Lecture 18

Splay-trees, hashing, skip-lists

TDDD86: DALP

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Content

Innehåll

1 Splay-trees

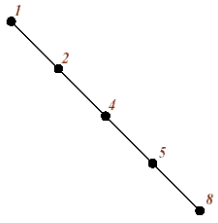
Binary search trees are not unique

Remember the binary search trees:

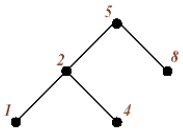
- Easy to insert and remove elements, but...
- "balance" is determined by the order of insertions and deletions.

Combine with heuristics "hold recently used elements first" for lists?

- Often, use elements close to the root!



insert: 1,2,4,5,8



insert: 5,2,1,4,8

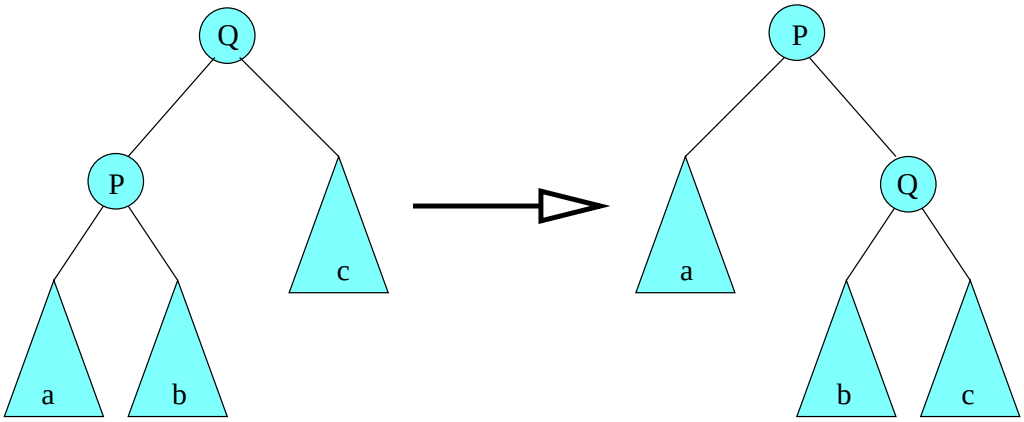
The operation splay(k)

A splay tree is a self-adjusting binary search tree with the additional property that recently accessed elements are quick to access again

- Perform a normal search for k, remember the nodes we pass...
- Mark the last node we examine with P
  - If k exists in T, it must be in node P,
  - otherwise, P is a parent of an empty tree.
- Return to the root and do a rotation at each node to move P upwards in the tree... (3 cases)

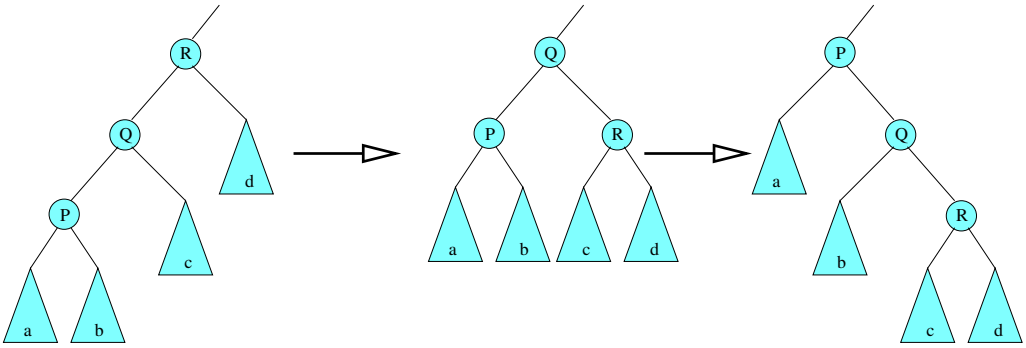
The operation splay(k)

- zig: parent(P) is the root: rotate around P



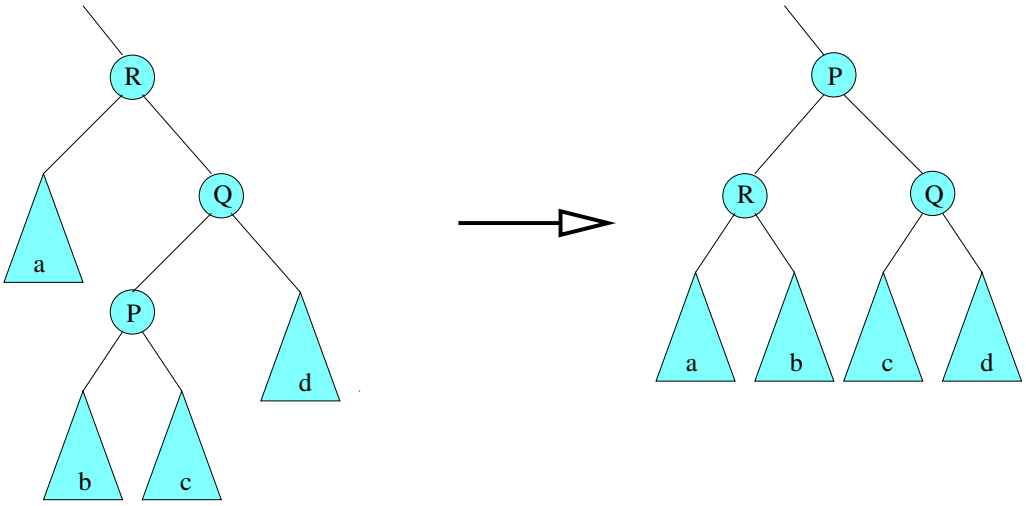
The operation splay(k)

- zig-zig: P and parent(P) are both left children (or both right children): perform 2 rotations to move up P



The operation splay(k)

- zig-zag: One of P and parent(P) is a left child and the other is a right child or vice versa: perform 2 rotations in different directions.



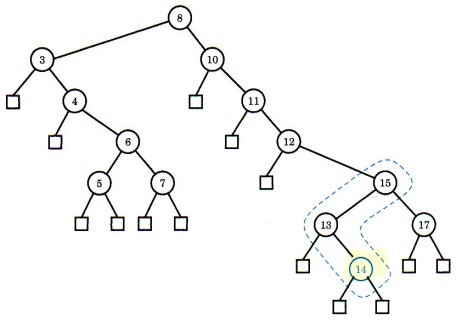
Note that these rotations can increase the height of the tree!

find and insert

function FIND(k, T)
  SPLAY(k, T)
  if KEY(ROOT$(T)) = k$ **then return** (k, v)
  else return null

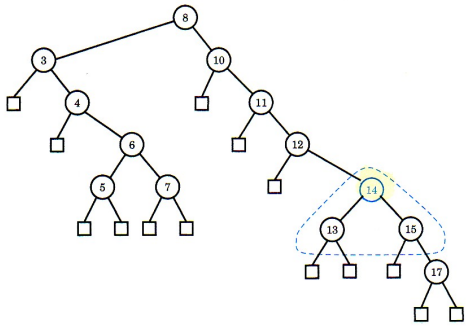
function INSERT(k, v, T)
  insert (k, v) in a binary search tree
  SPLAY(k, T)

Example: insertion of 14



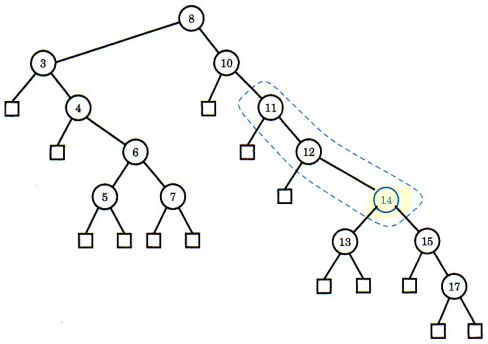
(a)

Example: insertion of 14



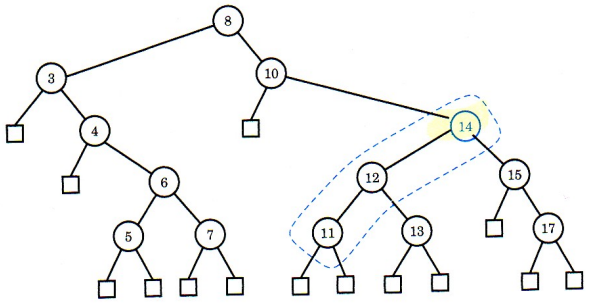
(b)

Example: insertion of 14



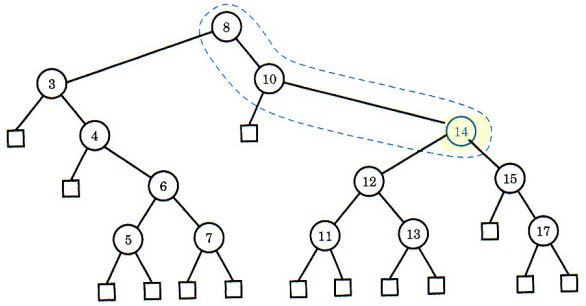
(c)

Example: insertion of 14



(d)

Example: insertion of 14



(e)

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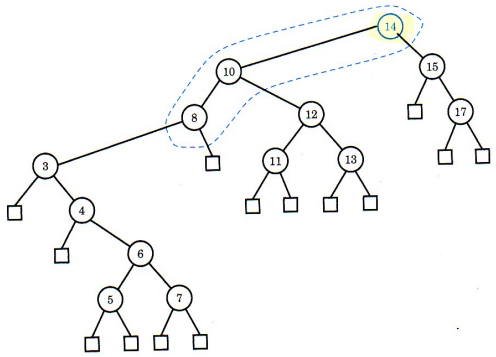
Example: insertion of 14



(f)

18.14

delete

function DELETE(k, T)
  if k is located in a leaf **then**
    do SPLAY to the parent of the leaf
  else if k is located in an internal node **then**
    replace the node with its predecessor in the order
    do SPLAY to the parent of the predecessor

It is of course possible to use the successor in the order also.

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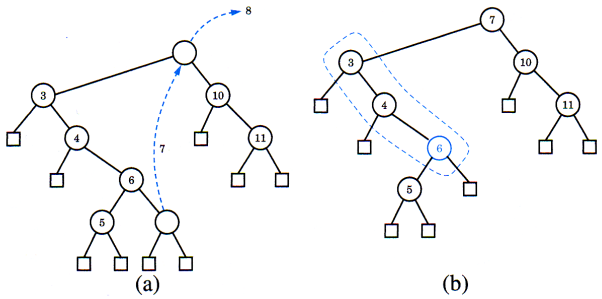
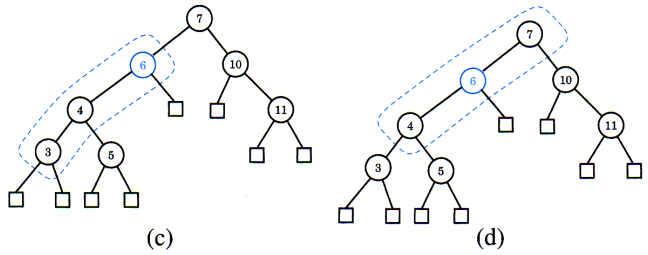
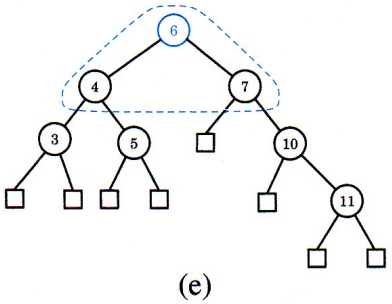
Example: removing 8



Example: removing 8



Example: removing 8



(e)

18.18

Performance

- Each operation may need to be carried out in a totally unbalanced tree
  - thus, no guarantee for $O(\log n)$ time in the worst case

- The amortized time is logarithmic
  - Each sequence of m operations performed on an initially empty tree lasts for $O(m \log m)$ time.
  - thus, the *amortized* cost/time for an operation is $O(\log n)$ although individual operations can behave much worse

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2 Hash tables

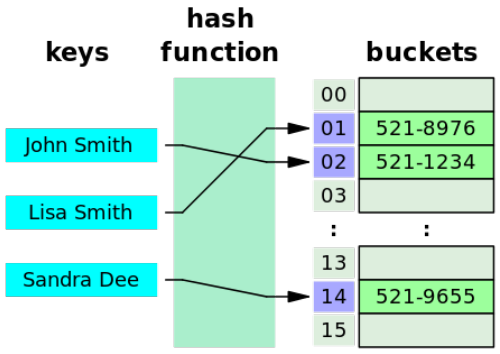
Can we find something better?

yes, with the help of *hash tables*

- Idea: given a table $T[0,\ldots,max]$ storing elements ... ...*find a suitable table index* for each element
- Find a function h such that $h(key) \in [0,\ldots,max]$ and (ideally) $k_1 \neq k_2 \Rightarrow h(k_1) \neq h(k_2)$
- Store each pair key-value (k,v) in $T[h(k)]$

18.20

Hash tables



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Hash tables

- In practice, hash functions are not given unique values (they are not *injective*)
- We need collision handling

... and

- We need to find a good hash function

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2.1 Collision Handling

Collision Handling

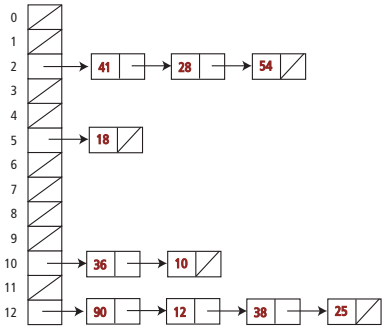
Collision occurs when at least 2 keys are hashed to the same slot. 2 principles to handle collisions:

- *Linking*: keep colliding data in linked lists
  - *Separate linking*: having the linked lists off the table
  - *Collective linking*: store all data *in* the table
- *Open adressing*: store all data *in* the table *and* let some algorithm to decide which index to use in a collision

[Eng: Separate Chaining, Coalesced Chaining, Open Addressing] 18.23

Example: hashing with separate linking

- Hash table of size 13
- Hash function h with $h(k) = k \bmod 13$
- Sore 10 integer keys: 54, 10, 18, 25, 28, 41, 38, 36, 12, 90



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Separate linking: find

Given: key k, hash table T, hash function h

- calculate $h(k)$
- look for k in the pointing out list $T[h(k)]$

Notation: probe= an access to the linked list

- 1 probe to access the list header (if non-empty)
- 1+1 probing to access the contents of the first list element
- 1+2 probing to access the contents of the second list element
- ...

A probe (to follow a pointer) takes a constant time. 18.25

Separate linking: failed lookups

- n data elements
- m entries in the table

Worst case:

- all data elements have the hash value: $P = 1 + n$

Average case:

- hash values uniformly distributed over m:
- average length α of the list: $\alpha = n/m$
- $P = 1 + \alpha$

18.26

Separate linking: successful lookups

Average case:

- accessing $T[h(k)]$ (the first element of a list L): 1
- traversing $L \Rightarrow k$ found after: $|L|/2$
- expected $|L|$ corresponding to α, thus: expected $P = \alpha/2 + 1$

18.27

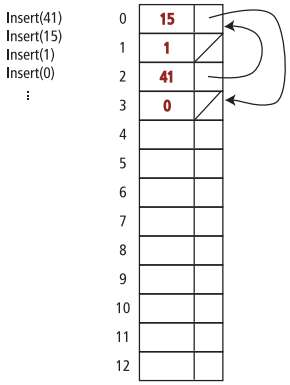
Collective linking: keep an element in the table

- Place the data element in the table
- Extend them with pointers
- Solve collisions by using the first free places

Chains may contain keys with different hash values. . . . . but all keys with the same hash value show up in the same chain

+ better memory utilization - The table can get full -Longer collision chains

Collective linking



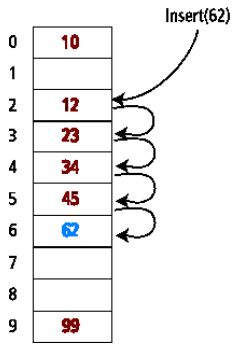
Open addressing

- Store all elements inside the table
- Use a fixed algorithm to find a vacant place

Sequential/linear probing

- desirable hash index $j = h(k)$ (mod 10 here)
- if conflict arises go to the *next* free place
- If the last element of the table reached, go to the first element. . .

- Positions near to each other fill up quickly (*primary clustering*)
- How to make remove(k)?



Open addressing — remove()

The element to be removed can be part of a collision chain – can we determine it?

If it is part of a chain, we can not only remove the element!

- Since all keys are stored, hash all of the the data left?
- Look through the elements after hashing or drawing together, when appropriate, and stop at the first available position. . .
- Ignore – insert a mark deleted"if the next place is not empty. . .

Double hashing – or what to do in collision?

- Second hash function h_2 calculates *increment* in case of conflicts
- When Increment value is out of the table, we use modulo $m = tableSize$

Linear probing is a double hashing with $h_2(k) = 1$ Requirements for h_2:

- $h_2(k) \neq 0$ for all k
- $h_2(k)$ has no common divisor with m for any $k \Rightarrow$ *all* table positions can be reached

A common choice $h_2(k) = q - (k \bmod q)$ for $q < m$, m prime number (i.e. choose a prime number less than the table size!)

2.2 Choosing a hash function

What is a good hash function?

Assuming that k is a natural number.

Hashing provides a uniform distribution of hash values, *but* this depends on the *distribution of keys* in the data to be hashed.

Example: Hashing of the surnames in a group of students

- hash function: ASCII-value of the last letter *bad choice:* the majority of names end with 'n'.

String hashing in Java

`hashCode()` for `String` i Java 1.1

- For long strings: examine just 8-9 evenly placed characters.

```
public int hashCode()
{
   int hash = 0;
   int skip = Math.max(1, length() / 8);
   for (int i = 0; i < length(); i += skip)
      hash = s[i] + (37 * hash);
   return hash;
}
```

- Advantage: save time
- Disadvantage: great potential for bad collision patterns

Suggestion for hash functions

- Memory address
  - Interpreting the memory address where the object to be hashed is stored in as an integer
  - Works well in general, but not good for example for numeric keys or string keys.
- Converting to integer
  - Interpreting the bits of the key as an integer
  - Suitable for keys of shorter length than the number of bits in integers
- Component sum
  - Divide the bits of the key into components of fixed length (eg 16 or 32 bits) and sum the components. (Ignore the overflow.)
  - Suitable for numeric keys of constant length greater than or equal to the number of bits in the integer type.

Suggestion for hash functions

- polynomial accumulation
  - Divide the bits of the key in a sequence of components of fixed length (e.g. 8, 16 or 32 bits)

$$a_0 a_1 .. a_{n-1}$$

  - Evaluate the polynom

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$$

    with a fixed value z. (Ignore overflow.)
  - Very suitable for hashing the strings. (e.g. $z = 33$ gives at most 6 collisions on an amount of 50000 English words.)
- Polynomial $p(z)$ can be evaluated in $O(n)$ time using Horners rule:
  - The following polynomial calculated successively. Each polynomial i the sequence can be calculated in $O(1)$ time from the previous polynomial in the sequence.

$$p_0(z) = a_{n-1}$$

$$p_i(z) = a_{n-i-1} + z p_{i-1}(z) \ (i = 1, 2, \ldots, n-1)$$

- We have $p(z) = p_{n-1}(z)$

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String hashing in Java

`hashCode()` for `String` in Java nowadays

```
public final class String
{
   private final char[] s;
   ...

   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

i:te tecknet i s

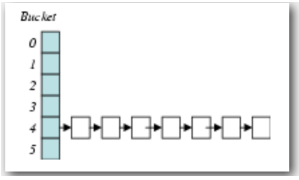
char	Unicode
...	...
'a'	97
'b'	98
'c'	99
...	...

18.37

Algorithmic complexity attacks

Player assumption of a uniform distribution of keys to insert any role in practice?

- Obvious situations: air traffic, nuclear power plants, pacemaker
- Surprisingly situations: denial-of-service-attacker



Elak motståndare lär sig din hashfunktion (t.ex. genom att läsa Javas API) och orsakar en lång kö i enstaka cell vilket påverkar prestanda drastiskt.

Real attack opportunities [Crosby-Wallach 2003]

- Bro server: send carefully selected packages to (DOS-) attack the server with less bandwidth than a dial-up modem.
- Perl 5.8.0: insert carefully selected strings in the associative array.
- Linux 2.4.20-core: save files with thoroughly selected names.

18.38

Algorithmic complexity attacks in Java

- Goal: Find a family of strings with the same hash value.
- Solution: Javas string-API uses 31-base code for string hashing

key	hashCode()
"Aa"	2112
"BB"	2112

key	hashCode()
"AaAaAaAa"	-540425984
"AaAaAaBB"	-540425984
"AaAaBBAa"	-540425984
"AaAaBBBB"	-540425984
"AaBBAaAa"	-540425984
"AaBBAaBB"	-540425984
"AaBBBBAa"	-540425984
"AaBBBBBB"	-540425984

key	hashCode()
"BBAaAaAa"	-540425984
"BBAaAaBB"	-540425984
"BBAaBBAa"	-540425984
"BBAaBBBB"	-540425984
"BBBBAaAa"	-540425984
"BBBBAaBB"	-540425984
"BBBBBBAa"	-540425984
"BBBBBBBB"	-540425984

2^N strängar av längd 2N som hashar till samma värde!

18.39

Hashing by integer division

Let m be the table size

$$h(k) = k \bmod m$$

Avoid

- $m = 2^d$: hashing gives the last d bits in k
- $m = 10^d$: hashing gives the last d numbers

It is usually suggested to use prime numbers for m Examine samples from real data to experiment with hashing parameters.

See `http://burtleburtle.net/bob/hash/doobs.html` for other opinions.

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3 Skip-lists

Skip-lists

- A hierarchical linked list...
- A randomized alternative to implement ADT Dictionary
- Insertion uses randomization (”coin toss”)
- Good performance in the expected case
- Worst case performance of skip-lists occurs very rarely (>250 data element, the risk that the search time is more than 3 times than expected is under 10^{-6})

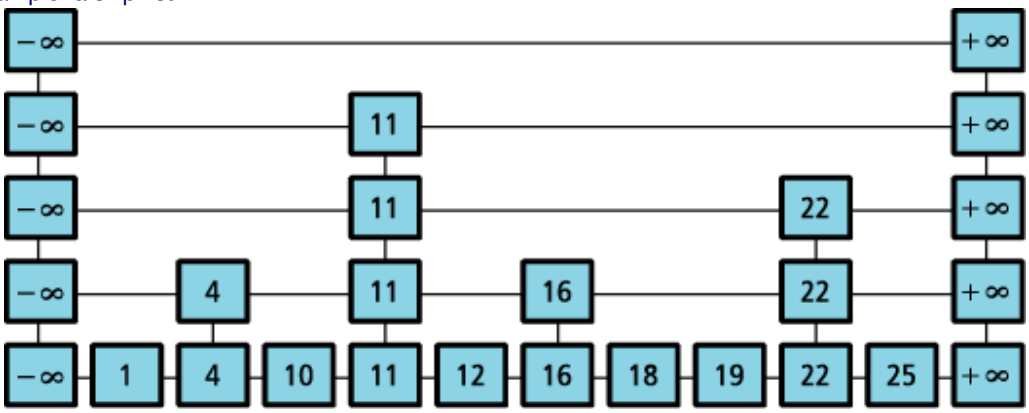
18.41

Skip-list data structure

- Levels $L_1, \ldots, L_h$ of nodes (keys, values)
- Same nodes exist in different -several- levels (tower)
- Special keys: $-\infty$ and $+\infty$ ...less/greater than each real key...
- Several *levels* of doubly linked lists, sparser higher
  - Level 1: all nodes in a doubly-linked list between $-\infty$ and $+\infty$ arranged according to ’<’-relation
  - In average, one half of the nodes exists in L_i and also in L_{i+1}
  - Special keys $-\infty$ and $+\infty$ exist in all levels
  - Only $-\infty$ and $+\infty$ exist in level L_h

18.42

Example: a skip-list



18.43

Search

Search for key k:

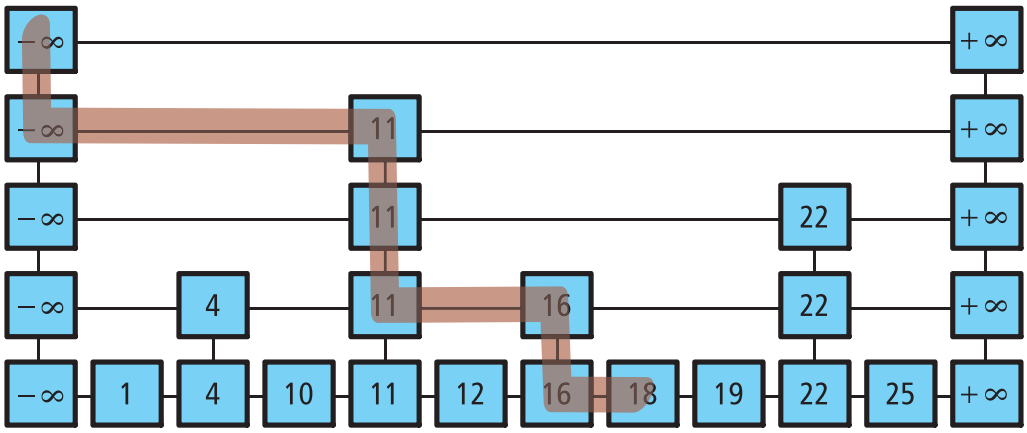
- Follow the list of highest level...
  - Stop before reaching any $k_i > k$ (we risk to miss what we're looking for)
  - If we found the right, return it, otherwise...
- We have been searching in one level:
  - Have we found the key?
  - No, change to next lower level (via "the last tower") and keep looking
  - Returns: the largest key $k_i \leq k$ (which can be $+\infty$)

18.44

Search

Search for key k:

- Similarities with binary search – but for lists
- Example: find(18)



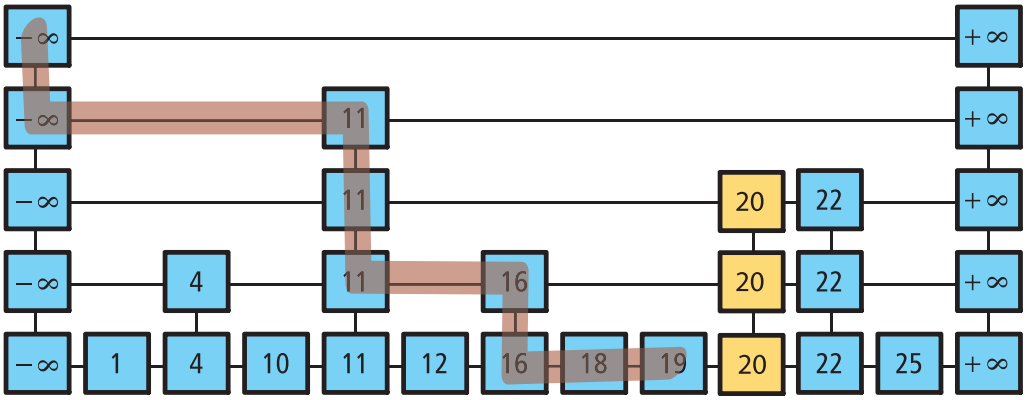
18.45

Insertion

function INSERT(x)
  $P \leftarrow$ FIND(x)
  if $P.value < x$ **then**
    insert a new list node after P
    "toss a coin" to determine how high the "tower" is:
    while "toss"=yes **do**
      increase the tower height with one step
      (increase possibly the height of the skip-list)

18.46

Example: insert(20)



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Removing... and characteristics

- Similar insert:
  - Search
  - if found, remove it and fix the links between the towers
- The worst case time for find, insert and remove in a skip-list with n inserted element is $O(n+h)$
- However, the expected execution time (assuming that the keys are uniformly distributed) is $O(\log n)$ if the search starts at height $\lfloor \log n \rfloor$

18.48