Artificial Intelligence

Planning 6: Solving MDPs

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based on slides by Thomas Keller and Malte Helmert (University of Basel)

Introduction

Computation of an Optimal Policy

there are various techniques to compute an optimal policy:

- (linear program)
- policy iteration
- value iteration

Computation of an Optimal Policy

there are various techniques to compute an optimal policy:

- (linear program)
- policy iteration
- value iteration

policy iteration consists of 2 components:

- policy evaluation: compute V_π for a given π
- policy improvement: use V_π to determine better policy

Policy Evaluation

Policy Evaluation: Implementations

computing the state value of all states for a given policy π (i.e., computing V_π) is called policy evaluation

there are several algorithms for policy evaluation:

1. linear program
2. backward induction
3. iterative policy evaluation

Iterative Policy Evaluation: Idea

- impossible to compute state-values
  in one sweep over the state space in presence of cycles
- start with arbitrary state-value function $\hat{V}_\pi^0$
- treat state-value function as update rule

$$\hat{V}_\pi^i(s) = \sum_{s' \in \text{succ}(s,\pi(s))} T(s,\pi(s),s') \cdot \Big(R(s,\pi(s),s') + \gamma \cdot V_\pi^{i-1}(s')\Big)$$

- apply update rule iteratively
- until state-values have converged
  (in practice: until maximal change is smaller than residual ϵ)

Iterative Policy Evaluation: Example

	1	2	3	4
5	$\Rightarrow$ 0.00	$\Rightarrow$ 0.00	$\Rightarrow$ 0.00	noop 0.00
4	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00
3	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00	$\Leftarrow$ 0.00
2	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00
1	$\Rightarrow^{s_I}$ 0.00	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^0$

- reward to `move` from striped cell is -3 (-1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Iterative Policy Evaluation: Example

5	⇒ −1.00	⇒ −1.00	⇒ −1.00	noop 0.00
4	⇒ −1.00	⇑ −1.00	⇑ −3.00	⇑ −1.00
3	⇒ −1.00	⇑ −1.00	⇐ −1.00	⇐ −1.00
2	⇑ −1.00	⇑ −1.00	⇑ −1.00	⇐ −1.00
1	⇒ s_I −1.00	⇒ −1.00	⇑ −1.00	⇐ −1.00
	1	2	3	4

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^1$

max diff: 3.0

- reward to `move` from striped cell is -3 (-1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Iterative Policy Evaluation: Example

	1	2	3	4
5	$\Rightarrow$ −1.90	$\Rightarrow$ −1.90	$\Rightarrow$ −1.00	noop 0.00
4	$\Rightarrow$ −1.90	$\Uparrow$ −1.90	$\Uparrow$ −4.98	$\Uparrow$ −1.54
3	$\Rightarrow$ −1.90	$\Uparrow$ −1.90	$\Leftarrow$ −1.90	$\Leftarrow$ −1.90
2	$\Uparrow$ −1.90	$\Uparrow$ −1.90	$\Uparrow$ −1.90	$\Leftarrow$ −1.90
1	$\Rightarrow$ s_I −1.90	$\Rightarrow$ −1.90	$\Uparrow$ −1.90	$\Leftarrow$ −1.90

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^2$

max diff: 1.98

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Iterative Policy Evaluation: Example

	1	2	3	4
5	⇒ −3.38	⇒ −1.90	⇒ −1.00	noop 0.00
4	⇒ −3.83	⇑ −2.71	⇑ −6.94	⇑ −2.07
3	⇒ −4.10	⇑ −3.44	⇐ −4.10	⇐ −4.10
2	⇑ −4.10	⇑ −4.10	⇑ −4.10	⇐ −4.10
1	⇒ s_I −4.10	⇒ −4.10	⇑ −4.10	⇐ −4.10

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^5$

max diff: 0.6561

- reward to move from striped cell is −3 (−1 everywhere else)
- move in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is noop for reward of 0

Iterative Policy Evaluation: Example

	1	2	3	4
5	⇒ −3.65	⇒ −1.90	⇒ −1.00	noop 0.00
4	⇒ −4.27	⇑ −2.71	⇑ −7.29	⇑ −2.17
3	⇒ −4.83	⇑ −3.44	⇐ −4.10	⇐ −5.32
2	⇑ −5.78	⇑ −4.83	⇑ −4.69	⇐ −5.74
1	⇒ s_I −6.13	⇒ −5.70	⇑ −5.22	⇐ −6.09

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^{10}$

max diff: 0.1395

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Iterative Policy Evaluation: Example

	1	2	3	4
5	⇒ −3.66	⇒ −1.90	⇒ −1.00	`noop` 0.00
4	⇒ −4.29	⇑ −2.71	⇑ −7.30	⇑ −2.17
3	⇒ −4.87	⇑ −3.44	⇐ −4.10	⇐ −5.38
2	⇑ −5.98	⇑ −4.87	⇑ −4.69	⇐ −5.84
1	⇒ s_I −6.13	⇒ −5.70	⇑ −5.22	⇐ −6.26

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}_\pi^{20}$

max diff: 0.0007

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Iterative Policy Evaluation: Properties

Theorem (Convergence of Iterative Policy Evaluation)

Let $\mathcal{T} = \langle S, A, R, T, s_I, \gamma \rangle$ be an MDP, π be a policy for $\mathcal{T}$ and $\hat{V}_\pi^0(s) \in \mathbb{R}$ arbitrarily for all $s \in S$.

Iterative policy evaluation converges to the true state-values, i.e.,

$$\lim_{i \to \infty} \hat{V}_\pi^i(s) = V_\pi(s) \text{ for all } s \in S.$$

In practice, iterative policy evaluation converges to true state-values if ϵ is small enough.

Policy Improvement

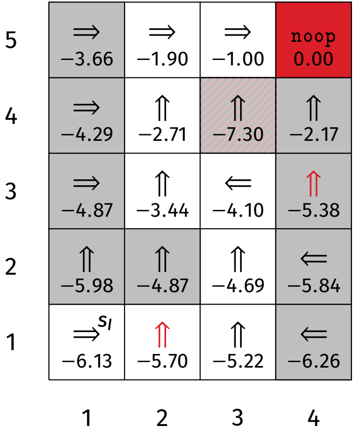
Greedy Action: Example

	1	2	3	4
5	$\Rightarrow$ −3.66	$\Rightarrow$ −1.90	$\Rightarrow$ −1.00	noop 0.00
4	$\Rightarrow$ −4.29	$\Uparrow$ −2.71	$\Uparrow$ −7.30	$\Uparrow$ −2.17
3	$\Rightarrow$ −4.87	$\Uparrow$ −3.44	$\Leftarrow$ −4.10	$\Leftarrow$ −5.38
2	$\Uparrow$ −5.98	$\Uparrow$ −4.87	$\Uparrow$ −4.69	$\Leftarrow$ −5.84
1	$\Rightarrow^{s_I}$ −6.13	$\Rightarrow$ −5.70	$\Uparrow$ −5.22	$\Leftarrow$ −6.26

$\gamma = 0.9$

- Can we learn more from this than the state-values of a policy?

Greedy Action: Example



$\gamma = 0.9$

- Can we learn more from this than the state-values of a policy?
- Yes! By evaluating all actions in each state, we can derive a better policy

Greedy Actions and Policies

Definition (Greedy Action)

Let s be a state of a MDP $\mathcal{T} = \langle S, A, R, T, s_I, \gamma \rangle$ and V be a state-value function for $\mathcal{T}$.

The set of greedy actions in s with respect to V is

$$A_V(s) := \arg\max_{a \in A(s)} \sum_{s' \in \mathrm{succ}(s,a)} T(s,a,s') \cdot (R(s,a,s') + \gamma \cdot V(s')) .$$

A policy π_V with $\pi_V(s) \in A_V(s)$ is a greedy policy.

Determining a greedy policy of a given state-value function is called policy improvement.

Policy Iteration

Policy Iteration

- policy iteration (PI) was first proposed by Howard in 1960
- based on the observation that greedy actions describe a better policy
- starts with arbitrary policy π_0
- alternates policy evaluation and policy improvement
- as long as policy changes

Example: Policy Iteration

	1	2	3	4
5	$\Rightarrow$ −3.66	$\Rightarrow$ −1.90	$\Rightarrow$ −1.00	noop 0.00
4	$\Rightarrow$ −4.29	$\Uparrow$ −2.71	$\Uparrow$ −7.30	$\Uparrow$ −2.17
3	$\Rightarrow$ −4.87	$\Uparrow$ −3.44	$\Leftarrow$ −4.10	$\Leftarrow$ −5.38
2	$\Uparrow$ −5.98	$\Uparrow$ −4.87	$\Uparrow$ −4.69	$\Leftarrow$ −5.84
1	$\Rightarrow$ s_I −6.13	$\Rightarrow$ −5.70	$\Uparrow$ −5.22	$\Leftarrow$ −6.26

$\gamma = 0.9$

π_0 and V_{π_0}

Example: Policy Iteration

5	$\Rightarrow$ −3.66	$\Rightarrow$ −1.90	$\Rightarrow$ −1.00	noop 0.00
4	$\Rightarrow$ −4.29	$\Uparrow$ −2.71	$\Uparrow$ −7.30	$\Uparrow$ −2.17
3	$\Rightarrow$ −4.87	$\Uparrow$ −3.44	$\Leftarrow$ −4.10	$\Uparrow$ −3.88
2	$\Uparrow$ −5.98	$\Uparrow$ −4.87	$\Uparrow$ −4.69	$\Leftarrow$ −5.84
1	$\Rightarrow^{s_I}$ −5.84	$\Uparrow$ −5.38	$\Uparrow$ −5.22	$\Leftarrow$ −6.26
	1	2	3	4

$\gamma = 0.9$

π_1 and V_{π_1}

Example: Policy Iteration

5	⇒ −3.66	⇒ −1.90	⇒ −1.00	noop 0.00
4	⇒ −4.29	⇑ −2.71	⇑ −7.30	⇑ −2.17
3	⇒ −4.87	⇑ −3.44	⇐ −4.10	⇑ −3.88
2	⇑ −5.98	⇑ −4.87	⇑ −4.69	⇑ −5.21
1	⇒ s_I −5.84	⇑ −5.38	⇑ −5.22	⇐ −6.26
	1	2	3	4

$\gamma = 0.9$

$\pi_2 = \pi_3$ and V_{π_2}

Computation of an Optimal Policy

there are various techniques to compute an optimal policy:

- ✓ (linear program)
- ✓ policy iteration
- value iteration

Value Iteration

From Policy Iteration to Value Iteration

- policy iteration:
  - search over policies
  - by evaluating their state-values
- value iteration:
  - search directly over state-values
  - optimal policy induced by final state-values

Value Iteration: Idea

- value iteration (VI) was first proposed by Bellman in 1957
- computes estimates $\hat{V}^0, \hat{V}^1, \ldots$ of $V_\star$ in an iterative process
- starts with arbitrary $\hat{V}^0$
- bases estimate $\hat{V}^{i+1}$ on values of estimate $\hat{V}^i$ by treating Bellman equation as update rule on all states:

$$\hat{V}^{i+1}(s) := \max_{a \in A(s)} \sum_{s' \in \text{succ}(s,a)} T(s,a,s') \cdot \left(R(s,a,s') + \gamma \hat{V}^i(s') \right)$$

- converges to state-values of optimal policy
- terminates when maximal change is smaller than residual ϵ

Value Iteration: Example

	1	2	3	4
5	0.00	0.00	0.00	noop 0.00
4	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00
1	s_I 0.00	0.00	0.00	0.00

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}^0$

- reward to `move` from striped cell is -3 (-1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

	1	2	3	4
5	−1.00	−1.00	−1.00	noop 0.00
4	−1.00	−1.00	−3.00	−1.00
3	−1.00	−1.00	−1.00	−1.00
2	−1.00	−1.00	−1.00	−1.00
1	s_I −1.00	−1.00	−1.00	−1.00

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}^1$

max diff: 3.0

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

5	−1.90	−1.90	−1.00	noop 0.00
4	−1.90	−1.90	−4.98	−1.54
3	−1.90	−1.90	−1.90	−1.90
2	−1.90	−1.90	−1.90	−1.90
1	s_I −1.90	−1.90	−1.90	−1.90
	1	2	3	4

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}^2$

max diff: 1.98

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

	1	2	3	4
5	−3.38	−1.90	−1.00	noop 0.00
4	−3.83	−2.71	−6.94	−2.07
3	−4.10	−3.44	−3.75	−3.36
2	−4.10	−4.10	−3.99	−3.93
1	s_I −4.10	−4.10	−4.10	−4.08

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}^5$

max diff: 0.6561

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

	1	2	3	4
5	−3.65	−1.90	−1.00	noop 0.00
4	−4.27	−2.71	−7.29	−2.17
3	−4.83	−3.44	−4.10	−3.84
2	−5.78	−4.83	−4.69	−5.05
1	s_I −5.74	−5.32	−5.22	−5.84

$\gamma = 0.9$

$\epsilon = 0.001$

$\hat{V}^{10}$

max diff: 0.187

- reward to `move` from striped cell is -3 (-1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

	1	2	3	4
5	−3.66	−1.90	−1.00	noop 0.00
4	−4.29	−2.71	−7.30	−2.17
3	−4.87	−3.44	−4.10	−3.88
2	−5.98	−4.87	−4.69	−5.21
1	s_I −5.84	−5.38	−5.22	−6.25

$\gamma = 0.9$

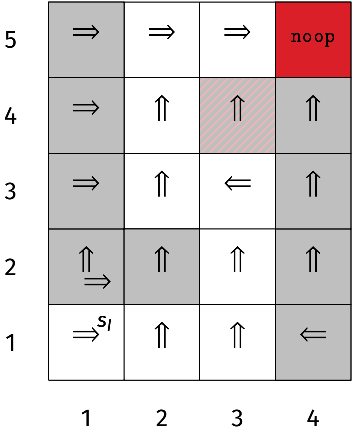
$\epsilon = 0.001$

$\hat{V}^{23}$

max diff: 0.0007

- reward to `move` from striped cell is −3 (−1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Value Iteration: Example

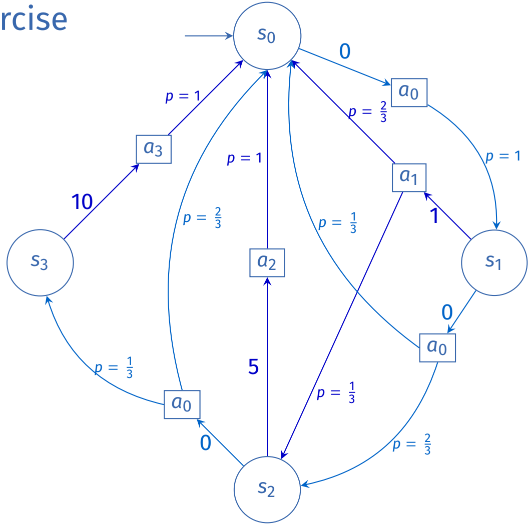


$\gamma = 0.9$

$\epsilon = 0.001$

- reward to `move` from striped cell is -3 (-1 everywhere else)
- `move` in gray cells unsuccessful with probability 0.6
- only applicable action in red cell is `noop` for reward of 0

Exercise



Using $\gamma = 0.9$ and the initial $\hat{V}^0$ values $\hat{V}^0(s_0) = 10$, $\hat{V}^0(s_1) = 0$, $\hat{V}^0(s_2) = 0$, $\hat{V}^0(s_3) = 0$, perform five iterations of value iteration.

Exercise: Solution

i	$\hat{V}^i(s_0)$	$\hat{V}^i(s_1)$	$\hat{V}^i(s_2)$	$\hat{V}^i(s_3)$
0	10	0	0	0
1	0	7	14	19
2	6.3	8.4	5.7	10
3	7.56	6.49	10.67	15.67
4	5.84	8.74	11.8	16.8
5	7.87	8.83	10.26	15.26

More about Planning

1. TDDD48 Automated Planning
2. MSc and PhD theses available in my group

Summary

Policy Iteration or Value Iteration?

- Policy evaluation is slightly cheaper than a VI iteration
  - PI faster than VI if few iterations required
- Asynchronous VI is basis of more sophisticated algorithm that can be applied in large MDPs and SSPs

Summary

- Policy iteration alternates policy evaluation and policy improvement.
- Value Iteration searches in the space of state-values and applies Bellman equation as update rule iteratively.

Quiz

Kahoot!