TDDD95 Algorithmic Problem Solving

Le 6 – Graphs II

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Outline

- Max Flow (lab 2.6)
- Min Cut (lab 2.7)
- Max Flow Min Cut Theorem
- Min Cost Max Flow (lab 2.8)
- Network Flow Variants

Network Flow

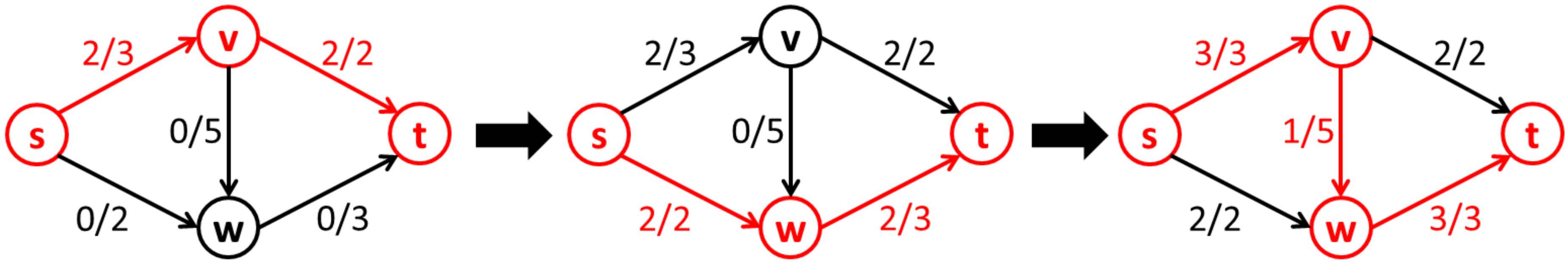
- A ***network*** is a directed graph $G = (V, E)$ with a ***source*** vertex $s \in V$ and a ***sink*** vertex $t \in V$. Also, each edge $(u, v) \in E$ has a ***capacity*** denoted by $c(u, v)$.

  If $(u, v) \notin E$, it is often useful to define $c(u, v) = 0$.

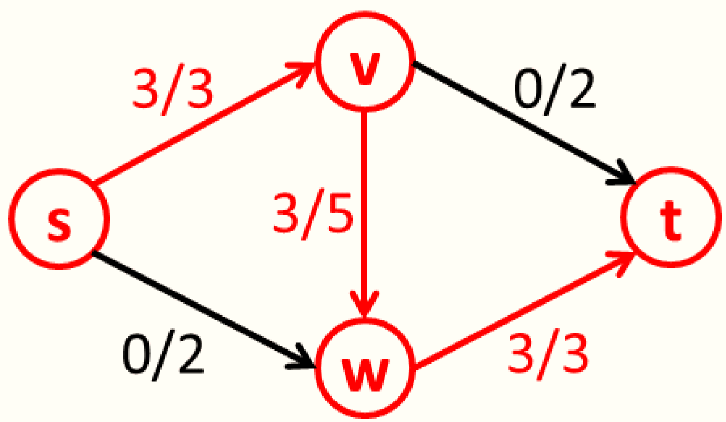
- In a network flow problem, we assign a ***flow*** $f(u, v)$ to all edges $(u, v) \in E$ that satisfy the following properties:
  - **Capacity constraint**: $f(u, v) \leq c(u, v)$ and $f(u, v) \geq 0$ for all $u, v \in V$.
  - **Conservation**: $\sum_{v \in V} f(u, v) = 0$ for all $u \in V \setminus \{s, t\}$.
- The ***flow value*** $F(s)$ from source s is defined as:

$$F(s) = \sum_{v \in V} f(s, v)$$

Maximum Flow



$P_1 = [s, v, t]$ $P_2 = [s, w, t]$ $P_3 = [s, v, w, t]$



Suboptimal solution with ***blocking flow.***

Ford Fulkerson's Method

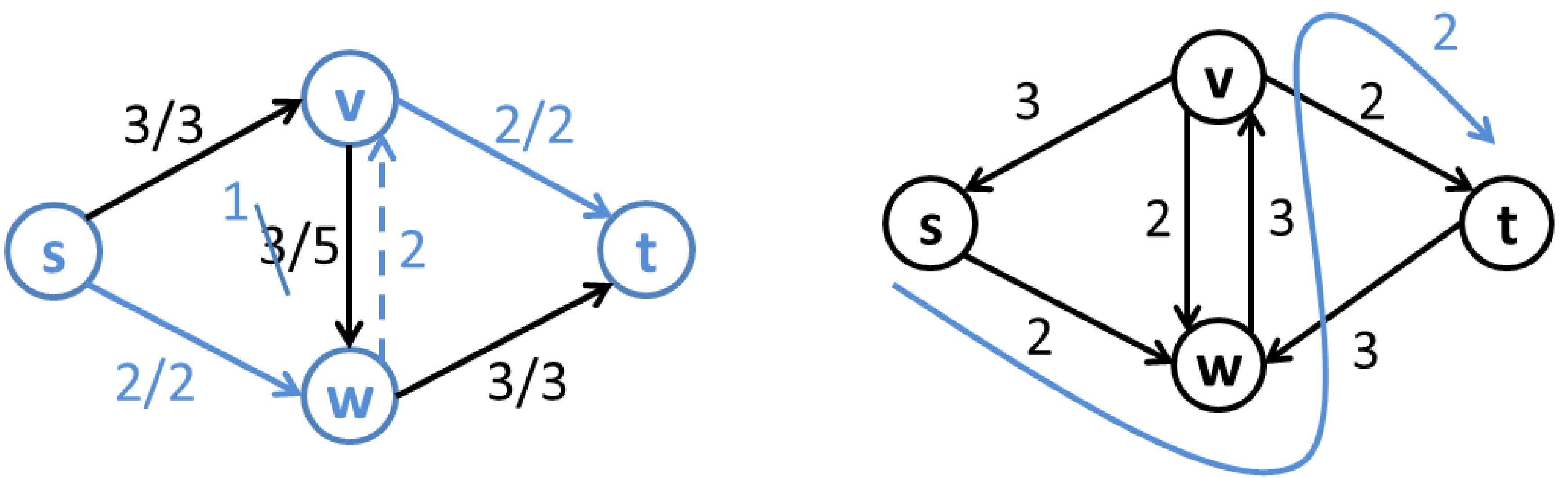
- One Solution: Ford Fulkerson's Method

  - A surprisingly **simple** *iterative* algorithm

Send a flow *f* through path *p* whenever there exists an **augmenting path** *p* from *s* to *t*

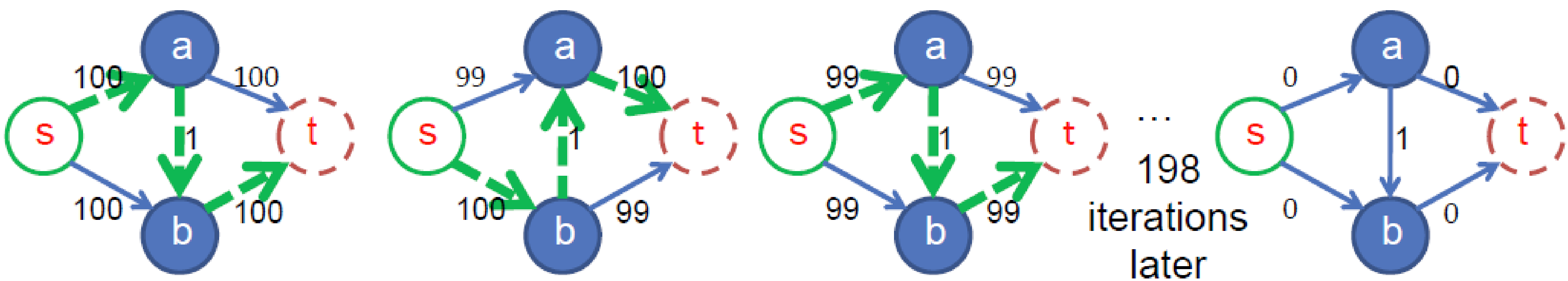
Ford Fulkerson's Method

- The ***residual capacity*** of an arc is the difference between an arc's capacity and its flow: $r(u,v) = c(u,v) - f(u,v)$
- To model this residual capacity, we introduce ***back edges*** that can reverse "bad choices of flow".
- A ***residual network*** is a network with back edges (and residual capacities).
- An ***augmenting path*** is a path from s to t in a residual network that we can add positive flow to.



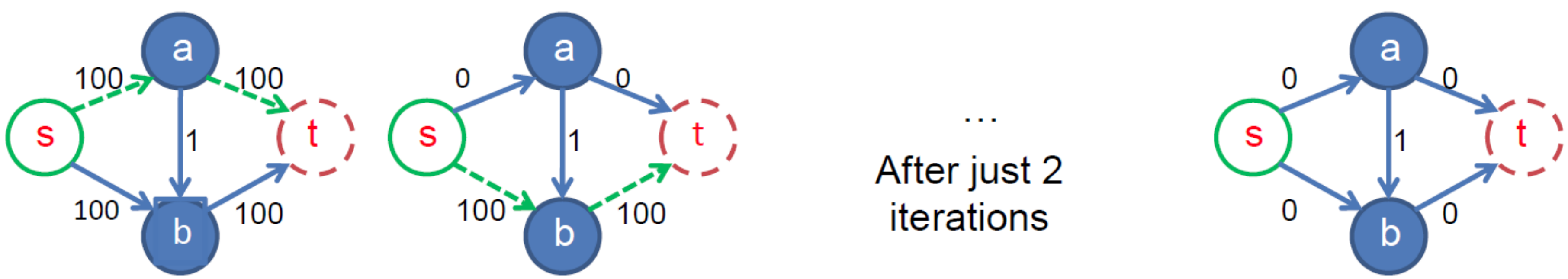
Ford Fulkerson's Method

- DFS implementation of Ford Fulkerson's runs in $O(f_{max}E)$.
- Very slow on certain types of graphs.



The Edmond-Karp Algorithm

- However, BFS implementation runs in $O(VE^2)$.



The Edmond-Karp Algorithm


```
bool hasAugPath(vvi& flow_graph, vvi& adj, vi& parent,
                int s, int t) {
    vector<bool> visited(flow_graph.size());
    queue<int> q;
    q.push(s);
    parent[s] = -1;
    visited[s] = true;

    while (!q.empty()) {
        int now = q.front(); q.pop();

        for (int n : adj[now]) {
            if (not visited[n] and flow_graph[now][n] > 0) {
                parent[n] = now;
                visited[n] = true;
                q.push(n);
                if (n == t) {
                    return true;
                }
            }
        }
    }

    return false;
}
```

The Edmond-Karp Algorithm


```
int MaxFlow(vvi& flow_graph, vvi& adjacency_list, int source, int sink) {
   vi parent(flow_graph.size());

   int maximum = 0;
   while (hasAugPath(flow_graph, adjacency_list, parent, source, sink)) {
      int flow = INF;

      // Search through path for limiting flow
      for (int current = sink; current != source; current = parent[current]) {
         int p = parent[current];
         flow = min(flow, flow_graph[p][current]);
      }

      // Fill path with limiting flow
      for (int current = sink; current != source; current = parent[current]) {
         int p = parent[current];
         flow_graph[p][current] -= flow;
         flow_graph[current][p] += flow;
      }

      maximum += flow;
   }

   return maximum;
}
```

Maximum Flow Algorithms

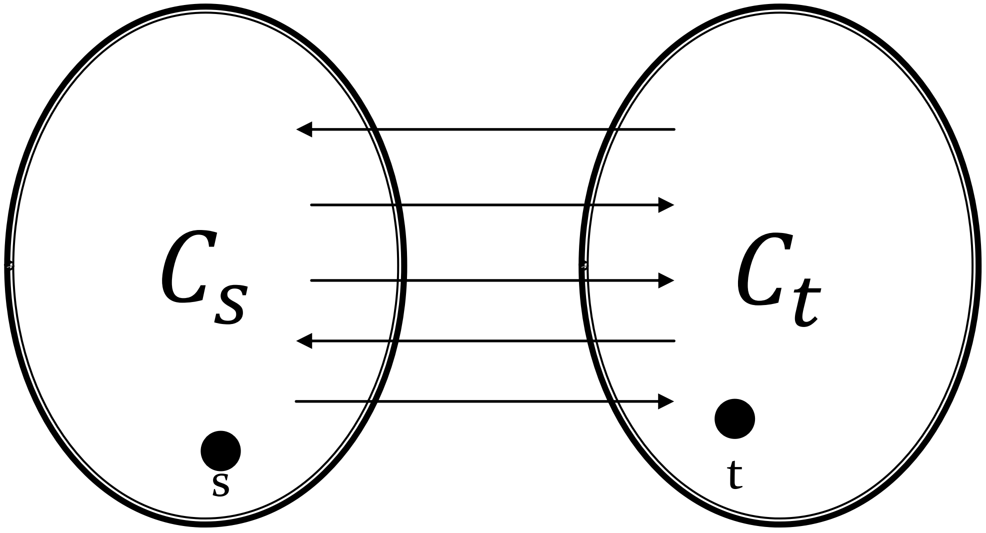
- Ford-Fulkerson with DFS $O(f_{max}E)$
- ***Edmond-Karp*** *(i.e. Ford-Fulkerson with BFS)* $O(VE^2)$
- Dinic's $O(V^2E)$
- Push-relabel (preflow-push) $O(V^3)$
- Binary blocking flow algorithm
  $O(min(V^{2/3}, E^{1/2})\, E\, log(V^2/E)\, log(f_{max}))$

Dinic's Algorithm

- The idea is to reduce our max flow problem to the simple case where all edge capacities are either 0 or 1 (Gabow in 1985 and Dinic in 1973):
  - Scale the problem down somehow by rounding off lower order bits.
  - Solve the rounded problem.
  - Scale the problem back up, add back the bits we rounded off, and fix any errors in our solution.
- In the specific case of the maximum flow problem, the algorithm is:
  - Start with all capacities in the graph at 0.
  - Shift in the higher-order bit of each capacity. Each capacity is then either 0 or 1.
  - Solve this maximum flow problem.
  - Repeat this process until we have processed all remaining bits.
- To scale back up:
  - Start with the maximum flow for the scaled-down problem. Shift the bit of each capacity by 1, doubling all the capacities. If we then double all our flow values, we still have a maximum flow.
  - Increment some of the capacities. This restores the lower order bits that we truncated. Find augmenting paths in the residual network to re-maximize the flow.

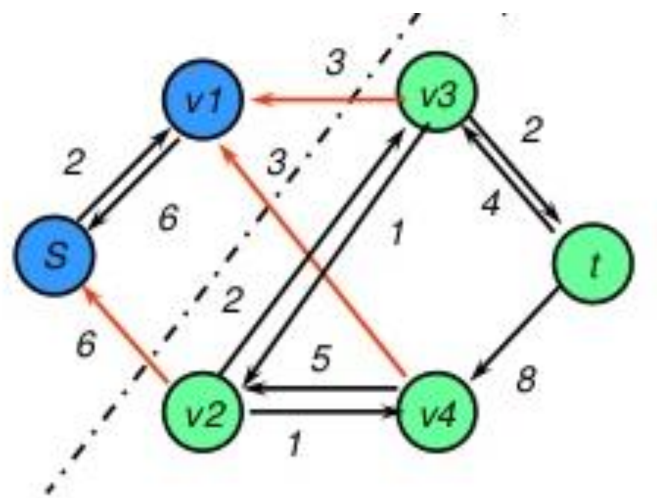
Minimum Cut

- An $s - t$ cut of a network is a partition of its vertices V into 2 groups: C_s and $C_t = V \setminus C_s$, such that $s \in C_s$ and $t \in C_t$.
  - The ***flow*** along cut $\mathrm{C} = (C_s, C_t)$ is defined as:
    $$f(C) = \sum v \in C_s \sum w \in C_t\, f(v,w)$$
  - The ***capacity*** of a cut is defined as:
    $$c(C) = \sum v \in C_s \sum w \in C_t\, c(v,w)$$
- A ***minimum cut*** is a cut with minimum possible capacity.



Minimum Cut

- To find a minimum $s - t$ cut $C = (C_s, C_t)$ of G, compute the maximum flow and find the set of vertices reachable from s in the residual graph, this is the set C_s.
- C_s represents the vertices that appear before the closest "bottleneck" (or "choke") that prevents us from adding more positive flow from s to t.
- The ***max-flow min-cut theorem*** states that the maximum flow is equal to the minimum capacity over all $s - t$ cuts.



Minimum Cost Maximum Flow

- Extend the definition of a network flow with a cost per unit of flow on each edge: $k(v, w) \in R$, where $(v, w) \in E$.
- The cost of a flow f is defined as: $k(f) = \sum e \in E\ f(e) \cdot k(e)$
- A minimum cost maximum flow of a network $G = (V, E)$ is a maximum flow with the smallest possible cost.
  - Note that edges in the residual graph of a network need to have their costs determined carefully. Consider an edge (v, w) with capacity $c(v, w)$, cost per unit flow $k(v, w)$. Let $f(v, w)$ be the flow of the edge. Then the residual graph has two edges corresponding to (v, w). The first edge is (v, w) with capacity $c(v, w) - f(v, w)$ and cost $k(v, w)$, and second edge is (w, v) with capacity $f(v, w)$ and cost $-k(v, w)$.
  - It's clear that minimum cost maximum flow generalizes maximum flow by assigning a cost to every edge.
  - It also generalizes shortest path: if we set each cost equal to its corresponding edge length while assigning the same capacity to every edge.
- The maximum flow with minimum cost can be found using a variation of Edmond-Karp's, in which we use Dijkstra instead of BFS.

Network Flow Variants

- Multi-source, multi-sink max flow
  - Create a super-source/sink with infinite capacity edges to the sources/sinks.
- Vertex capacities
  - Split each vertex into two vertices and add a bi-directional edge with the vertex capacity between them. Remember to change the edges to the vertex.

Summary

- Residual Graphs, Augmenting Paths, Flow Networks
- Max Flow (lab 2.6)
- Min Cut (lab 2.7)
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