Artificial Intelligence

Logic 1: Syntax, Semantics and CNF

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based on slides by Thomas Keller and Malte Helmert (University of Basel)

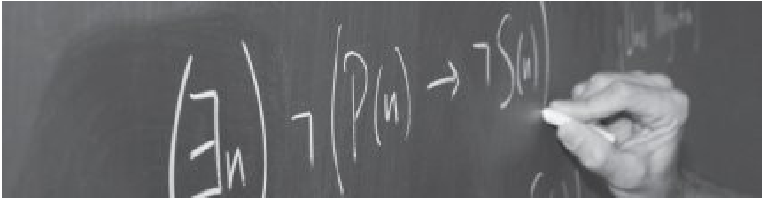
Questions?

post feedback and ask questions anonymously at

`https://padlet.com/jendrikseipp/tddc17`

Introduction

Motivation: Logic



- allows to model problems and represent knowledge
- allows to derive conclusions from knowledge (reasoning)
- basics for general problem descriptions and solving strategies
  e.g., automated planning

we restrict to the (simple) form of propositional logic

Reasoning Example

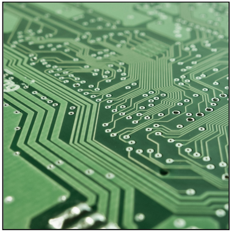
Adam and Berta are in a pizzeria.

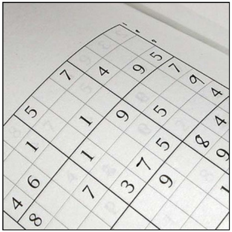
Waiter: “Who ordered Margherita?”
Adam: “It’s mine.”
$\leadsto$ Waiter gives Adam the pizza.

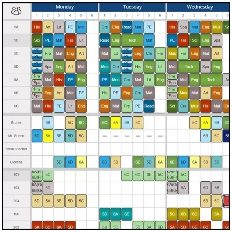
Waiter: “Who ordered Quattro Stagioni?”

...

Reasoning Examples







Syntax

Propositions and Sentences

- a proposition is an atomic statements over the world, e.g.
  - AM
  - BM
  - cell-1-1-is-1
  - cell-1-1-is-2
- propositions with logical connectives
  like "and" ($\land$), "or" ($\lor$), "not" ($\neg$)
  form sentences (or propositional formulas), e.g.,
  - $(\texttt{AM} \lor \texttt{BM})$
  - $(\neg\texttt{AQ} \rightarrow \texttt{BQ})$
  - $\neg(\texttt{cell-1-1-is-1} \land \texttt{cell-1-2-is-1}) \land$
    $\neg(\texttt{cell-1-1-is-1} \land \texttt{cell-1-3-is-1}) \land \ldots$

Syntax

let Σ be an alphabet of propositions
(e.g., $\{P, Q, R\}$ or $\{X_1, X_2, X_3, \dots\}$)

Definition (sentences)

- $\top$ ("always-true") and $\bot$ ("always-false") are sentences
- every proposition in Σ is an atomic sentence
- if φ is a sentence, then $\neg\varphi$ is a sentence (negation)
- if φ and ψ are sentences, then so are
  - $(\varphi \land \psi)$ (conjunction)
  - $(\varphi \lor \psi)$ (disjunction)
  - $(\varphi \rightarrow \psi)$ (implication)
  - $(\varphi \leftrightarrow \psi)$ (biconditional)

binding strength: $(\neg) > (\land) > (\lor) > (\rightarrow) > (\leftrightarrow)$
(redundant parentheses may be omitted)

Reasoning Example

[each pizza has exactly one owner]

[each person ordered exactly one pizza]

Reasoning Example

[each pizza has exactly one owner]

$\mathtt{AM} \leftrightarrow \neg\mathtt{BM}$

$\mathtt{AQ} \leftrightarrow \neg\mathtt{BQ}$

[each person ordered exactly one pizza]

Reasoning Example

[each pizza has exactly one owner]

$AM \leftrightarrow \neg BM$

$AQ \leftrightarrow \neg BQ$

[each person ordered exactly one pizza]

$AM \leftrightarrow \neg AQ$

$BM \leftrightarrow \neg BQ$

Semantics

Semantics: Intuition

a sentence can be true or false,
depending on the truth values of the propositions

- a proposition p is either true or false, and
  the truth value of p is determined by an interpretation
- the truth value of a sentence follows from
  the truth values of the propositions

Example

$\varphi = (P \lor Q) \land R$

- if P and Q are false, then φ is false
  (independent of the truth value of R)
- if P and R are true, then φ is true
  (independent of the truth value of Q)

Semantics: Formally

- defined over interpretations $I : \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$
- interpretation I: assignment of propositions in Σ
- when is a sentence φ true under interpretation I?
  symbolically: When does $I \models \varphi$ hold?

Note: The AIMA book calls all interpretations "models", but we want to say "I is a model of φ" or "I models φ".

Semantics: Formally

Definition ($I \models \varphi$)

- $I \models \top$
- $I \not\models \bot$
- $I \models P$ iff $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \not\models P$ iff $I(P) = \mathbf{F}$ for $P \in \Sigma$
- $I \models \neg\varphi$ iff $I \not\models \varphi$
- $I \models (\varphi \land \psi)$ iff $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \lor \psi)$ iff $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ iff $I \not\models \varphi$ or $I \models \psi$
- $I \models (\varphi \leftrightarrow \psi)$ iff $I \models \varphi$ and $I \models \psi$ or $I \not\models \varphi$ and $I \not\models \psi$
- $I \models \Phi$ for a set of sentences Φ iff $I \models \varphi$ for all $\varphi \in \Phi$

Examples

Example (Interpretation I)

$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$

Which sentences are true under I?

- $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

Terminology

Definition (satisfiable etc.)

a sentence φ is called

- satisfiable if there is an interpretation I such that $I \models \varphi$
- unsatisfiable if φ is not satisfiable
- falsifiable if there is an interpretation I such that $I \not\models \varphi$
- valid (= a tautology) if $I \models \varphi$ for all interpretations I

Definition (logical equivalence)

sentences φ and ψ are called logically equivalent ($\varphi \equiv \psi$) if for all interpretations I: $I \models \varphi$ iff $I \models \psi$.

Truth Tables

How to determine automatically if a given sentence is (un)satisfiable, falsifiable, valid?

$\rightsquigarrow$ simple method: truth tables

example: Is $\varphi = ((P \lor H) \land \neg H) \rightarrow P$ valid?

P	H	$P \lor H$	$((P \lor H) \land \neg H)$	$((P \lor H) \land \neg H) \rightarrow P$
F	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**
T	**T**	**T**	**F**	**T**

$I \models \varphi$ for all interpretations $I \rightsquigarrow \varphi$ is valid.

What about satisfiability, falsifiability, unsatisfiability?

Drawback of truth tables: exponential size in the number of propositions

Example

Fill out the truth table for the following formula:

$$\varphi = (A \vee \neg B) \longrightarrow B$$

A	B	$A \vee \neg B$	$(A \vee \neg B) \longrightarrow B$
T	**T**		
T	**F**		
F	**T**		
F	**F**		

Example

Fill out the truth table for the following formula:

$$\varphi = (A \lor \neg B) \longrightarrow B$$

A	B	$A \lor \neg B$	$(A \lor \neg B) \longrightarrow B$
T	**T**	**T**	**T**
T	**F**	**T**	**F**
F	**T**	**F**	**T**
F	**F**	**T**	**F**

Conjunctive Normal Form

CNF: Terminology

Definition (literal)

If $P \in \Sigma$, then the sentences P and $\neg P$ are called literals.

P is called positive literal, $\neg P$ is called negative literal.

The complementary literal to P is $\neg P$ and vice versa.
For a literal ℓ, the complementary literal to ℓ is denoted with $\bar{\ell}$.

Definition (clause)

A disjunction of 0 or more literals is called a clause.
The empty clause $\bot$ is also written as $\square$.
Clauses consisting of only one literal are called unit clauses.

Conjunctive Normal Form

Definition (conjunctive normal forms)

A sentence φ is in conjunctive normal form (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

Normal Forms

for every sentence, there is a logically equivalent sentence in CNF

Conversion to CNF

important rules for conversion to CNF:

- $(\varphi \leftrightarrow \psi) \equiv (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$ (($\leftrightarrow$)-elimination)
- $(\varphi \rightarrow \psi) \equiv (\lnot\varphi \lor \psi)$ (($\rightarrow$)-elimination)
- $\lnot(\varphi \land \psi) \equiv (\lnot\varphi \lor \lnot\psi)$ (De Morgan)
- $\lnot(\varphi \lor \psi) \equiv (\lnot\varphi \land \lnot\psi)$ (De Morgan)
- $\lnot\lnot\varphi \equiv \varphi$ (double negation)
- $((\varphi \land \psi) \lor \eta) \equiv ((\varphi \lor \eta) \land (\psi \lor \eta))$ (distributivity)

there are sentences φ for which every logically equivalent sentence in CNF is exponentially longer than φ

Example

Convert the following formula into CNF:

$$\varphi = (\neg P \vee Q) \longrightarrow R$$

Example

Convert the following formula into CNF:

$$\varphi = (\neg P \vee Q) \longrightarrow R$$

1. $(\neg P \vee Q) \longrightarrow R$
2. $\neg(\neg P \vee Q) \vee R$ [($\longrightarrow$)-elimination]
3. $(\neg\neg P \wedge \neg Q) \vee R$ [De Morgan]
4. $(P \wedge \neg Q) \vee R$ [double negation]
5. $(P \vee R) \wedge (\neg Q \vee R)$ [distributivity]

Summary

Summary

- Propositional logic forms the basis for a general representation of problems and knowledge.
- Propositions (atomic formulas) are statements over the world which cannot be divided further.
- Propositional formulas combine atomic formulas with $\neg$, $\wedge$, $\vee$, $\rightarrow$ or $\leftrightarrow$ to form more complex statements.
- Interpretations determine which atomic formulas are true and which ones are false.
- important terminology:
  - model
  - satisfiable, unsatisfiable, falsifiable, valid
  - logically equivalent
  - conjunctive normal form