Algorithm Design and Analysis

CSE 565

LECTURE 14

Divide and Conquer

- Fast Fourier Transform

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9/24/2008

A. Smith; based on slides by E. Demaine, C. Leiserson, S. Raskhodnikova, K. Wayne

Midterm Exam #1

- Willard Building Room 76
- Tuesday night, September 30, 8:15pm

- You may bring: one (1)
  double-sided,
  hand-written 8.5” x 11” sheet of notes
  on colored paper
  - **Hint:** use its preparation as a study aid

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Fast Fourier Transform: Applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, control systems, signal processing, speech recognition, data compression, image processing.
- DVD, JPEG, MP3, MRI, CAT scan.
- Numerical solutions to Poisson's equation.

> The FFT is one of the truly great computational developments of this [20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. -Charles van Loan

Fast Fourier Transform: Brief History

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

Importance not fully realized until advent of digital computers.

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add: O(n) arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate: O(n) using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

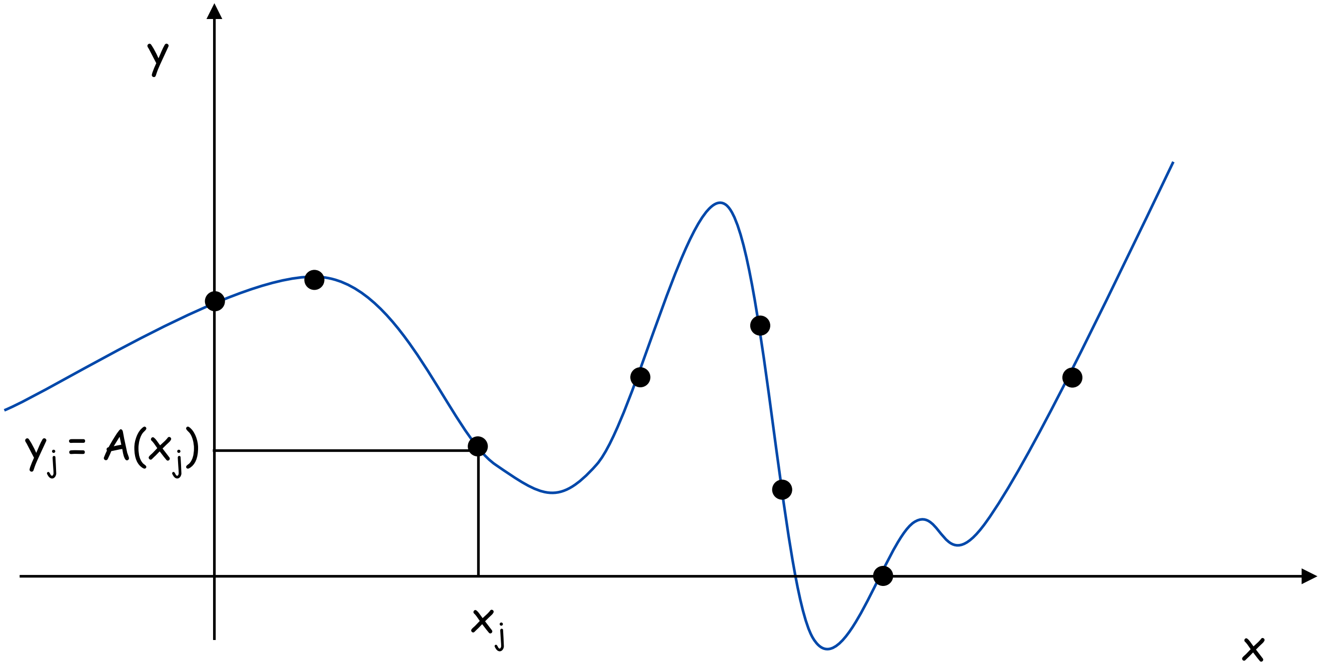
Multiply (convolve): $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

Fundamental theorem of algebra. [Gauss, PhD thesis] *A degree n polynomial with complex coefficients has n complex roots.*

Corollary. *A degree n-1 polynomial A(x) is uniquely specified by its evaluation at n distinct values of x.*



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): \; (\mathrm{x}_0, y_0), \; \ldots, (\mathrm{x}_{\text{n-1}}, y_{n-1})$$
$$B(x): \; (\mathrm{x}_0, \; z_0), \; \ldots, (\mathrm{x}_{\text{n-1}}, z_{n-1})$$

Add: O(n) arithmetic operations.

$$A(x) + B(x): \;\; (\mathrm{x}_0, y_0 + z_0), \ldots, (\mathrm{x}_{\text{n-1}}, y_{n-1} + z_{n-1})$$

Multiply: O(n), but need 2n-1 points.

$$A(x) \times B(x): \;\; (\mathrm{x}_0, y_0 \times z_0), \ldots, (\mathrm{x}_{\text{2n-1}}, y_{2n-1} \times z_{2n-1})$$

Evaluate: O(n^2) using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

Representation	Multiply	Evaluate
Coefficient	$O(n^2)$	$O(n)$
Point-value	$O(n)$	$O(n^2)$

Goal. Make all ops fast by efficiently converting between two representations.

$$\begin{tikzcd}
a_0, a_1, \ldots, a_{n-1} \arrow[r, shift left] & (x_0, y_0), \ldots, (x_{n-1}, y_{n-1}) \arrow[l, shift left]
\end{tikzcd}$$

coefficient representation

point-value representation

Converting Between Two Polynomial Representations: Brute Force

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots , x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$O(n^2)$ for matrix-vector multiply

$O(n^3)$ for Gaussian elimination

Vandermonde matrix is invertible iff x_i distinct

Point-value to coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x A_{odd}(x^2)$.

Intuition. Choose two points to be ±1.

- $A(1) = A_{even}(1) + 1 A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1 A_{odd}(1)$.

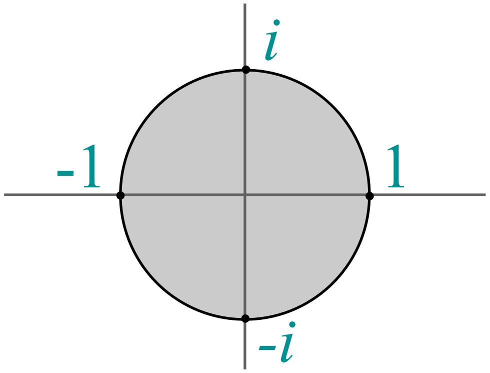
Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.



Intuition. Choose four points to be ±1, ±i.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1)$.
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1)$.

Can evaluate polynomial of degree ≤ n at 4 points by evaluating two polynomials of degree ≤ $\frac{1}{2}$n at 2 points.

Discrete Fourier Transform

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots , x_{n-1}$.

Key idea: choose $x_k = \omega^k$ where ω is principal n^th^ root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$\uparrow$ Discrete Fourier transform

$\uparrow$ Fourier matrix F_n

Roots of Unity

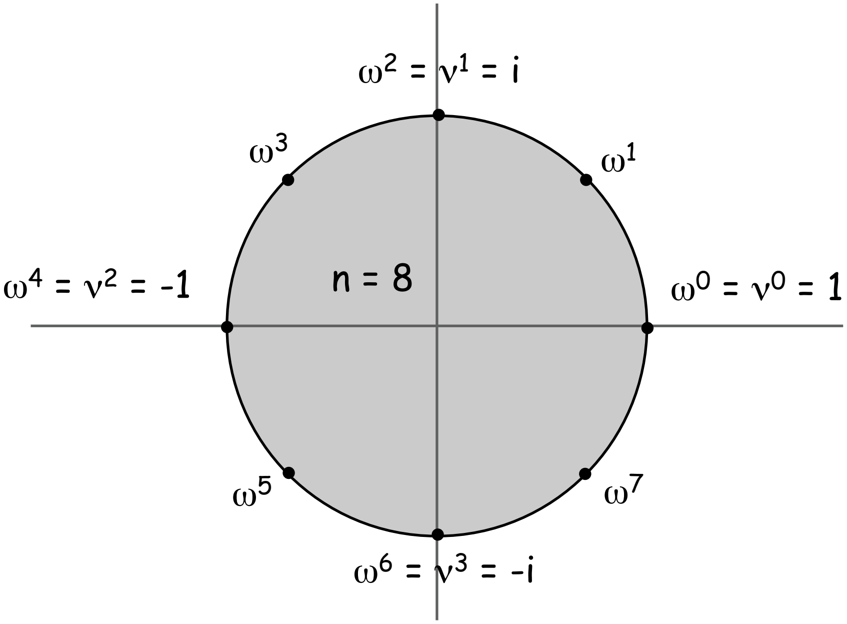
Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.

Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at its nth roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n/2-2}\, x^{(n-1)/2}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n/2-1}\, x^{(n-1)/2}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate degree $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}$nth roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+n/2}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\uparrow$

$\omega^{k+n/2} = -\omega^k$

$\nu^k = (\omega^k)^2 = (\omega^{k+n/2})^2$

FFT Algorithm

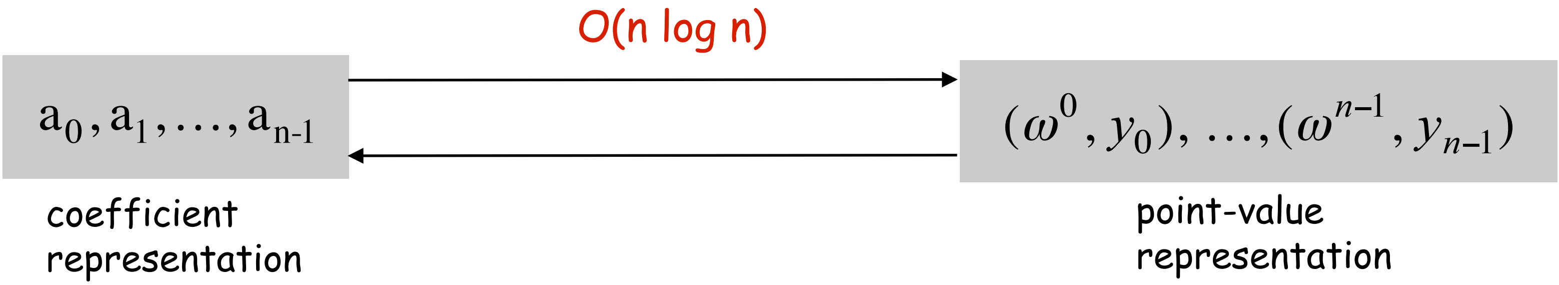
```
fft(n, $a_0,a_1,\ldots,a_{n-1}$) {
   if (n == 1) return $a_0$

   $(e_0,e_1,\ldots,e_{n/2-1}) \leftarrow$ FFT(n/2, $a_0,a_2,a_4,\ldots,a_{n-2}$)
   $(d_0,d_1,\ldots,d_{n/2-1}) \leftarrow$ FFT(n/2, $a_1,a_3,a_5,\ldots,a_{n-1}$)

   for k = 0 to n/2 - 1 {
      $\omega^k \leftarrow e^{2\pi ik/n}$
      $y_k \leftarrow e_k + \omega^k d_k$
      $y_{k+n/2} \leftarrow e_k - \omega^k d_k$
   }

   return $(y_0,y_1,\ldots,y_{n-1})$
}
```

FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the nth roots of unity in O(n log n) steps.

assumes n is a power of 2

Running time. $T(2n) = 2T(n) + O(n) \Rightarrow T(n) = O(n \log n)$.



O(n log n)

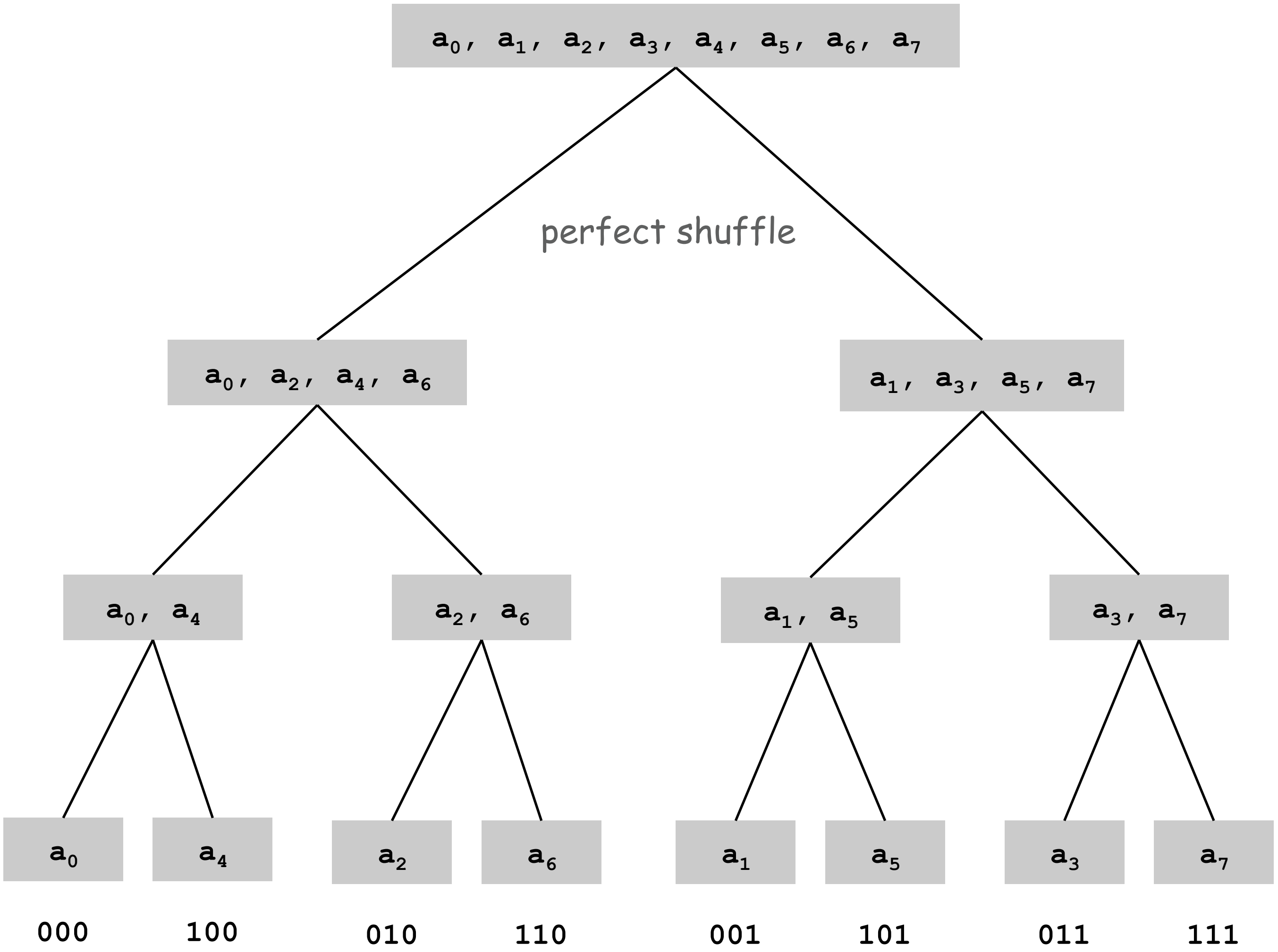
$a_0, a_1, \ldots, a_{n-1}$

coefficient representation

$(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$

point-value representation

Recursion Tree



$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$

perfect shuffle

a_0, a_2, a_4, a_6

a_1, a_3, a_5, a_7

a_0, a_4

a_2, a_6

a_1, a_5

a_3, a_7

a_0 a_4 a_2 a_6 a_1 a_5 a_3 a_7

000 100 010 110 001 101 011 111

"bit-reversed" order

Point-Value to Coefficient Representation: Inverse DFT

Goal. Given the values $y_0, \ldots, y_{n-1}$ of a degree n-1 polynomial at the n points $\omega^0, \omega^1, \ldots, \omega^{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

↑ Inverse DFT

↑ Fourier matrix inverse $(F_n)^{-1}$

Inverse FFT

Claim. Inverse of Fourier matrix is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal nth root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n \, G_n\right)_{k\,k'} \;=\; \frac{1}{n} \sum_{j=0}^{n-1} \omega^{k\,j}\, \omega^{-j\,k'} \;=\; \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} \;=\; \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal nth root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{\,k\,j} \;=\; \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \;\Rightarrow\;$ sums to n.
- Each nth root of unity ω^k is a root of $x^n - 1 = (x - 1)\,(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \;\Rightarrow\;$ sums to 0. ▪

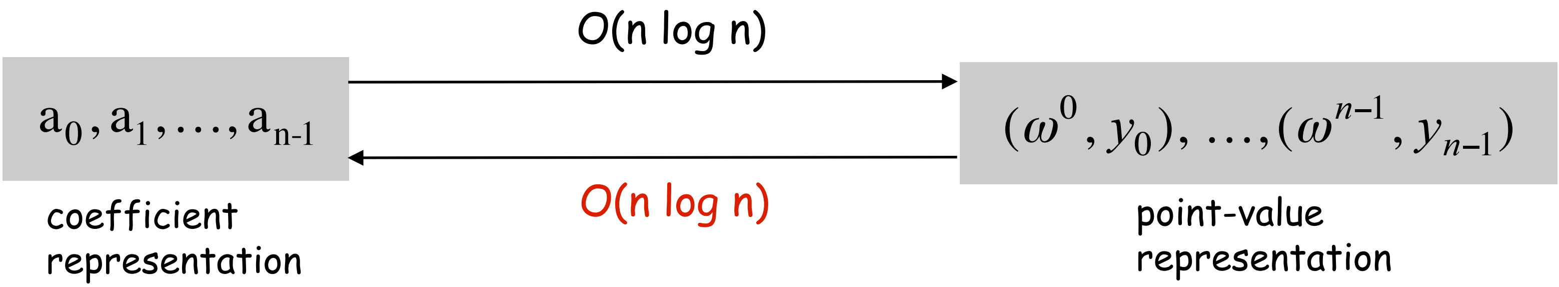
Inverse FFT: Algorithm

```
ifft(n, $a_0, a_1, \ldots, a_{n-1}$) {
   if (n == 1) return $a_0$

   $(e_0, e_1, \ldots, e_{n/2-1})$ ← FFT(n/2, $a_0, a_2, a_4, \ldots, a_{n-2}$)
   $(d_0, d_1, \ldots, d_{n/2-1})$ ← FFT(n/2, $a_1, a_3, a_5, \ldots, a_{n-1}$)

   for k = 0 to n/2 - 1 {
      $\omega^k$ ← $e^{-2\pi ik/n}$
      $y_k$ ← $(e_k + \omega^k d_k)$ / n
      $y_{k+n/2}$ ← $(e_k - \omega^k d_k)$ / n
   }

   return $(y_0, y_1, \ldots, y_{n-1})$
}
```

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the nth roots of unity in O(n log n) steps.

assumes n is a power of 2



O(n log n)

$a_0, a_1, \ldots, a_{n-1}$

$(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$

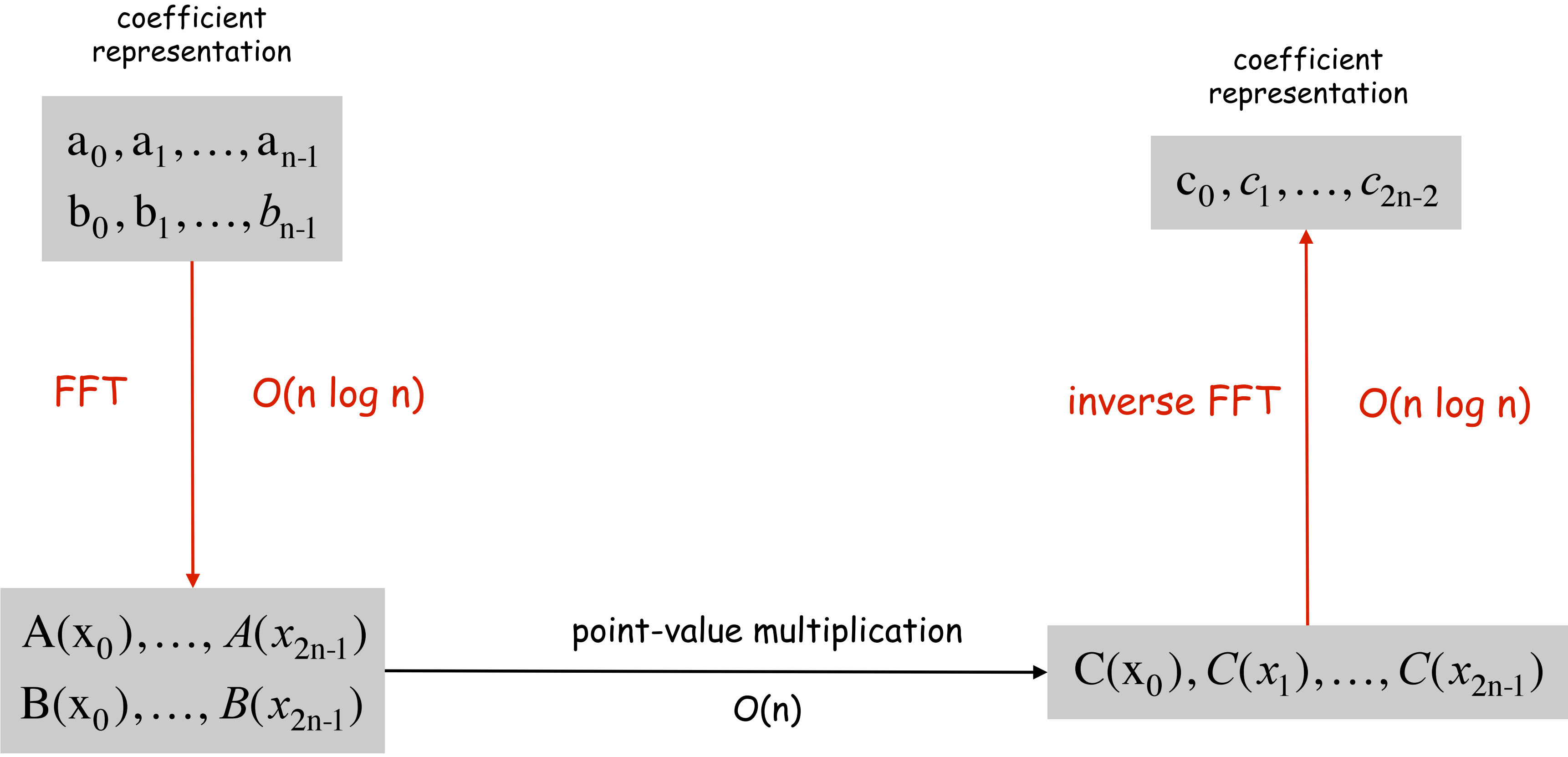
O(n log n)

coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in O(n log n) steps.



coefficient representation

$a_0, a_1, \ldots, a_{n-1}$

$b_0, b_1, \ldots, b_{n-1}$

FFT

O(n log n)

$A(x_0), \ldots, A(x_{2n-1})$

$B(x_0), \ldots, B(x_{2n-1})$

point-value multiplication

O(n)

$C(x_0), C(x_1), \ldots, C(x_{2n-1})$

inverse FFT

O(n log n)

coefficient representation

$c_0, c_1, \ldots, c_{2n-2}$

Integer Multiplication

Integer multiplication. Given two n bit integers a = $a_{n-1} \ldots a_1 a_0$ and b = $b_{n-1} \ldots b_1 b_0$, compute their product c = a × b.

Convolution algorithm.

- Form two polynomials.
- Note: a = A(2), b = B(2).
- Compute C(x) = A(x) × B(x).
- Evaluate C(2) = a × b.
- Running time: O(n log n) complex arithmetic steps.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Theory. [Schönhage-Strassen 1971] O(n log n log log n) bit operations.
[Martin Fürer (Penn State) 2007] O(n log n $2^{\log^* n}$) bit operations.

Practice. [GNU Multiple Precision Arithmetic Library] GMP proclaims to be "the fastest bignum library on the planet." It uses brute force, Karatsuba, and FFT, depending on the size of n.