

Artificial Intelligence

Logic 1: Syntax, Semantics and CNF

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Questions?

post **feedback** and ask **questions** anonymously at

`https://padlet.com/jendrikseipp/tddc17`

Introduction

Motivation: Logic



- allows to **model problems** and **represent knowledge**
- allows to derive **conclusions** from knowledge (**reasoning**)
- basics for **general** problem descriptions and solving strategies
e.g., **automated planning**

we restrict to the (simple) form of **propositional logic**

Reasoning Example

Adam and Berta are in a pizzeria.

Waiter: “Who ordered Margherita?”

Adam: “It’s mine.”

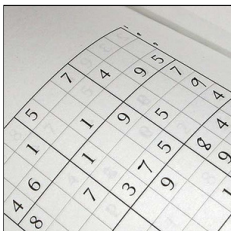
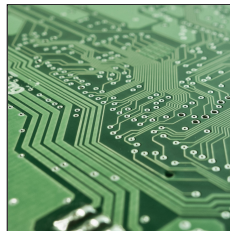
↪ Waiter gives Adam the pizza.

Waiter: “Who ordered Quattro Stagioni?”

...



Reasoning Examples



	Monday	Tuesday	Wednesday
1A	1	2	3
2A	4	5	6
3A	7	8	9
4A	10	11	12
5A	13	14	15
6A	16	17	18
7A	19	20	21
8A	22	23	24
9A	25	26	27
10A	28	29	30
11A	31	32	33
12A	34	35	36
13A	37	38	39
14A	40	41	42
15A	43	44	45
16A	46	47	48
17A	49	50	51
18A	52	53	54
19A	55	56	57
20A	58	59	60
21A	61	62	63
22A	64	65	66
23A	67	68	69
24A	70	71	72
25A	73	74	75
26A	76	77	78
27A	79	80	81
28A	82	83	84
29A	85	86	87
30A	88	89	90
31A	91	92	93
32A	94	95	96
33A	97	98	99
34A	100	101	102
35A	103	104	105
36A	106	107	108
37A	109	110	111
38A	112	113	114
39A	115	116	117
40A	118	119	120
41A	121	122	123
42A	124	125	126
43A	127	128	129
44A	130	131	132
45A	133	134	135
46A	136	137	138
47A	139	140	141
48A	142	143	144
49A	145	146	147
50A	148	149	150
51A	151	152	153
52A	154	155	156
53A	157	158	159
54A	160	161	162
55A	163	164	165
56A	166	167	168
57A	169	170	171
58A	172	173	174
59A	175	176	177
60A	178	179	180
61A	181	182	183
62A	184	185	186
63A	187	188	189
64A	190	191	192
65A	193	194	195
66A	196	197	198
67A	199	200	201
68A	202	203	204
69A	205	206	207
70A	208	209	210
71A	211	212	213
72A	214	215	216
73A	217	218	219
74A	220	221	222
75A	223	224	225
76A	226	227	228
77A	229	230	231
78A	232	233	234
79A	235	236	237
80A	238	239	240
81A	241	242	243
82A	244	245	246
83A	247	248	249
84A	250	251	252
85A	253	254	255
86A	256	257	258
87A	259	260	261
88A	262	263	264
89A	265	266	267
90A	268	269	270
91A	271	272	273
92A	274	275	276
93A	277	278	279
94A	280	281	282
95A	283	284	285
96A	286	287	288
97A	289	290	291
98A	292	293	294
99A	295	296	297
100A	298	299	300
101A	301	302	303
102A	304	305	306
103A	307	308	309
104A	310	311	312
105A	313	314	315
106A	316	317	318
107A	319	320	321
108A	322	323	324
109A	325	326	327
110A	328	329	330
111A	331	332	333
112A	334	335	336
113A	337	338	339
114A	340	341	342
115A	343	344	345
116A	346	347	348
117A	349	350	351
118A	352	353	354
119A	355	356	357
120A	358	359	360
121A	361	362	363
122A	364	365	366
123A	367	368	369
124A	370	371	372
125A	373	374	375
126A	376	377	378
127A	379	380	381
128A	382	383	384
129A	385	386	387
130A	388	389	390
131A	391	392	393
132A	394	395	396
133A	397	398	399
134A	400	401	402
135A	403	404	405
136A	406	407	408
137A	409	410	411
138A	412	413	414
139A	415	416	417
140A	418	419	420
141A	421	422	423
142A	424	425	426
143A	427	428	429
144A	430	431	432
145A	433	434	435
146A	436	437	438
147A	439	440	441
148A	442	443	444
149A	445	446	447
150A	448	449	450
151A	451	452	453
152A	454	455	456
153A	457	458	459
154A	460	461	462
155A	463	464	465
156A	466	467	468
157A	469	470	471
158A	472	473	474
159A	475	476	477
160A	478	479	480
161A	481	482	483
162A	484	485	486
163A	487	488	489
164A	490	491	492
165A	493	494	495
166A	496	497	498
167A	499	500	501
168A	502	503	504
169A	505	506	507
170A	508	509	510
171A	511	512	513
172A	514	515	516
173A	517	518	519
174A	520	521	522
175A	523	524	525
176A	526	527	528
177A	529	530	531
178A	532	533	534
179A	535	536	537
180A	538	539	540
181A	541	542	543
182A	544	545	546
183A	547	548	549
184A	550	551	552
185A	553	554	555
186A	556	557	558
187A	559	560	561
188A	562	563	564
189A	565	566	567
190A	568	569	570
191A	571	572	573
192A	574	575	576
193A	577	578	579
194A	580	581	582
195A	583	584	585
196A	586	587	588
197A	589	590	591
198A	592	593	594
199A	595	596	597
200A	598	599	600
201A	601	602	603
202A	604	605	606
203A	607	608	609
204A	610	611	612
205A	613	614	615
206A	616	617	618
207A	619	620	621
208A	622	623	624
209A	625	626	627
210A	628	629	630
211A	631	632	633
212A	634	635	636
213A	637	638	639
214A	640	641	642
215A	643	644	645
216A	646	647	648
217A	649	650	651
218A	652	653	654
219A	655	656	657
220A	658	659	660
221A	661	662	663
222A	664	665	666
223A	667	668	669
224A	670	671	672
225A	673	674	675
226A	676	677	678
227A	679	680	681
228A	682	683	684
229A	685	686	687
230A	688	689	690
231A	691	692	693
232A	694	695	696
233A	697	698	699
234A	700	701	702
235A	703	704	705
236A	706	707	708
237A	709	710	711
238A	712	713	714
239A	715	716	717
240A	718	719	720
241A	721	722	723
242A	724	725	726
243A	727	728	729
244A	730	731	732
245A	733	734	735
246A	736	737	738
247A	739	740	741
248A	742	743	744
249A	745	746	747
250A	748	749	750
251A	751	752	753
252A	754	755	756
253A	757	758	759
254A	760	761	762
255A	763	764	765
256A	766	767	768
257A	769	770	771
258A	772	773	774
259A	775	776	777
260A	778	779	780
261A	781	782	783
262A	784	785	786
263A	787	788	789
264A	790	791	792
265A	793	794	795
266A	796	797	798
267A	799	800	801
268A	802	803	804
269A	805	806	807
270A	808	809	810
271A	811	812	813
272A	814	815	816
273A	817	818	819
274A	820	821	822
275A	823	824	825
276A	826	827	828
277A	829	830	831
278A	832	833	834
279A	835	836	837
280A	838	839	840
281A	841	842	843
282A	844	845	846
283A	847	848	849
284A	850	851	852
285A	853	854	855
286A	856	857	858
287A	859	860	861
288A	862	863	864
289A	865	866	867
290A	868	869	870
291A	871	872	873
292A	874	875	876
293A	877	878	879
294A	880	881	882
295A	883	884	885
296A	886	887	888
297A	889	890	891
298A	892	893	894
299A	895	896	897
300A	898	899	900
301A	901	902	903
302A	904	905	906
303A	907	908	909
304A	910	911	912
305A	913	914	915
306A	916	917	918
307A	919	920	921
308A	922	923	924
309A	925	926	927
310A	928	929	930
311A	931	932	933
312A	934	935	936
313A	937	938	939
314A	940	941	942
315A	943	944	945
316A	946	947	948
317A	949	950	951
318A	952	953	954
319A	955	956	957
320A	958	959	960
321A	961	962	963
322A	964	965	966
323A	967	968	969
324A	970	971	972
325A	973	974	

Syntax

Propositions and Sentences

- a **proposition** is an atomic statements over the world, e.g.
 - AM
 - BM
 - cell-1-1-is-1
 - cell-1-1-is-2
- propositions with **logical connectives**
like “and” (\wedge), “or” (\vee), “not” (\neg)
form **sentences** (or **propositional formulas**), e.g.,
 - $(AM \vee BM)$
 - $(\neg AQ \rightarrow BQ)$
 - $\neg(\text{cell-1-1-is-1} \wedge \text{cell-1-2-is-1}) \wedge$
 $\neg(\text{cell-1-1-is-1} \wedge \text{cell-1-3-is-1}) \wedge \dots$

Syntax

let Σ be an **alphabet of propositions**
(e.g., $\{P, Q, R\}$ or $\{X_1, X_2, X_3, \dots\}$)

Definition (sentences)

- \top (“**always-true**”) and \perp (“**always-false**”) are sentences
- every proposition in Σ is an **atomic sentence**
- if φ is a sentence, then $\neg\varphi$ is a sentence (**negation**)
- if φ and ψ are sentences, then so are
 - $(\varphi \wedge \psi)$ (**conjunction**)
 - $(\varphi \vee \psi)$ (**disjunction**)
 - $(\varphi \rightarrow \psi)$ (**implication**)
 - $(\varphi \leftrightarrow \psi)$ (**biconditional**)

binding strength: $(\neg) > (\wedge) > (\vee) > (\rightarrow) > (\leftrightarrow)$
(redundant parentheses may be omitted)

Reasoning Example

[each pizza has exactly one owner]

[each person ordered exactly one pizza]



Reasoning Example

[each pizza has exactly one owner]

$AM \leftrightarrow \neg BM$

$AQ \leftrightarrow \neg BQ$

[each person ordered exactly one pizza]



Reasoning Example

[each pizza has exactly one owner]

$AM \leftrightarrow \neg BM$

$AQ \leftrightarrow \neg BQ$

[each person ordered exactly one pizza]

$AM \leftrightarrow \neg AQ$

$BM \leftrightarrow \neg BQ$



Semantics

Semantics: Intuition

a sentence can be **true** or **false**,
depending on the **truth values** of the propositions

- a proposition p is either true or false, and the truth value of p is determined by an **interpretation**
- the truth value of a sentence **follows** from the truth values of the propositions

Example

$$\varphi = (P \vee Q) \wedge R$$

- if P and Q are false, then φ is false
(independent of the truth value of R)
- if P and R are true, then φ is true
(independent of the truth value of Q)

Semantics: Formally

- defined over **interpretations** $I : \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$
- interpretation I : **assignment** of propositions in Σ
- when is a sentence φ true under interpretation I ?
symbolically: When does $I \models \varphi$ hold?

Note: The AIMA book calls all interpretations “models”, but we want to say “ I is a model of φ ” or “ I models φ ”.

Semantics: Formally

Definition ($I \models \varphi$)

- $I \models \top$
- $I \not\models \perp$
- $I \models P$ iff $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \not\models P$ iff $I(P) = \mathbf{F}$ for $P \in \Sigma$
- $I \models \neg\varphi$ iff $I \not\models \varphi$
- $I \models (\varphi \wedge \psi)$ iff $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \vee \psi)$ iff $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ iff $I \not\models \varphi$ or $I \models \psi$
- $I \models (\varphi \leftrightarrow \psi)$ iff $I \models \varphi$ and $I \models \psi$ or $I \not\models \varphi$ and $I \not\models \psi$
- $I \models \Phi$ for a set of sentences Φ iff $I \models \varphi$ for all $\varphi \in \Phi$

Examples

Example (Interpretation I)

$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$

Which sentences are true under I ?

- $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

Terminology

Definition (satisfiable etc.)

a sentence φ is called

- **satisfiable** if there is an interpretation I such that $I \models \varphi$
- **unsatisfiable** if φ is not satisfiable
- **falsifiable** if there is an interpretation I such that $I \not\models \varphi$
- **valid** (= a **tautology**) if $I \models \varphi$ for all interpretations I

Definition (logical equivalence)

sentences φ and ψ are called **logically equivalent** ($\varphi \equiv \psi$) if for all interpretations I : $I \models \varphi$ iff $I \models \psi$.

Truth Tables

How to determine automatically if a given sentence is (un)satisfiable, falsifiable, valid?

↪ simple method: **truth tables**

example: Is $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$ valid?

P	H	$P \vee H$	$((P \vee H) \wedge \neg H)$	$((P \vee H) \wedge \neg H) \rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

$I \models \varphi$ for all interpretations $I \rightsquigarrow \varphi$ is valid.

What about **satisfiability**, **falsifiability**, **unsatisfiability**?

Drawback of truth tables: **exponential size** in the number of propositions

Example

Fill out the truth table for the following formula:

$$\varphi = (A \vee \neg B) \rightarrow B$$

A	B	$A \vee \neg B$	$(A \vee \neg B) \rightarrow B$
T	T		
T	F		
F	T		
F	F		

Example

Fill out the truth table for the following formula:

$$\varphi = (A \vee \neg B) \rightarrow B$$

A	B	$A \vee \neg B$	$(A \vee \neg B) \rightarrow B$
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	F

Conjunctive Normal Form

CNF: Terminology

Definition (literal)

If $P \in \Sigma$, then the sentences P and $\neg P$ are called **literals**.

P is called **positive literal**, $\neg P$ is called **negative literal**.

The **complementary literal** to P is $\neg P$ and vice versa.

For a literal ℓ , the complementary literal to ℓ is denoted with $\bar{\ell}$.

Definition (clause)

A disjunction of 0 or more literals is called a **clause**.

The **empty clause** \perp is also written as \square .

Clauses consisting of only one literal are called **unit clauses**.

Conjunctive Normal Form

Definition (conjunctive normal forms)

A sentence φ is in **conjunctive normal form** (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^n \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

Normal Forms

for every sentence, there is a logically equivalent sentence in CNF

Conversion to CNF

important rules for conversion to CNF:

- $(\varphi \leftrightarrow \psi) \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ ((\leftrightarrow)-elimination)
- $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ ((\rightarrow)-elimination)
- $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ (De Morgan)
- $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$ (De Morgan)
- $\neg\neg\varphi \equiv \varphi$ (double negation)
- $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$ (distributivity)

there are sentences φ for which every logically equivalent sentence in CNF is exponentially longer than φ

Example

Convert the following formula into CNF:

$$\varphi = (\neg P \vee Q) \rightarrow R$$

Example

Convert the following formula into CNF:

$$\varphi = (\neg P \vee Q) \rightarrow R$$

① $(\neg P \vee Q) \rightarrow R$

② $\neg(\neg P \vee Q) \vee R$

[(\rightarrow)-elimination]

③ $(\neg\neg P \wedge \neg Q) \vee R$

[De Morgan]

④ $(P \wedge \neg Q) \vee R$

[double negation]

⑤ $(P \vee R) \wedge (\neg Q \vee R)$

[distributivity]

Summary

Summary

- **Propositional logic** forms the basis for a general representation of problems and knowledge.
- **Propositions** (atomic formulas) are statements over the world which cannot be divided further.
- **Propositional formulas** combine atomic formulas with \neg , \wedge , \vee , \rightarrow or \leftrightarrow to form more complex statements.
- **Interpretations** determine which atomic formulas are true and which ones are false.
- important terminology:
 - model
 - satisfiable, unsatisfiable, falsifiable, valid
 - logically equivalent
 - conjunctive normal form