TDDD14 / TDDD85 – Lecture 11

Closure Properties and Pumping Lemma

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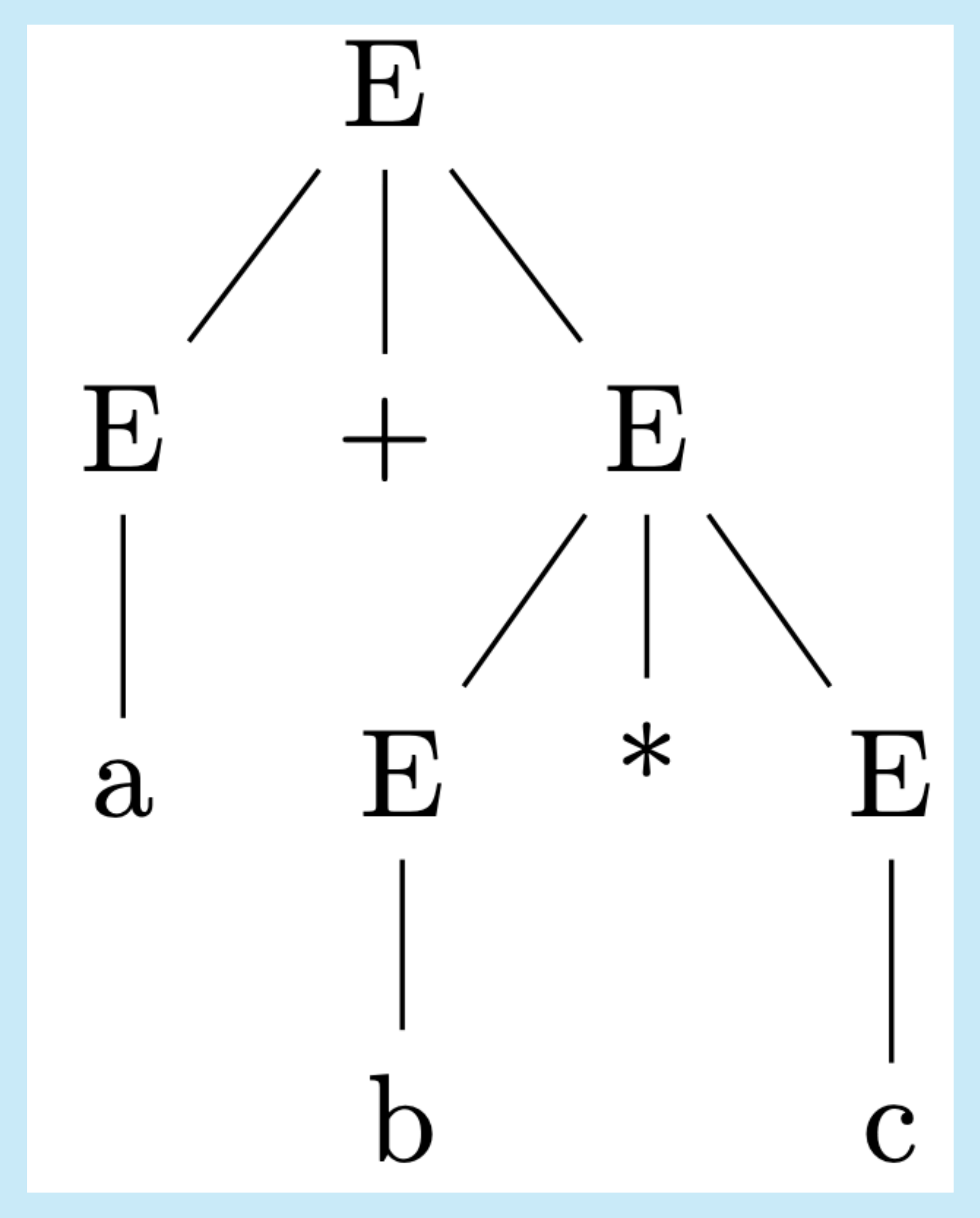
From previous lectures

Derivation trees

- A *derivation tree* is a tree such that:
  - The *root* of a derivation tree is S.
  - Each *leaf* of a derivation tree $\in \Sigma$.
  - Each *inner node* of a derivation tree $\in$ N.
  - **If** the node A has the children p, q, r, …
    - **then** there is a rule A $\rightarrow$ pqr… $\in$ P

Derivation trees

- Example grammar: $E \rightarrow E{*}E \mid E{+}E \mid a \mid b \mid c$
- Example derivation:
  - $E \Rightarrow E{+}E \Rightarrow E{+}E{*}E \Rightarrow E{+}E{*}c \Rightarrow E{+}b{*}c \Rightarrow a{+}b{*}c$
- Derivation (parse) tree:



Chomsky normal form

- A grammar is in Chomsky normal form
  - if all rules have the form A → a or A → BC.

- Example: **Grammar G6**
  - S → aSb | ab | pTq | pq
  - T → pTq | pq

- **Grammar G7**
  - S → ASB | AB | PTQ | PQ
  - T → PTQ | PQ

- **Grammar G8**
  - S → AE |AB | PF | PQ
  - T → PF | PQ
  - E → SB
  - F → TQ
  - A → a
  - B → b
  - P → p
  - Q → q

Let’s start!

Today's topic

- In lecture 4, some *closure properties* for regular languages were presented.
- In lecture 6, the *pumping lemma* for regular languages was discussed.

- In this lecture: corresponding features for *context-free languages*.
  1. Some closure properties,
  2. The pumping lemma,
  3. Some more closure properties that need the pumping lemma for their proofs.

Some closure properties

- The set of context-free languages are closed under
  - **Theorem 1**: Union
  - **Theorem 2**: Concatenation
  - **Theorem 3**: Star operation

- Proofs: Coming up.
  - Denote languages L_1 and L_2
  - With the grammars $G_1 = \langle N_1, \Sigma_1, P_1, S_1 \rangle$ and $G_2 = \langle N_2, \Sigma_2, P_2, S_2 \rangle$.

Closure under union

- **Theorem 1**. CFLs are closed under union.
- **Proof**. Construct a new grammar:
  - $G_3 = \langle N_1 \cup N_2 \cup \{ S_3 \}, \quad \Sigma_1 \cup \Sigma_2, \quad P_1 \cup P_2 \cup \{ S_3 \rightarrow S_1 \mid S_2 \}, \quad S_3 \rangle$.
  - (If N_1 and N_2 are not disjoint sets, rename the nonterminals in N_2)
  - Now, starting from S_3 all derivations starting with either S_1 or S_2 can be performed
    - All strings in $L(G_1)$ and $L(G_2)$ can be derived
    - $L(G_3) = L(G_1) \cup L(G_2)$

Closure under concatenation

- **Theorem 2**. CFLs are closed under concatenation.
- **Proof**. Construct a new grammar:
  - $G_4 = \langle\, N_1 \cup N_2 \cup \{\, S_4 \,\}, \quad \Sigma_1 \cup \Sigma_2, \quad P_1 \cup P_2 \cup \{\, S_4 \to S_1S_2 \,\}, \quad S_4 \,\rangle$.
  - (If N_1 and N_2 are not disjoint sets, rename the nonterminals in N_2)
  - Now, starting from S_4 all derivations consist of one part derived from S_1 in $L(G_1)$ followed by one part derived from S_2 in $L(G_2)$,
    - $L(G_4) = L(G_1)L(G_2)$.

Closure under *

- **Theorem 3**. CFLs are closed under the star operation.
- **Proof**. Construct a new grammar:
  - $G_5 = \langle\ N_1 \cup \{ S_5 \}, \quad \Sigma_1, \quad P_1 \cup \{ S_5 \rightarrow S_1S_5 \mid \varepsilon \}, \quad S_5\ \rangle$.
  - Now, starting from S_5
    - either ε can be derived
    - or one string of $L(G_1)$ followed once again by what can be derived from S_5.
    - $L(G_5) = (L(G_1))*$

Pumping lemma for CFLs – Motivation and proof sketch

- On the board.
- See lecture notes and course book for similar figures to what was shown during the lecture.

Pumping lemma for CFLs – Direct form

- From the ideas in the figures, this form of the pumping lemma can be stated
- L is CF $\rightarrow \exists k > 0$:
  - $\forall z \in L : |z| \geq k \rightarrow$
    - $\exists u, v, w, x, y \in \Sigma^* : z = uvwxy \wedge vx \neq \varepsilon \wedge |vwx| \leq k \rightarrow$
      - $\forall i \geq 0 : uv^i wx^i y \in L$

1. If L is context-free then there is some constant (pumping length) such that
2. if a string in L is of enough length then
3. it can be split into uvwxy—such that vx isn't empty and uvw isn't too long—so that
4. the vx parts can be repeated arbitrarily number of times and the string still is in L.

Pumping lemma for CFLs – Inverted form

- Like for regular pumping instead of using the lemma on the form "if a then b" we will use the inverted form "if not b then not a".
- $(\forall k > 0 :$
  - $\exists z \in L : |z| \geq k \land$
    - $\forall u, v, w, x, y \in \Sigma* : z = uvwxy \land vx \neq \varepsilon \land |vwx| \leq k \rightarrow$
      - $\exists i \geq 0 : uv^iwx^iy \notin L) \rightarrow L$ is not CF

1. For all constants (pumping lengths)
2. there is a string of enough length and
3. for all ways to split it into uvwxy—such that vx isn't empty and uvw isn't too long—
4. you can pump the vx parts to get the string outside of the language.

Example 1

- *Prove that $L_6 = \{ a^n b^n c^n \mid n \geqslant 0 \}$ is not context-free.*
- For the pumping length k let $z = a^k b^k c^k$.
- Now, since vwx cannot contain more than k symbols, vx cannot contain both a:s, b:s, and c:s.
- They have the same number of occurences in z, but not all of them will change when pumping so the pumped string will be out of L_6.
- E.g. if $u = a^p$, $v = a^q$, $w = a^r$, $x = a^s$, $y = a^{k-p-q-r-s} b^k c^k$,
  - then pumping to uv^2wx^2y will give the string $a^{k+q+s} b^k c^k$.
- So L_6 is <u>not context-free</u>.

Example 2

- *Prove that the language L_7 = { $a^m b^n a^m b^n$ | $m, n \geqslant 0$ } is not context-free.*
- For the pumping length k, let $z = a^k b^k a^k b^k$.
- We call each of a^k and b^k a *block*.
  - If v or x contains both a and b then pumpig up (uv^2wx^2y) will in v^2 or x^2 result in a mixture of two symbols and z will break the original $a{*}b{*}a{*}b{*}$ pattern in L_7.
  - If v and x are in the same block then pumping up (uv^2wx^2y) will make that block longer than the other block with the same symbol.
  - If v and x are in neighbouring blocks then there will be different numbers of both a:s and b:s in the blocks. E.g. if $u = a^p$, $v = a^q$, $w = a^{k-p-q}b^r$, $x = b^s$, $y = b^{k-r-s}a^k b^k$ then $uv^2wx^2y = a^{k+q}b^{k+s}a^k b^k$.
- So, for every possible split of z it is pumpable out of L_7. <u>L_7 is not context-free.</u>

More closure properties

- The set of context-free languages are
  - **Theorem 1**: not closed under intersection
  - **Theorem 2**: are closed under intersection with regular languages
  - **Theorem 3**: are not closed under complement

- Proofs: Coming up.
  - In this section we will introduce a number of languages (L8, L9, . . .), their
  - grammars(G8, G9, . . .), and a number of automata (M10, M11, . . .)

Theorem 4: Not closed under intersection

- **Theorem 4**. CFLs are **not** closed under intersection.
- **Proof**. Let $L_8 = \{ a^m b^m c^n \mid m, n \geq 0 \}$ and $L_9 = \{ a^m b^n c^n \mid m, n \geq 0 \}$.
  - Both languages are context-free.
    They can be defined with the following grammars:
    - G_8: $S \rightarrow AC$ $\quad$ $A \rightarrow aAb \mid \varepsilon$ $\quad$ $C \rightarrow cC \mid \varepsilon$
    - G_9: $S \rightarrow AC$ $\quad$ $A \rightarrow aA$ $\quad$ $C \rightarrow bCc \mid \varepsilon$
  - Now, $L_8 \cap L_9 = \{ a^n b^n c^n \mid n \geq 0 \}$
  - Since L_8 and L_9 are context-free, and the result was proved not context-free in Example 1, context-free languages are not closed under intersection.

Theorem 5: Closure under intersection with regular languages

- **Theorem 5**. CFLs are closed under intersection **with regular languages**.
- **Proof**. Let CFL L_{10} be accepted by a PDA M_{10} and regular L_{11} by a DFA M_{11}.
  - A PDA M_{12} can be constructed where each state is a pair of one state in M_{10} and one state in M_{11}.
  - The transitions in M_{12} follow simultaneously the transitions in M_{10} and M_{11}.
  - A string is accepted in M_{12} if it is accepted in both M_{10} and M_{11}.
  - The stack is handled as in M_{10}—since M_{11} doesn't use the stack the M_{10} stack handling won't be disturbed.

Theorem 6: Not closed under complement

- **Theorem 6**. CFLs are not closed under complement.
- **Proof**. $L_{13} \cap L_{14} = \sim (\sim L_{13} \cup \sim L_{14})$
  - If CFLs were closed under complement:
    - then the union of two CFLs would have been constructed,
      - which would be context-free,
    - and then the complement of that result should have been context-free.
  - But the result is the intersection that, according to Theorem 4, does not preserve context-freeness.

Example 3

- *L_{15} = { ww | w ∈ {a, b}∗ } is not context-free.*
  - L_{15} contains strings consisting of two equal parts,
    - e.g. aabaab, ababaababa, and bbbbabbbba.
- Some of the strings in L_{15} are the same as some strings of the form a∗b∗a∗b∗.
- $L_{15} \cap a{*}b{*}a{*}b{*} = L_7$, with L_7 from Example 2.
- Since CFLs are closed under intersection with regular languages, L_7 would be context-free if L_{15} were context-free.

To think about

- Can we use the pumping lemma for context-free languages to prove that a given language is context-free?
- Can every language which is not context-free be proven so by using the pumping lemma introduced in this lecture?

Coming up soon …

- Next week:
  - **Monday**: LL(1) and LR(0) parsing
  - **Friday**: LR(1) parsing

Thanks for today!