TDDD95 Algorithmic Problem Solving

Le 5 – Graphs I

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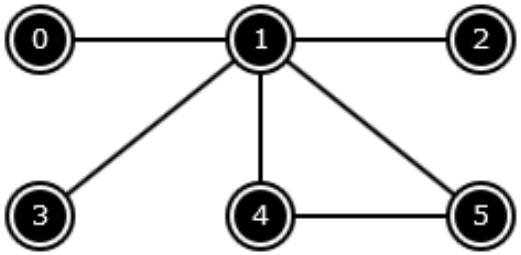
Outline

- Graph Representations (adjacency matrix, adjacency list, edge list)
- Graph Traversal (BFS, DFS, Best First Search)
- Single-Source Shortest Path (Dijkstra, Bellman-Ford; lab 2.1-2.3)
- All-Pairs Shortest Path (Floyd-Warshall, lab 2.4)
- Minimum Spanning Tree (Prim, Kruskal; lab 2.5)

Graph Representations

Graph



Tree | Star | K5 | CP2.2 | CP4.2 | CP4.5 | CP4.8 | Clear

Adjancency Matrix

```
  0 1 2 3 4 5
0 0 1 0 0 0 0
1 1 0 1 1 1 1
2 0 1 0 0 0 0
3 0 1 0 0 0 0
4 0 1 0 0 0 1
5 0 1 0 0 1 0
```

Adjacency List

```
0: 1
1: 0 2 3 4 5
2: 1
3: 1
4: 1 5
5: 1 4
```

Edge List

```
0:  0 1
1:  1 0
2:  1 2
3:  1 3
4:  1 4
5:  1 5
6:  2 1
7:  3 1
8:  4 1
9:  4 5
10: 5 1
11: 5 4
```

Graph Representations

- Adjacency Matrix ($O(V^2)$ space, $O(V)$ to enumerate neighbors)
  - A 2D matrix of weights: int AdjMat[V][V]
  - Good for small or dense graphs
  - Cannot represent multi-graphs
- Adjacency List ($O(V+E)$ space, $O(k)$ to enumerate neighbors)
  - A vector of lists of node and weight pairs:
    typedef vector< vector< pair<int, int> > > AdJList
  - Good for large and sparse graphs
  - Can represent multi-graphs
- Edge List ($O(E)$ space, $O(E)$ to enumerate neighbors)
  - A vector of triples from node, to node, weight:
    typedef vector< pair<int, pair<int, int> > > EdgeList
  - The list of edges is usually sorted
  - Useful for some algorithms, like Kruskal's algorithm for finding MSTs

Graph Traversal

- Depth-First Search (O(V+E) for adjacency lists and O(V^2) for adjacency matrices)
  - Keep a stack of unvisited nodes, initialize it with the start node.
  - Visit the first node in the stack. If it has been visited before then there is a cycle, otherwise add all its children to the stack.
- Breadth-First Search (*O(V+E)* for adjacency lists and *O(V^2)* for adjacency matrices)
  - Keep a queue of unvisited nodes, initialize it with the start node.
  - Visit the first node in the queue. If it has been visited before then there is a cycle, otherwise add all its children to the queue.
  - Visits the nodes of a graph in the order of the distance to the start node.
- Best-First Search
  - Keep a priority queue of unvisited nodes, initialize it with the start node.
  - Visit the first node in the queue (the best next node). If it has been visited before then there is a cycle, otherwise add all its children to the queue.
  - Visits the nodes of a graph in the shortest weighted distance from the start node.
- A* Search
  - Like Best-First Search but with a heuristic that underestimates the distance from the current node to the goal node.

Applications of DFS/BFS

- Finding Connected Components – undirected graph
  - Repeatedly select an unvisited node u and run DFS(u) (or BFS(u)) to find all reachable nodes from u. The number of repetitions is the number of components.
- Flood Fill
  - Adaptation of DFS to count the number of cells in a 2D grid with a particular color/property. Usually on implicit graphs, i.e. 2D grids.
- Topological Sorting of Directed Acyclic Graphs
  - Creates a linear ordering of the nodes in a graph such that a node u comes before the node v if there is an edge $u \rightarrow v$.
  - Tarjan's algorithm is based on DFS
    - After visiting all children, add the current node to the topological list of nodes.
  - Kahn's algorithm is based on BFS
    - When visiting a node, remove it and all its outgoing edges. Add every node v that now have 0 incoming edges.

Applications of DFS/BFS

- Bipartite Graph Check
- Graph Edge Property Check via DFS Spanning Tree
  - Tree edge
  - Back edge
  - Forward/cross edge
- Finding Articulation Points and Bridges in Undirected Graphs
  - An *articulation point* in a graph G is a node whose removal disconnects G.
  - A graph without articulation points is called *biconnected*.
  - A *bridge* is an edge in a graph G whose removal disconnects G.
- Finding Strongly Connected Components in Directed Graphs
  - Tarjan's algorithm
  - 2-SAT

Single Source Shortest Path

- Given a weighted graph *G* and a starting source node *s*, what are the shortest paths from *s* to every other node of *G*?
- SSSP on Unweighted Graphs (or all edges have equal weight)
  - Use BFS (*O(V+E)*)
  - To reconstruct the shortest path keep a vector<int> p with the parent node of each node and generate the path starting from the destination.
- SSSP on Weighted Graphs
  - Dijkstra (*O((V+E) log V)*)
  - Maintain a priority queue with reachable nodes sorted on their total distance from the source (increasing).
  - Greedily select the node *u* with shortest distance *d* from the source and update the shortest distance to *u* according to dist[*u*]=min(dist[*u*], *d*).
  - If dist[*u*] is decreased add all neighboring nodes to the priority queue.

Single Source Shortest Path

- SSSP on Weighted Graphs with Negative Cycles
  - The Dijkstra version described above works even with negative weights, but not with negative cycles.
  - Bellman-Ford (*O(VE)* for adjacency lists)
  - Idea: Relax all *E* edges *V*-1 times.
  - Basically do a DP over every edge *(u,v)* *V*-1 times and update as follows:
    - dist[v] = min(dist[v], dist[u] + w[u][v])
  - It is possible to determine if there are negative cycles since if there are no negative cycles then after *V*-1 iterations no relaxations should be possible. This means that if on the *V*th iteration relaxations are possible, there is a negative cycle.

All-Pairs Shortest Path

- Given a weighted graph G, find the shortest path between every pair of nodes u and v.
- Floyd-Warshall ($O(V^3)$)
  - For every triple k, i, j compute
    AdjMat[i][j] = min(AdjMat[i][j], AdjMat[i][k] + AdjMat[k][j])
- Applications of APSP
  - Solving the SSSP on small weighted graphs
  - Transitive closure
  - Minimax and maximin
    AdjMat[i][j] = min(AdjMat[i][j], max(AdjMat[i][k], AdjMat[k][j]))
  - Finding the cheapest/negative cycle
  - Finding the diameter of a graph
  - Finding the strongly connected components of a graph

Summary SSSP/APSP

Graph Criteria	BFS O(V+E)	Dijkstra O((V+E) log V)	Bellman-Ford O(VE)	Floyd Warshall $O(V^3)$
Max size	V, E <= 10M	V, E <= 300K	VE <= 10M	V <= 400
Unweighted	Best	Ok	Bad	Bad in general
Weighted	WA	Best	Ok	Bad in general
Negative weight	WA	Our variant ok	Ok	Bad in general
Negative cycle	Cannot detect	Cannot detect	Can detect	Can detect
Small graph	WA if weighted	Overkill	Overkill	Best

Minimum Spanning Tree

- Given a connected, undirected and weighted graph *G*, select a subset of edges $E' \in E$ such that the graph *G* is connected and the total weight of the selected edges *E'* is minimal.
- This corresponds to finding a *minimal spanning tree* of G, i.e. a tree which connects all the nodes in the graph G and whose total weight is minimal.
- Kruskal's Algorithm (*O(E log V)*)
  - Sort the edges based on non-decreasing weight (use an EdgeList).
  - Greedily add the next edge unless if forms a cycle (use UnionFind)
- Prim's Algorithm (*O(E log V)*)
  - When visiting a node, add all next nodes sorted by the weight of the edge in a priority queue.
  - Repeatedly take the minimum weight node, which hasn't been visited before (keep track of taken nodes in an array).
- Variants
  - Maximum Spanning Tree
  - "Minimum" Spanning Subgraph
  - Minimum Spanning Forest (with K trees)
  - Minimax and Maximin path between nodes *i* and *j*

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