Problem Set for Tutorial 1 — TDDD08*

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1. Which of the following expressions are well-formed formulas in first-order logic?

   (a) $\forall x(p(x) \rightarrow q(x))$ where p and q are predicate symbols.
   (b) $p(x) \rightarrow q(x)$ where p and q are predicate symbols.
   (c) $\exists x(p(f(x)) \land q(x))$ where p and q are predicate symbols and f a function symbol.
   (d) $\exists x(p(f(x)) \land \exists x q(x))$ where p and q are predicate symbols and f a function symbol.
   (e) $\forall x(f(x) \land p(x))$ where f is a function symbol and p a predicate symbol.
   (f) $\forall x p(f(x))$ where p is a predicate symbol and f a function symbol.
   (g) $\forall x f(p(x))$ where p is a predicate symbol and f a function symbol.

   Solution.

   (a) Yes.
   (b) Yes.
   (c) Yes.
   (d) Yes.
   (e) No.
   (f) Yes.
   (g) No.

2. Translate the following sentences into first-order logic:

   (a) "All employees have income."
   (b) "Some employees are on holidays."
   (c) "No employees are unemployed."
   (d) "Some employees are not satisfied by their salary policy."

   Solution. For example:

   (a) $\forall x(\text{employee}(x) \rightarrow \text{income}(x))$.

*Some exercises taken from TDDD72.
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(b) $\exists x(\text{employee}(x) \wedge \text{holiday(x)})$.
(c) $\forall x(\text{employee}(x) \rightarrow \neg\text{unemployed}(x))$.
(d) $\exists x(\text{employee}(x) \wedge \neg\text{satisfied}(x, \text{salary\_policy}))$.

3. Translate the following formulas into natural language:

   (a) $\forall x((\text{strongEngine}(x) \wedge \text{car}(x) \wedge \text{wheels}(x, 4)) \rightarrow \text{fast}(x))$.
   (b) $\forall x \forall y((\text{parent}(x, y) \wedge \text{ancestor}(y)) \rightarrow \text{ancestor}(x))$.
   (c) $\forall x \forall y((\text{car}(x) \wedge \text{onRoad}(x, y) \wedge \text{highway}(y) \wedge \text{normalConditions}(y)) \rightarrow \text{fastSpeedAllowed}(x))$.

   Solution. For example:

   (a) Four-wheel cars with strong engines are fast.
   (b) A parent of an ancestor is also an ancestor.
   (c) Under normal conditions, cars on highways can drive fast.

4. Consider the formula
$$\forall x(\text{shaves}(barber, x) \leftrightarrow \neg\text{shaves}(x, x)),$$
   which in natural language can be read as "The barber is a man in a town who shaves all those, and only those, men in town who do not shave themselves.". Does it have a model, i.e., is it satisfiable?

   Solution. The formula does not have a model. Assume there exists a model M over a universe $\{d_0, d_1, \ldots\}$, and assume for simplicity that $\text{barber}_M = d_0$, i.e., the constant symbol barber is mapped to the element d_0 from the universe D. Assume first that $\text{shaves}(d_0, d_0)$ is true in M. Then $\neg\text{shaves}(d_0, d_0)$ must be true, too, which is a contradiction. Similarly, assume that $\text{shaves}(d_0, d_0)$ is *not* true. Then $\neg\text{shaves}(d_0, d_0))$ is true, and we conclude that $\text{shaves}(d_0, d_0)$ must also be true, which is again a contradiction.

5. Let a and b be constant symbols, f a binary function symbol, and l a unary predicate symbol. Consider the first-order sentence $l(a) \wedge \forall x(l(x) \rightarrow l(f(b, x)))$. Provide two interpretations: one which is a model and one which is not.

   Solution. Two possible interpretations are:

   (a) A model: an interpretation I where the universe $D = \{a, b, f(a, a), f(a, b), f(b, a), f(b, b), f(f(a, a), a), \ldots\}$ contains all ground terms, where $a_I = a$, $b_I = b$, f_I is the function $f_I(t, t') = f(t, t')$ for $t, t' \in D$ which simply maps $t, t' \in D$ to the ground term obtained by taking t and t' and combining them with the symbol f, and where $l_I = \{a, f(b, a), f(b, f(b, a)), \ldots\}$[1].
   (b) A non-model: there ary many possible choices. If we build upon the previous answer we could e.g. define $l_I = \{a\}$. Then the subformula $\forall x(l(x) \rightarrow l(f(b, x)))$ is not satisfied (why?).

6. Consider the formula $\text{nat}(zero) \wedge \forall x\,(\text{nat}(x) \rightarrow \text{nat}(f(x)))$. Provide two interpretations: one which is a model and one which is not.

   Solution. Two possible interpretations are:

[1] An interpretation of this form is known as an *Herbrand interpretation* and will play an important role later on in the course.

(a) A model: an interpretation I with universe $D = \mathbb{N} = \{0, 1, 2, \ldots\}$, where $\text{zero}_I = 0$, $\text{nat}_I = D$, and f_I is the function $f_I(x) = x + 1$ over $\mathbb{N}$.

(b) A non-model: an interpretation I with universe $D = \mathbb{N} = \{0, 1, 2, \ldots\}$, where nat_I is finite, and f_I is the function $f_I(x) = x + 1$ over $\mathbb{N}$.