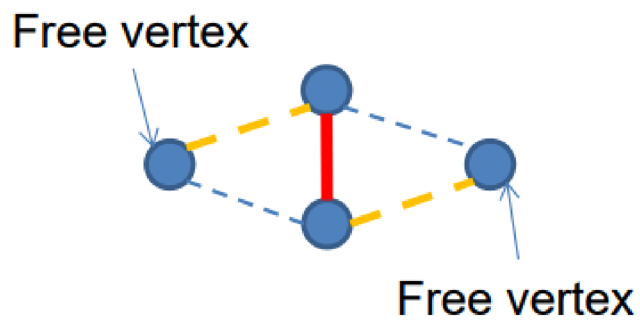
All edges have capacity = 1

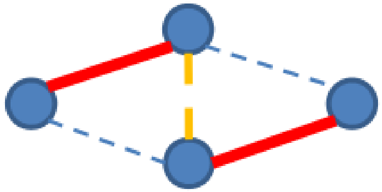
Augmenting Path

- A path $P = v_1, v_2, \ldots, v_k$ is ***alternating*** if the edges v_{i-1},v_i and v_i,v_{i+1} alternate between ***matched*** and ***unmatched***.
- P is **augmenting** if it is alternating and v_1 and v_k are unmatched.

Augmenting Path

- In this graph, the path colored orange(unmatched)-red(matched)-orange is an augmenting path
- We can flip the edge status to red-orange-red and the number of edges in the matching set increases by 1





The Augmenting Path Algorithm

- Lemma (Claude Berge 1957):

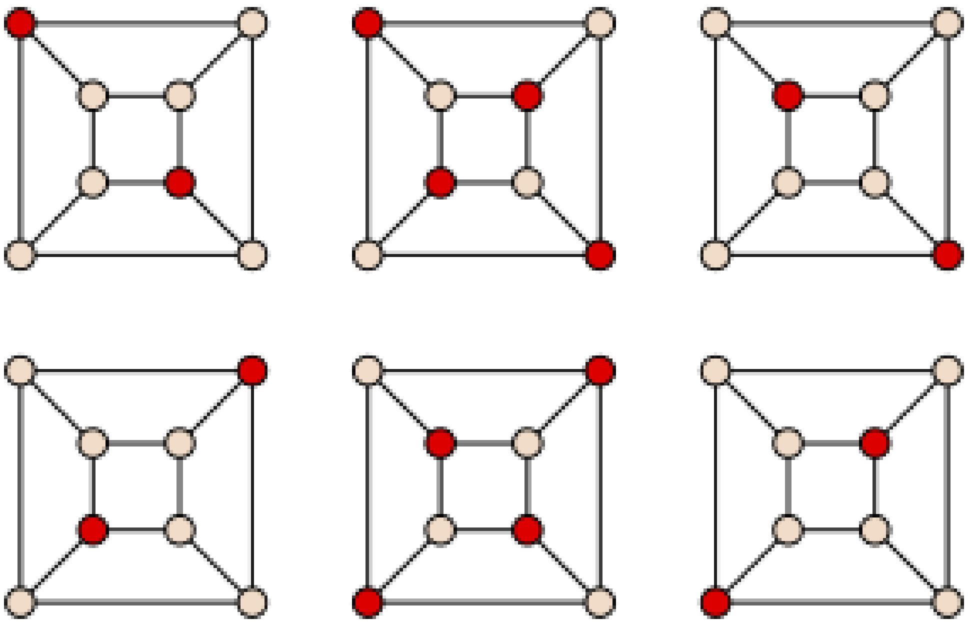
> A matching M in G is maximum
> iff there is no more augmenting path in G

- Augmenting Path Algorithm is a simple $O(V*(V+E)) = O(V^2 + VE) \sim= O(VE)$ implementation of that lemma

Recall Edmond-Karp $O(VE^2)$.

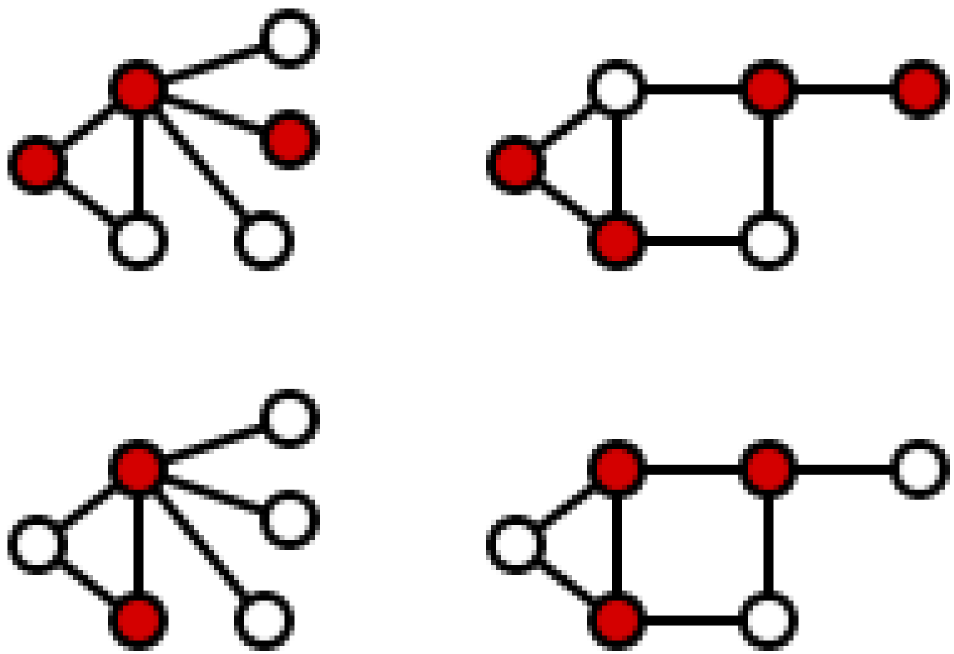
Independent Set

- An ***independent set*** (IS) is a set of vertices in a graph for which no two vertices are adjacent.
- A ***maximal independent set*** (MIS) is such a set that we cannot add additional vertices to.
- A ***maximum independent set*** is a maximum MIS.



Vertex Cover

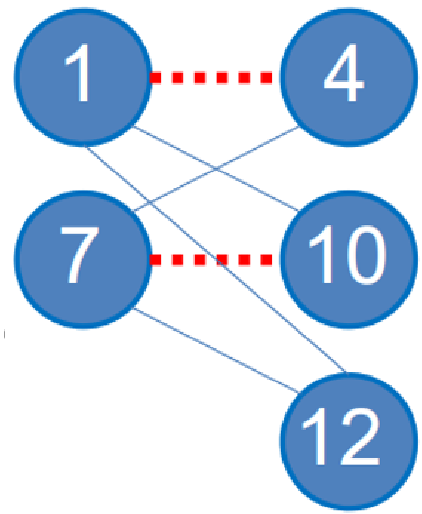
- A ***vertex cover*** in a graph is a set of vertices that includes at least one endpoint of every edge.
- A ***minimum vertex cover*** is a vertex cover of smallest possible size.



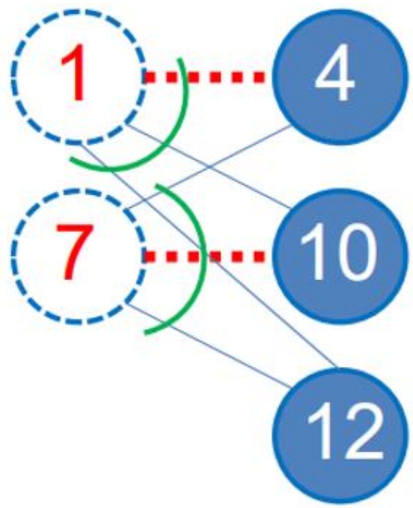
König's Theorem

- ***König's theorem:*** in any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.
  (can be derived from the max-flow min-cut theorem)
- In a bipartite graph, the complement of a maximum independent set is a minimum vertex cover.

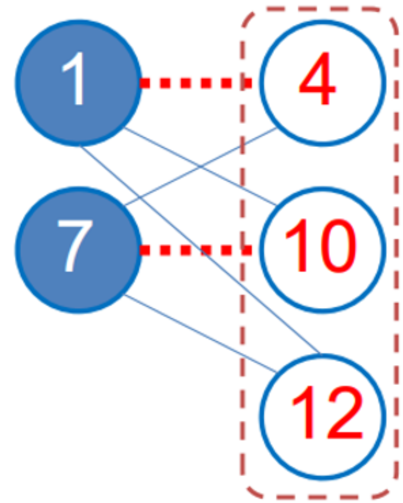
Applications of König’s Theorem



Maximum Cardinality Bipartite Matching



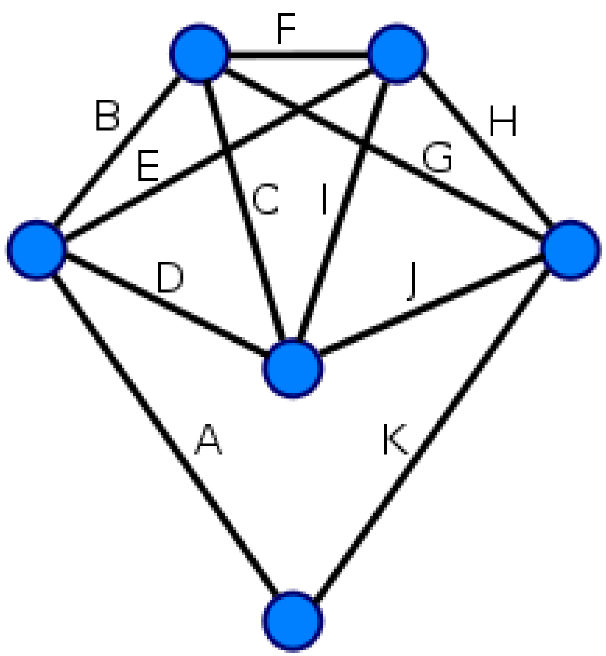
Minimum Vertex Cover
(König’s Theorem)

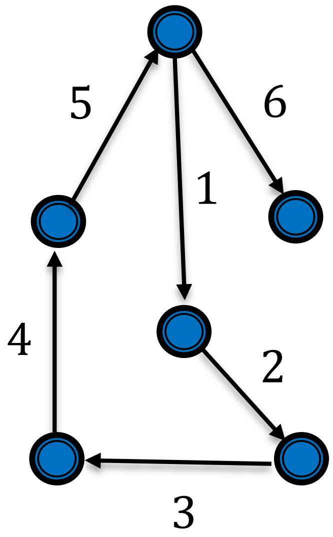


Maximum Independent Set

Eulerian Path

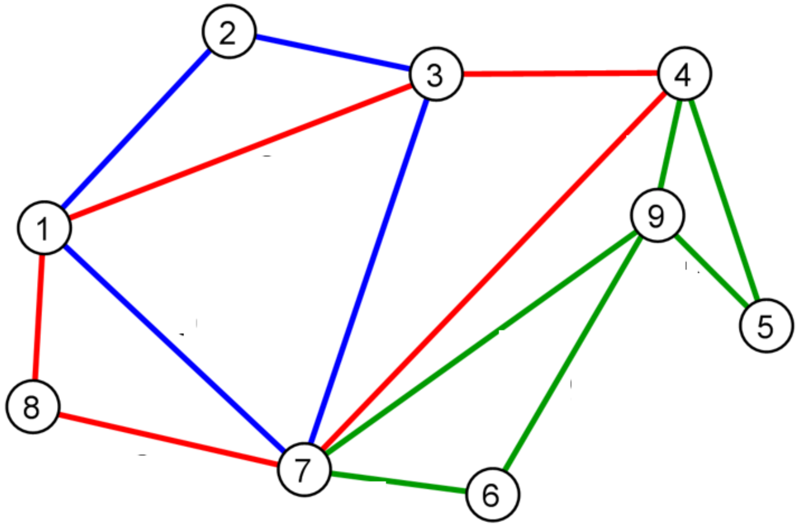
- A **<u>Eulerian path</u>** is a path in a graph that visits every edge exactly once.





Eulerian Path

- **Hierholzer's algorithm**
  - Choose any starting vertex v, and follow a trail of edges from that vertex until returning to v. The tour formed in this way is a closed tour, but may not cover all the vertices and edges of the initial graph.
  - As long as there exists a vertex v that belongs to the current tour but that has adjacent edges not part of the tour, start another trail from v, following unused edges until returning to v, and join the tour formed in this way to the previous tour.
- Time complexity $\boldsymbol{O}(\boldsymbol{E})$.



Source: https://www.geeksforgeeks.org/hierholzers-algorithm-directed-graph/

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  - Hopcroft-Karp's Algorithm
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