Artificial Intelligence

Logic 4: First-Order Logic

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First-Order Logic

What is First-Order Logic?

- First-order logic (FOL) is a formal system used in mathematics, philosophy, linguistics, and computer science.
- It allows for the expression of statements involving objects and their relationships.
- FOL extends propositional logic by introducing quantifiers, predicates, and functions.

Example 1: Universal Quantification

- Statement: "All humans are mortal."
- Formalization: $\forall x(\text{Human}(x) \longrightarrow \text{Mortal}(x))$

Syntax of First-Order Logic

Definition (Formulas)

- Every constant and variable is a term.
- If $t_1, t_2, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, t_2, \ldots, t_n)$ is a term.
- If $t_1, t_2, \ldots, t_n$ are terms, then $P(t_1, t_2, \ldots, t_n)$ is an atomic formula.
- If φ is a formula, then $\neg\varphi$ is a formula (negation).
- If φ and ψ are formulas, then so are:
  - $(\varphi \land \psi)$ (conjunction)
  - $(\varphi \lor \psi)$ (disjunction)
  - $(\varphi \rightarrow \psi)$ (implication)
  - $(\varphi \leftrightarrow \psi)$ (biconditional)
- If φ is a formula and x is a variable, then the following are formulas:
  - $\forall x \varphi$ (universal quantifier)
  - $\exists x \varphi$ (existential quantifier)

Semantics of First-Order Logic

Let I be an interpretation for a first-order language $\mathcal{L}$, including:

- A domain D of discourse (a non-empty set).
- An interpretation c^I, f^I, P^I for each constant symbol c, n-ary function symbol f, and n-ary predicate symbol P over D.

Definition ($I \models \varphi$)

- $I \models P(t_1, t_2, \ldots, t_n)$ iff $(t_1^I, t_2^I, \ldots, t_n^I) \in P^I$.
- $I \models \neg\varphi$ iff $I \not\models \varphi$.
- $I \models \varphi \wedge \psi$ iff $I \models \varphi$ and $I \models \psi$.
- $I \models \varphi \vee \psi$ iff $I \models \varphi$ or $I \models \psi$.
- $I \models \varphi \rightarrow \psi$ iff $I \not\models \varphi$ or $I \models \psi$.
- $I \models \varphi \leftrightarrow \psi$ iff $I \models \varphi$ and $I \models \psi$, or $I \not\models \varphi$ and $I \not\models \psi$.
- $I \models \forall x \varphi$ iff for every $d \in D$, $I[x \mapsto d] \models \varphi$.
- $I \models \exists x \varphi$ iff there exists $d \in D$ such that $I[x \mapsto d] \models \varphi$.

Comparison with Propositional Logic

Propositional Logic vs. First-Order Logic

- Propositional Logic:
  - Deals with propositions that are either true or false.
  - Example: $P \wedge Q$, where P and Q are propositions.
- First-Order Logic:
  - Deals with predicates, quantifiers, and functions.
  - Example: $\forall x(P(x) \longrightarrow Q(x))$, where P and Q are predicates.

Why Use First-Order Logic?

- FOL is more expressive than propositional logic.
- Allows for reasoning about the properties of objects and their relationships.
- Essential for formalizing mathematical theorems, computer programs, and linguistic statements.

Grounding First-Order Logic to Propositional Logic

We can convert FOL formulas into propositional logic by replacing variables with all possible values from the domain.

Example

- FOL Statement: $\forall x(\text{Human}(x) \longrightarrow \text{Mortal}(x))$
- Domain: $\{a, b\}$ (representing specific humans)
- Grounding Steps:
  - Replace x by each element in the domain.
  - $\text{Human}(a) \longrightarrow \text{Mortal}(a)$
  - $\text{Human}(b) \longrightarrow \text{Mortal}(b)$
- Grounded Propositional Logic:
  - $P \longrightarrow Q$ where $P = \text{Human}(a)$ and $Q = \text{Mortal}(a)$
  - $R \longrightarrow S$ where $R = \text{Human}(b)$ and $S = \text{Mortal}(b)$

Examples

Example 2: Existential Quantification

- **Statement:** "There exists a person who is a genius."

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- Formalization: $\exists x(\text{Person}(x) \wedge \text{Genius}(x))$

Example 3: Relations

- **Statement:** "Some people love everyone."

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- Formalization: $\exists x \forall y(\text{Loves}(x, y))$

Example 4: Combining Quantifiers

- **Statement:** "Every student in the class has submitted an answer."

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- **Statement:** "Every student in the class has submitted an answer."
- **Formalization:**
  $\forall x(\text{Student}(x) \longrightarrow \exists y(\text{Answer}(y) \land \text{Submitted}(x, y)))$

Example 5: Multiple Relations

- Statement: "Every dog has a master who loves it."

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- Statement: "Every dog has a master who loves it."
- Formalization:
  $\forall x(\mathrm{Dog}(x) \longrightarrow \exists y(\mathrm{Human}(y) \wedge \mathrm{MasterOf}(y, x) \wedge \mathrm{Loves}(y, x)))$

Example 6: Negation with Quantifiers

- Statement: "Not every student likes math."

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- Statement: "Not every student likes math."
- Formalization: $\neg\forall x(\text{Student}(x) \longrightarrow \text{LikesMath}(x))$
- Equivalent: $\exists x(\text{Student}(x) \land \neg\text{LikesMath}(x))$

Example 7: Using Functions

- **Statement:** "The mother of every child is a woman."

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- **Statement:** "The mother of every child is a woman."
- **Formalization:** $\forall x(\mathrm{Child}(x) \longrightarrow \mathrm{Woman}(\mathrm{motherOf}(x)))$