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9. Suppose that a fair die is thrown until a six appears. What is the probability that this will take more than three throws?

10. Illustrate on a Venn diagram that $P(A \cap B) \leq P(A) \leq P(A \cup B)$ for any events A and B. Also, give a brief intuitive explanation of this rule of probability.

11. A famous eighteenth-century mathematician, d'Alembert, argued that in two tosses of a coin, heads could appear once, twice, or not at all, and thus each of these three events should be assigned a probability of one-third. Do you agree with him? Comment on the issues involved and on the implications of the situation with regard to the use of symmetry assumptions to determine probabilities.

12. It is claimed by some that defining probability in terms of Equation 2.2.1 is a circular definition because it involves "equally likely" events, thereby using the notion of probability in the definition of probability. Discuss this claim.

13. The so-called "gambler's fallacy" goes something like this: In a dice game, for example, a seven has not turned up in quite a few rolls of a pair of honest dice. Therefore, a seven is said to be "due" to come up. Why is this a fallacy?

14. The law of large numbers has occasionally been called "the link between the mathematical concept of probability and the real world about us." Discuss this proposition.

15. Take a fair die and throw it 300 times, recording the number that appears on each throw. Find the frequency of each of the six possible events after 2, 5, 10, 20, 50, 100, and 300 throws. Repeat this experiment several times and comment on the results in light of the law of large numbers and the concept of statistical regularity.

16. In what sense is the relative-frequency interpretation of probability a conceptual interpretation rather than an operational interpretation?

17. Find the probability of the event E if the odds in favor of E are

(a) 2 to 1, (c) 3 to 7,
(b) 1 to 2, (d) 9 to 2.

18. Find the odds in favor of the event E if the probability of E is

(a) 0.50, (c) 0.875,
(b) 0.20, (d) 0.001.

19. Explain why the subjective interpretation of probability can be thought of as an extension of the frequency interpretation. Are there any restrictions on the types of situations for which subjective probabilities can be used?

20. Discuss the following statement: "If a person feels subjectively that an event will occur with relative frequency p in a long series of identical, independent trials, then his subjective probability of the event occurring on any single trial should be p."

21. Explain why the basic axioms of probability presented in Section 2.1 must be obeyed under both (a) the relative-frequency interpretation of probability and (b) the subjective interpretation of probability.

22. (a) What is your subjective probability that it will rain tomorrow?
(b) What, in your opinion, are the odds in favor of rain tomorrow?

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(b) Although psychological experiments have indicated that many individuals are *not* indifferent between the two lotteries, it is claimed by some that they *should* be indifferent. Discuss.

32. Discuss each of the following situations with regard to the different interpretations of probability:
    (a) A coin has been tossed 1000 times and exactly 500 heads have been observed. What is the probability of heads on the next toss?
    (b) A brand-new coin is obtained at a bank. What is the probability of heads on the next toss?
    (c) A coin was tossed at the beginning of the 1970 Stanford-Washington football game to determine choice of kicking versus receiving or choice of goal. What is the probability that the coin came up heads?
    (d) A coin is tossed, placed on a table, and covered with a sheet of paper. What is the probability that "heads" is up?
    (e) A stranger produces a coin and tosses it 10 times. Each time it comes up heads. What is the probability of heads on the next toss?
    (f) You take a brand new coin and toss it 10 times. Each time it comes up heads. What is the probability of heads on the next toss?

33. What, in your opinion, is the probability that Duke will win the next basketball game they play against North Carolina? What is the probability that Duke beat North Carolina in the basketball game they played in Chapel Hill in 2001?

34. Suppose that you are contemplating a picnic on the coming Fourth of July. If you are concerned about the weather on that day, how might you define the relevant sample space, taking into account any factors about the weather that might have some effect on your decision concerning the picnic?

35. If $P(B)=0$, does it make sense to consider the conditional probability $P(A|B)$? Explain.

36. Discuss the statement "In a sense, all probabilities are conditional."

37. Explain the difference between mutually exclusive events and independent events. Is it possible for two events A and B to be both mutually exclusive and independent? Explain.

38. Suppose that you are offered the following choice of lotteries:

    Lottery A: If the winner of the next presidential election in the United States is a member of the Democratic party, then you win \$10 with probability 1/2 and you lose \$10 with probability 1/2. If the winner of the election is not a Democrat, you win \$0.

    Lottery B: If the winner of the next presidential election in the United States is a member of the Democratic party, then you win \$10 if the Senate has more Democrats than Republicans at the beginning of the president's term of office, and you lose \$10 if the Senate does not have more Democrats than Republicans at that time. If the winner of the election is not a Democrat, you win \$0.

    Which lottery would you select? What canonical probabilities in Lottery A would make you indifferent between the two lotteries? What subjective probability of yours is being assessed here?

39. Combining the results of Exercises 28 and 38, what is your joint probability for the event that the next presidential election is won by a Democrat *and* the Senate has more Democrats than

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47. In the first medical example in Section 2.7, the new information, which consists of a positive reading on the tuberculin skin test, increases the probability of tuberculosis from 0.01 to 0.165. This result is surprising to some people, who claim that they expected a greater increase in the probability of tuberculosis. Explain why the increase is no greater, despite the fact that the tuberculin skin test results in only a 0.02 chance of a "false negative" reading for a person with tuberculosis and a 0.05 chance of a "false positive" reading for a person without tuberculosis.

48. The probability that 1 percent of the items produced by a certain process are defective is 0.80, the probability that 5 percent of the items are defective is 0.10, and the probability that 10 percent of the items are defective is 0.10. An item is randomly chosen, and it is defective. *Now* what is the probability that 1 percent of the items are defective? That 5 percent are defective? That 10 percent are defective? Suppose that a *second* item is randomly chosen from the output of the process, and it too is defective. Following this second observation, what are the probabilities that 1, 5, and 10 percent, respectively, of the items produced by the process are defective?

49. In a football game, suppose that a team has time for one more play and that they need to score a touchdown on this play to win the game. As a fan, you feel that there are only three possible plays that can be used, and that the three plays are equally likely to be used. The plays are a long pass, a screen pass, and an end run. Based on past experience in similar situations, you feel that the probabilities of getting a touchdown with these plays are as follows:

Play	*P(touchdown | play)*
Long pass	0.50
Screen pass	0.30
End run	0.10

The team scores a touchdown on the play. What is the probability that they used the long pass? The screen pass? The end run?

50. Suppose that Urn *A* is filled with 700 red balls and 300 green balls and that Urn *B* is filled with 700 green balls and 300 red balls. One of the two urns is selected at random (that is, the two urns are equally likely to be selected). The experiment consists of selecting a ball at random from the chosen urn, recording its color, and then replacing it and thoroughly mixing the balls again. This experiment is conducted three times, and each time the ball chosen is red. What is the probability that the urn chosen is Urn *A*?

51. You are interested in the stock of the XYZ corporation. You feel that if the stock market goes up in the next year, the probability is 0.9 that the price of XYZ stock will go up. If the market goes down, the probability is 0.4 that XYZ will go up. Finally, if the market remains steady, the probability is 0.7 that XYZ will go up. Furthermore, you think that the probabilities are 0.5, 0.3, and 0.2 for the market to go up, go down, or remain steady. At the end of the year, the price of XYZ stock has not gone up. What is the probability that the stock market as a whole went up?

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15. Complete the following table, given that $P(\tilde{x} = 1 \mid \tilde{y} = 2) = 1/3$ and $P(\tilde{y} = 3 \mid \tilde{x} = 2) = 1/2$.

$\tilde{x}$ \ $\tilde{y}$	1	2	3	4	
1				0.15	0.60
2				0.05	
	0.10		0.50	0.20	

16. Complete the following table of probabilities, given that $\tilde{x}$ and $\tilde{y}$ are independent.

$\tilde{x}$ \ $\tilde{y}$	10	20	30	
-1	0.05			
0			0.20	0.50
1				
	0.20			

17. From the joint distribution of $\tilde{x}$ and $\tilde{y}$ presented in Exercise 15,
    (a) find the conditional distribution of $\tilde{x}$, given that $\tilde{y} = 1$,
    (b) find the conditional distribution of $\tilde{x}$, given that $\tilde{y} = 3$,
    (c) find the conditional distribution of $\tilde{y}$, given that $\tilde{x} = 1$,
    (d) find the conditional distribution of $\tilde{y}$, given that $\tilde{x} = 2$,
    (e) find $E(\tilde{x} \mid \tilde{y} = 1)$, $E(\tilde{x} \mid \tilde{y} = 3)$, $E(\tilde{y} \mid \tilde{x} = 1)$, and $E(\tilde{y} \mid \tilde{x} = 2)$,
    (f) find $V(\tilde{x} \mid \tilde{y} = 1)$, $V(\tilde{x} \mid \tilde{y} = 3)$, $V(\tilde{y} \mid \tilde{x} = 1)$, and $V(\tilde{y} \mid \tilde{x} = 2)$,
    (g) given the marginal distributions of $\tilde{x}$ and $\tilde{y}$, what would the joint distribution be if $\tilde{x}$ and $\tilde{y}$ were independent?

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$V(\tilde{p})$. The researcher conducts an experiment in which 20 persons are exposed to the disease and 2 catch it. On the basis of this new information, find the researcher's posterior distribution for $\tilde{p}$ and find the mean and the variance of this posterior distribution.

27. A veteran public official is up for reelection and is concerned about $\tilde{p}$, the proportion of the votes cast that will be for him. In the past, he has run for office 20 times; he received 48 percent of the vote four times, 50 percent of the vote four times, 52 percent of the vote five times, 54 percent of the vote four times, 56 percent of the vote twice, and 58 percent of the vote once. In the upcoming election, however, his opponent is young, vigorous, and has a large campaign budget. As a result, the veteran official feels that the past frequencies slightly overstate his chances. He feels that this opponent will obtain about 2 percent more of the votes than his past opponents, so that his prior distribution consists of the probabilities $P(\tilde{p} = 0.46) = 4/20$, $P(\tilde{p} = 0.48) = 4/20$, $P(\tilde{p} = 0.50) = 5/20$, and so on. He then conducts a small survey of 100 voters, 49 of whom state that they will vote for him and 51 of whom state that they will vote for his young opponent. Find the posterior distribution of $\tilde{p}$ and use this distribution to calculate the expected proportion of votes the veteran public official will obtain.

28. In sampling from a Bernoulli process, it is possible to sample with a fixed r and a variable $\tilde{n}$ instead of with a fixed n and a variable $\tilde{r}$. The former sampling procedure is called Pascal sampling, while the latter procedure is called binomial sampling. For an example of Pascal sampling, a quality–control statistician might decide to observe items from a production process until he finds five defectives, so that $r = 5$ and $\tilde{n}$ is an uncertain quantity (it may only take five trials to find five defectives, or it may take, say, 1000 trials). The conditional probability distribution of $\tilde{n}$, given r and $\tilde{p}$, which is known as the Pascal distribution, is of the form

$$P(\tilde{n} = n|r, p) = \binom{n-1}{r-1} p^r (1-p)^{n-r} = \frac{(n-1)!}{(r-1)!(n-r)!} p^r (1-p)^{n-r},$$

where $\tilde{n}$ can take on the values $r, r+1, r+2, \ldots$ For the example presented in Section 3.3, calculate the posterior probabilities under the assumption that the sampling was conducted with r fixed rather than with n fixed (that is, for the first sample, assume Pascal sampling with $r = 1$; for the second sample, assume Pascal sampling with $r = 2$). Compare your results with the posterior probabilities calculated in Section 3.3 and explain the relationship between the two sets of results.

29. Explain the similarities and differences between the Bernoulli process and the Poisson process. Give some examples of realistic situations in which the Bernoulli model might be applicable, and do the same for the Poisson model. Explain the terms "stationary" and "independent" with regard to a Poisson process.

30. If $\tilde{p}$ is small and n is large, binomial probabilities may be approximated by Poisson probabilities with $\tilde{\lambda} t = n\tilde{p}$. For $n = 100$ and $\tilde{p} = 0.05$, compare the binomial distribution with the Poisson approximation.

31. Suppose that cars arrive at a toll booth according to a Poisson process with intensity five per minute. What is the probability that no cars will arrive in a particular minute? What is the probability that more than eight cars will arrive in a particular minute? What is the probability that

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number found in any other cubic millimeter. You feel that the "intensity of occurrence" of these microorganisms is either 50 per cubic millimeter, in which case the body of water is not polluted; 75 per cubic millimeter, in which case it is mildly polluted; or 100 per cubic millimeter, in which case it is highly polluted. Furthermore, on the basis of your current information about the body of water, you think that (a) the odds are 3 to 1 that it is polluted and (b) given that it is polluted, the odds are even that it is highly polluted. You obtain sample information in the form of two cubic millimeters of water from the body of water, and these two cubic millimeters contain 180 microorganisms. What is your posterior distribution for the intensity of occurrence of the microorganisms?

39. Suppose that $\tilde{p}$ represents the proportion (rounded to the nearest tenth) of males among the full-time students at the University of Chicago. Thus, the possible values of $\tilde{p}$ are 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0. Assess your subjective probability distribution for $\tilde{p}$. If you take a random sample of 10 students and observe 6 males, use this sample to revise your distribution.

40. Do the first part of Exercise 39, letting $\tilde{p}$ represent (a) the proportion of students at the University of Chicago who are undergraduates, (b) the proportion of students at the University of Chicago who are United States citizens, and (c) the proportion of 21 year–olds in the United States who are *full–time* students at colleges or universities.

41. In Exercise 31, suppose that after seeing a sample in which 8 cars arrive at the toll booth in a two–minute period, your posterior distribution for $\tilde{\lambda}$, intensity per minute, is $P(\tilde{\lambda} = 4) = .40$, $P(\tilde{\lambda} = 5) = 0.50$, and $P(\tilde{\lambda} = 6) = 0.10$. What was your prior distribution for $\tilde{\lambda}$ before seeing this sample?

42. Suppose that $\tilde{\lambda}$ represents the number of planes per minute landing at O'Hare Field in Chicago during the hours of 5 P.M. to 7 P.M. on a Friday night in June. Assuming for simplicity that $\tilde{\lambda}$ can only take on integer values, assess your prior distribution for $\tilde{\lambda}$. If you are then told that, in a randomly chosen 30–second period during the above–stated hours, exactly four planes landed, revise your distribution for $\tilde{\lambda}$ intuitively without using any formulas. Then, assuming a Poisson process, revise your distribution formally according to Bayes' theorem. Explain any differences in the two revised distributions; what are the possible implications of these differences with respect to this example and with respect to Bayesian inference in general?

43. Suppose that you are interested in $\tilde{\theta}$, the sales (in thousands) of a particular product, and that $\tilde{\theta}$ depends on whether or not a competing firm introduces a new product. Let $\tilde{\phi} = 1$ if the new product is introduced and let $\tilde{\phi} = 0$ otherwise. From your knowledge of the competing firm, you assess $P(\tilde{\phi} = 1) = 0.30$. Furthermore, suppose that the possible values of $\tilde{\theta}$ are taken to be 200, 250, 300, and 350. You feel that if the competitor introduces a new product, your probabilities for the four possible values of $\tilde{\theta}$ are 0.30, 0.50, 0.15, and 0.05, respectively. If the new product is not introduced, your probabilities for $\tilde{\theta}$ are 0.10, 0.10, 0.40, and 0.40, respectively. Express these probability assessments on a tree diagram and calculate the joint probability distribution of $\tilde{\phi}$ and $\tilde{\theta}$. From this joint distribution, find the marginal distribution of $\tilde{\theta}$.

44. Suppose that $\tilde{\theta} = 1$ if a member of the Democratic party is elected president of the United States in the next presidential election, $\tilde{\theta} = 2$ if a member of the Republican party is elected president, and $\tilde{\theta} = 3$ if a person who is a member of neither party is elected president. Further-

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more, let $\tilde{\phi}$ represent the number of senators from the Democratic party after the next presidential election. Assess your subjective probability distribution of $\tilde{\theta}$, and assess your subjective conditional probability distribution of $\tilde{\phi}$, given that $\tilde{\theta} = 1$, given that $\tilde{\theta} = 2$, and given that $\tilde{\theta} = 3$. Use these distributions to find the joint distribution of $\tilde{\theta}$ and $\tilde{\phi}$ and the marginal distribution of $\tilde{\phi}$.

45. Why is the assessment of a joint probability distribution of two random variables greatly simplified if it can be assumed that the two variables are independent?

46. Explain the differences among the binomial, multinomial, and hypergeometric distributions, and for each, give examples of realistic situations in which the model might be applicable. When might the binomial distribution serve as a good approximation to the hypergeometric distribution? Explain your answer.

47. A university committee consists of six faculty members and four students. If a subcommittee of four persons is to be chosen randomly from the committee, what is the probability that it will consist of *at least* two faculty members? Compute this probability from the hypergeometric distribution *and* from the binomial approximation with $\tilde{p} = \tilde{R}/\tilde{N}$. How good is the approximation?

48. If 12 cards are to be drawn at random without replacement from a standard, 52–card deck, what is the probability that there will be exactly three cards of each suit? What is this probability if the 12 cards are drawn *with* replacement? (Assume that the deck is thoroughly shuffled before each draw.)

49. For any given match, the probabilities that a soccer team will win, tie, or lose are 0.3, 0.4, and 0.8. If the team plays 10 matches and if the outcomes of the different matches are assumed to be independent, what is the probability that they will win five, tie three, and lose two? What is the probability that they will win exactly eight matches?

50. What is the probability that in 24 tosses of a fair die, each face will occur exactly four times? What is the probability that in six tosses, each face will occur exactly once?

51. An instructor in an introductory statistics class knows that out of 50 students, 12 are freshmen, 25 are sophomores, 11 are juniors, and only 2 are seniors. He assigns these students to 5 extra review sessions randomly, 10 to a session. What is the probability that the first session consists of 1 freshman, 1 sophomore, and 8 juniors? What is the probability that the first session consists only of sophomores *and* that the second session consists of 8 juniors and 2 seniors?

52. Why are statistical models such as the Bernoulli and the Poisson models useful with regard to the assessment of likelihoods? What are the advantages and disadvantages of using such models?

53. Suppose that a judge is presiding over an antitrust case and that he is concerned with the effect of a particular merger on competition in the brewing industry. Let $\tilde{\theta} = 2$ if the merger has a severe effect on competition, $\tilde{\theta} = 1$ if it has a minor effect, and $\tilde{\theta} = 0$ if it has no effect. The judge feels that $\tilde{\theta} = 0$ and $\tilde{\theta} = 1$ are equally likely and that each is three times as likely as $\tilde{\theta} = 2$. An economist is called in as a witness to provide additional information, and the gist of his tes-

(b) Find the marginal density functions of $\tilde{x}$ and $\tilde{y}$.
(c) Find the conditional density function of $\tilde{x}$, given that $\tilde{y} = 1$.
(d) Find the conditional density function of $\tilde{x}$, given that $\tilde{y} = 1/2$.
(e) Are $\tilde{x}$ and $\tilde{y}$ independent?
(f) Find $E(\tilde{x})$ and $E(\tilde{y})$.

7. In Bayes' theorem, why is it necessary to divide by Σ(prior probability)(likelihood) in the discrete case and by $\int$ (prior density)(likelihood) in the continuous case?

8. Suppose that $\tilde{\theta}$ represents the rate of return (expressed in decimal form, not in percentage form) for a particular investment and that your uncertainty about $\tilde{\theta}$ can be expressed in terms of the following probability distribution:

$$f(\theta) = \begin{cases} 100(\theta + 0.10)/3 & \text{if } -0.10 \le \theta \le 0.10, \\ 200(0.20 - \theta)/3 & \text{if } \ \ 0.10 \le \theta \le 0.20, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Graph this prior distribution and discuss what it implies about your judgments concerning $\tilde{\theta}$.
(b) An investment analyst is trying to convince you that this is a good investment (he will receive a commission if you make the investment), and he claims that the return on the investment will be 0.15. Treating this claim as new information and denoting it by y, you decide that your likelihood function (as a function of θ) is

$$f(y \mid \theta) = 5 \quad \text{if} -0.10 \le \theta \le 0.20 .$$

Graph this likelihood function and comment on its implications concerning the new information.
(c) On the basis of the new information y, revise your distribution of $\tilde{\theta}$.
(d) The posterior distribution in part (c) applies only to this specific problem. However, can you generalize this result? Explain.

9. In Exercise 8, suppose that your likelihood function is

$$f(y \mid \theta) = \begin{cases} 8(\theta + 0.10) & \text{if } -0.10 \le \theta \le 0.12, \\ 22(0.20 - \theta) & \text{if } \ \ 0.12 \le \theta \le 0.20, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Graph this likelihood function and comment on its implications concerning the new information.
(b) On the basis of the new information, revise your distribution of $\tilde{\theta}$.

10. Suppose that you are interested in $\tilde{p}$, the proportion of station wagons among the registered vehicles in a particular state. Your prior distribution for $\tilde{p}$ is a normal distribution with mean 0.05 and variance 0.0004. To obtain more information, a random sample of 50 registered vehicles is taken, and three are station wagons.

(a) In using Equation 4.2.3 to revise your distribution of $\tilde{p}$, what difficulties are encountered?
(b) How might you avoid such difficulties in this situation? Can they always be avoided?

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11. Discuss the role of conjugate families of distributions in Bayesian statistics.

12. Find the mean and the variance of the beta distribution with parameters $r' = 2$ and $n' = 6$, and graph the density function. Do the same for the following beta distributions:

    $r' = 4, n' = 6$; $r' = 4, n' = 12$; $r' = 8, n' = 12$.

    Explain how the different values of r' and n' affect the shape and the location of these four distributions.

13. If the mean and the variance of a beta distribution with parameters r' and n' are 2/3 and 1/72, respectively, find r' and n'.

14. If the mean and the 0.05 fractile of a beta distribution with parameters r' and n' are 0.20 and 0.13, respectively, find r' and n'.

15. In Exercise 23 in Chapter 3, suppose that the prior distribution could be represented by a beta distribution with $r' = 4$ and $n' = 10$. Find the posterior distribution. Also, find the posterior distribution corresponding to the following beta prior distributions:

    $r' = 2, n' = 5$; $r' = 8, n' = 20$; $r' = 6, n' = 15$.

    In each of the four prior distributions considered in this exercise, the mean of the prior distribution is 0.40. How, then, do you explain the differences in the means of the posterior distributions?

16. In Exercise 24 in Chapter 3, suppose that you feel that the mean of your prior distribution is 1/2 and that the variance of the distribution is 1/20. If your prior distribution is a member of the beta family, find r' and n' and determine the posterior distribution following the sample of size six. Graph the density functions and find the mean and the variance of the posterior distribution.

17. In sampling from a Bernoulli process, the posterior distribution is the same whether one samples with n fixed (binomial sampling) or with r fixed (Pascal sampling). Explain why this is true. Suppose that a statistician merely samples until he is tired and decides to go home. Would the posterior distribution still be the same (that is, is the stopping rule noninformative)?

18. Try to assess a subjective distribution of $\tilde{p}$, the probability of rain tomorrow in the city where you live. Can you find a beta distribution that expresses your judgments reasonably well?

19. In Exercise 26 in Chapter 3, suppose that the medical researcher decides to treat $\tilde{p}$ as a continuous random variable. She subjectively assesses the mean, the 0.25 fractile, and the 0.75 fractile of her distribution to be 0.167, 0.118, and 0.187, respectively. Can you find a beta distribution satisfying these assessments? If not, explain (in terms of the general "shape" of the distribution) why not.

20. Suppose that $\tilde{x}$ is normally distributed with mean 3 and variance 16.

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(a) Find $P(1 \le \tilde{x} \le 7)$, $P(\tilde{x} \le 5)$, $P(-2 \le \tilde{x} \le 1.5)$.
(b) Find a number c such that $P[(\tilde{x} - 2) < c] = 0.95$.
(c) Find the 0.05, 0.25, 0.67, and 0.99 fractiles of the distribution of $\tilde{x}$.
(d) Graph the density function of $\tilde{x}$.

21. If the 0.35 fractile of a normal distribution is 105 and the 0.85 fractile is 120, find the mean and the standard deviation of the distribution.

22. The normal model of a data-generating process cannot be conveniently explained in terms of assumptions such as stationarity and independence, as can the Bernoulli and Poisson models. How, then, can the use of the normal model be justified in any specific application? Give some examples of realistic situations in which the normal model might be a suitable representation of a data-generating process.

23. For a random sample of size n from a data-generating process with mean μ and variance σ^2, show that

$$E(\tilde{m} \mid \mu, \sigma^2) = \mu$$

and

$$V(\tilde{m} \mid \mu, \sigma^2) = \frac{\sigma^2}{n},$$

where $\tilde{m}$ is the sample mean. If $n = 5$ and the sample results are 12, 18, 13, 15, and 19, find the sample mean and the sample variance.

24. A sample of size 500 is taken from a Bernoulli process with $\tilde{p} = 0.4$. Using the normal approximation to the binomial distribution, find the probability of observing at least 180 "successes" and the probability of observing no more than 210 "successes."

25. For each of the following Bernoulli situations, determine and graph the appropriate binomial distribution, the normal approximation to the binomial, and the Poisson approximation to the binomial.

(a) $n = 10$, $\tilde{p} = 0.05$.
(b) $n = 20$, $\tilde{p} = 0.05$.
(c) $n = 10$, $\tilde{p} = 0.40$.
(d) $n = 20$, $\tilde{p} = 0.40$.

Compare the two approximations in each case.

26. In Exercise 24 in Chapter 3, suppose that the sample information consists of 85 heads in 200 tosses of the coin. Revise your distribution of $\tilde{p}$ using the normal approximation to the binomial distribution to determine the likelihoods.

27. A production manager is interested in the mean weight of items turned out by a particular process. He feels that the weight of items from the process is normally distributed with mean $\tilde{\mu}$ and that $\tilde{\mu}$ is either 109.4, 109.7, 110.0, 110.3, or 110.6. The production manager assesses prior probabilities of $P(\tilde{\mu} = 109.4) = 0.05$, $P(\tilde{\mu} = 109.7) = 0.20$, $P(\tilde{\mu} = 110.0) = 0.50$, $P(\tilde{\mu} = 110.3) = 0.20$, and $P(\tilde{\mu} = 110.6) = 0.05$. From past experience, he is willing to assume that the process variance

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is $\sigma^2 = 4$. He randomly selects five items from the process and weighs them, with the following results: 108, 109, 107.4, 109.6, and 112. Find the production manager's posterior distribution and compute the means and the variances of the prior and posterior distributions.

28. In Exercise 27, if $\tilde{\mu}$ is assumed to be continuous and if the prior distribution for $\tilde{\mu}$ is a normal distribution with mean 110 and variance 0.4, find the posterior distribution.

29. You are attempting to assess a prior distribution for the mean of a process, and you decide that the 0.25 fractile of your distribution is 160 and the 0.60 fractile is 180. If your prior distribution is normal, determine the mean and the variance.

30. In reporting the results of a statistical investigation, a statistician reports that his posterior distribution for $\tilde{\mu}$ is a normal distribution with mean 52 and variance 10 and that his sample of size four with sample mean 55 was taken from a normal population with variance 100. On the basis of this information, determine the statistician's *prior* distribution.

31. Explain the parametrization of a normal prior distribution in terms of n' and m', as given in Section 4.5. How does this parametrization make it easier to see the relative weights of the prior and the sample information in computing the mean of the posterior distribution? For the prior and posterior distributions in Exercise 28, express the distributions in terms of n', m', n'', and m''. How could you interpret this prior distribution in terms of an equivalent sample? Also, express the distributions in Exercise 28 in terms of the parametrization involving the measures of information I', I, and I''.

32. In assessing a distribution for the mean height of a certain population of college students, a physical-education instructor decides that his distribution is normal, the median is 70 inches, the 0.20 fractile is 67 inches, and the 0.80 fractile is 72 inches. Can you find a normal distribution with these fractiles? Comment on the ways in which the instructor could make his assessments more consistent.

33. Suppose that a data-generating process is a normal process with unknown mean $\tilde{\mu}$ and with known variance $\sigma^2 = 225$. A sample of size $n = 9$ is taken from this process, with the sample results 42, 56, 68, 56, 48, 36, 45, 71, and 64. If your prior judgments about $\tilde{\mu}$ can be represented by a normal distribution with mean 50 and variance 14, what is your posterior distribution for $\tilde{\mu}$? From this distribution, find $P(\tilde{\mu} \geq 50)$ and $P(\tilde{\mu} \geq 55)$.

34. The number of customers entering a certain store on a given day is assumed to be normally distributed with unknown mean $\tilde{\mu}$ and unknown variance $\tilde{\sigma}^2$. As the store manager, you feel that your prior distribution for $\tilde{\mu}$ is a normal distribution with mean 1000 and that $P(900 \leq \tilde{\mu} \leq 1100) = 0.95$. You then take a random sample of 10 days, observing a sample mean of $m = 900$ customers and a sample variance of $s^2 = 50{,}000$. Find your approximate posterior distribution for $\tilde{\mu}$. Aside from the usual argument concerning the applicability of the normal model, why is your posterior distribution only approximate?

35. Using Equation 4.2.3, prove that if the data-generating process of interest is a normal process with unknown mean $\tilde{\mu}$ and known variance σ^2; if the prior distribution of $\tilde{\mu}$ is a normal distribution with mean m' and variance σ^2/n'; and if the sample information consists

of a sample of size n from the process with sample mean m; then the posterior distribution of $\tilde{\mu}$ is a normal distribution with mean m'' and variance σ^2/n'', where n'' and m'' are given by Equations 4.5.15 and 4.5.16.

36. In Exercise 34, suppose that your joint prior distribution for $\tilde{\mu}$ and $\tilde{\sigma}^2$ is a normal-gamma distribution with $m' = 1000$, $n' = 20$, $v' = 60{,}000$, and $d' = 11$.
    (a) Find the conditional prior distribution of $\tilde{\mu}$, given that $\tilde{\sigma}^2 = 60{,}000$.
    (b) Find the marginal prior distribution of $1/\tilde{\sigma}^2$.
    (c) Find the joint posterior distribution of $\tilde{\mu}$ and $\tilde{\sigma}^2$.

37. In Exercise 27, suppose that the production manager is unwilling to assume that $\tilde{\sigma}^2$ is known. Instead, he assesses a normal-gamma prior distribution for $\tilde{\mu}$ and $\tilde{\sigma}^2$ with parameters $m' = 110$, $n' = 10$, $v' = 4$, and $d' = 6$. Find the posterior distribution of $\tilde{\mu}$ and $\tilde{\sigma}^2$ and compute $E(\tilde{\mu}|\sigma^2)$ and $E(1/\tilde{\sigma}^2)$ from this distribution.

38. In Exercises 38 and 39 in Chapter 3, assume that $\tilde{p}$ is continuous and assess a continuous distribution for $\tilde{p}$ in each case. Try to fit beta distributions to the subjectively assessed distributions.

39. Try to assess a probability distribution for $\tilde{T}$, the maximum temperature tomorrow in the city where you live. In assessing the distribution, use two or three of the methods proposed in the text and compare the results. Save the distribution and look at it again after you find out the true value of $\tilde{T}$. Do this for three or four consecutive days and comment on any difficulties that you encounter in attempting to express your subjective judgments in probabilistic form.

40. Follow the procedure in Exercise 39 for the variable $\tilde{D}$, the daily change in the price of one share of IBM common stock.

41. Suppose that a date is to be chosen randomly from next year's calendar, and let $\tilde{\theta}$ represent the maximum official temperature (in degrees Fahrenheit) on that date in Chicago. Assuming that $\tilde{\theta}$ is continuous, assess a probability distribution for $\tilde{\theta}$. Furthermore, let $\tilde{\phi}$ represent the maximum official temperature in Chicago on the day after the chosen date. Assuming that $\tilde{\phi}$ is continuous, assess a conditional distribution for $\tilde{\phi}$, given that $\tilde{\theta} = 60$. Repeat this process for $\tilde{\theta} = 80$, $\tilde{\theta} = 40$, and $\tilde{\theta} = 20$.

42. In Exercise 41, can you approximate your subjectively assessed probability distributions by members of any of the statistical models that have been discussed in this book (or by any other statistical models)? Also, although your conditional distribution of $\tilde{\phi}$, given $\tilde{\theta}$, no doubt varies considerably for different values of $\tilde{\theta}$, can you express your conditional distribution of $\tilde{\phi}$, given $\tilde{\theta}$, as a function of $\tilde{\theta}$ [for example, can you express $E(\tilde{\phi}|\tilde{\theta} = \theta)$ as a function of θ, and so on]? Discuss the use of approximations such as this and the use of statistical models in inferential problems.

43. In assessing a prior distribution for $\tilde{p}$, the proportion of votes that will be cast for a particular candidate in a statewide election, a political analyst feels that the mean of her prior distribution is 0.45. Furthermore, she feels that if she observes a random sample of 2000 voters, 960 of whom state that they will definitely vote for the candidate and 1040 of whom state that they will *not* vote for the candidate (the sample includes no undecided voters), the mean of her posterior

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distribution would be 0.47. Assuming that the process behaves approximately as a Bernoulli process, that stated voting intentions are representative of actual voting behavior, and that the political analyst's prior distribution is a member of the beta family of distributions, find the exact form of the prior distribution.

44. Suppose that a statistician decides that his prior distribution for an uncertain quantity $\tilde{\theta}$ is an exponential distribution and that the 0.67 fractile of this distribution is 2. If the density function for the exponential distribution is given by

$$f(\theta) = \begin{cases} \lambda e^{-\lambda\theta} & \text{for } \theta \geq 0, \\ 0 & \text{for } \theta < 0, \end{cases}$$

find the exact form of the statistician's prior distribution for $\tilde{\theta}$.

45. In Exercise 33 in Chapter 3, suppose that your prior distribution of $\tilde{\lambda}$, the intensity of accidents per week, is a gamma distribution with mean 3.5 and variance 0.5. Find the posterior distribution of $\tilde{\lambda}$ and determine the mean and the variance of this posterior distribution.

46. Comment on the following statement: "One cannot speak of sensitivity except in connection with a particular decision-making situation." Can you think of an example in which the decision-making procedure would be quite insensitive to changes in the prior distribution? Can you think of an example in which it would be quite sensitive?

47. Discuss the importance of discrete approximations to continuous prior distributions in Bayesian analysis. Since the concept of conjugate prior distributions greatly simplifies the analysis, why is it ever necessary to use discrete approximations?

48. In Exercise 26 in Chapter 3, suppose that the medical researcher assesses a normal prior distribution for $\tilde{p}$ with mean 0.16 and variance 0.0009. Using a discrete approximation to this prior distribution with intervals of width 0.01, find the approximate posterior distribution of $\tilde{p}$ following the sample of size 20 with $\tilde{r} = 2$. Repeat the procedure with intervals of width 0.05 and comment on the differences in the approximate posterior distributions.

49. In Exercise 28, use a discrete approximation to the prior distribution with intervals of width 0.2 and compare the resulting posterior distribution with the posterior distribution found in Exercise 28.

50. In Exercise 10,
    (a) try to find a beta distribution to approximate the normal prior distribution (for example, you might find the beta distribution with the same mean and variance as this normal distribution), discuss the "closeness" of the approximation, and find the posterior distribution;
    (b) use a discrete approximation to the normal prior distribution with intervals of width 0.01, discuss the "closeness" of the approximation, and find the posterior distribution.

51. Comment on the following statement: "A diffuse prior state of information is not informationless in the usual meaning of the word, but rather in a relative sense."

52. Give a few examples of situations in which your prior distribution would effectively be diffuse and a few examples in which it would definitely not be diffuse relative to a given sample.

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53. The beta distribution with $r' = n' = 0$ and the beta distribution with $r' = 1$ and $n' = 2$ have both been used by statisticians as "diffuse" beta distributions. Discuss the advantages and disadvantages of each.

54. In Exercise 28, find predictive distributions (based on the prior distribution) for the sample mean $\tilde{m}$ for samples of sizes 1, 2, 5, 10, 50, and 100.

55. In Exercise 33, find the predictive distribution (based on the prior distribution) for the sample mean $\tilde{m}$ if $n = 9$. Find $P(\tilde{m} \geq 50)$ and $P(\tilde{m} \geq 55)$.

56. After revising your distribution in Exercise 16, you contemplate the possibility of taking another sample of size six. What is your predictive distribution for $\tilde{r}$? From this distribution, find $E(\tilde{r})$ and $V(\tilde{r})$.

57. If a data-generating process is assumed to follow the Bernoulli model and if the prior distribution of $\tilde{p}$ is a beta distribution with $r' = 1$ and $n' = 2$, find the predictive distribution for $\tilde{r}$, the number of successes in a sample of size n. For $n = 5$, graph the PMF of $\tilde{r}$.

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Exercises

1. Explain the difference between decision making under certainty and decision making under uncertainty.

2. You are given the following payoff table.

STATE OF THE WORLD

ACTION	A	B	C	D	E
1	-50	80	20	100	0
2	30	40	70	20	50
3	10	30	-30	10	40
4	-10	-50	-70	-20	200

(a) Are any of the actions inadmissible? If so, eliminate them from further consideration.
(b) Find the loss table corresponding to the above payoff table.

3. Consider the following *loss* table.

STATE OF THE WORLD

ACTION	I	II	III
1	0	3	6
2	1	1	0
3	4	0	1

Given this loss table, complete the corresponding *payoff* table:

		STATE OF THE WORLD I	II	III
ACTION	1	12	7	9
	2			
	3			

Are any of the actions inadmissible?

4. The owner of a clothing store must decide how many men's shirts to order for the new season. For a particular type of shirt, he must order in quantities of 100 shirts. If he orders 100 shirts, his cost is \$30 per shirt; if he orders 200 shirts, his cost is \$27 per shirt; and if he orders 300 or more shirts, his cost is \$25.50 per shirt. His selling price for the shirt is \$36, but if any are left unsold at the end of the season, they will be sold in his famous "half-price, end-of-the-season sale." For the sake of simplicity, he is willing to assume that the demand for this type of shirt will be either 100, 150, 200, 250, or 300 shirts. Of course, he cannot sell more shirts than he stocks. He is also willing to assume that he will suffer no loss of goodwill among his customers if he understocks and the customers cannot buy all the shirts they want. Furthermore, he must place his order today for the entire season; he cannot wait to see how the demand is running for this type of shirt.
   (a) Given these details of the clothing-store owner's problem, list the set of actions available to him (including both admissible and inadmissible actions).
   (b) Construct a payoff table and eliminate any inadmissible actions.
   (c) Represent the problem in terms of a tree diagram.
   (d) If the owner decides that there is a loss of goodwill that is roughly equivalent to \$1.50 for each person who wants to buy the shirt but cannot because it is out of stock, make the appropriate changes in the payoff table. (Assume that the goodwill cost is \$1.50 *per shirt* if a customer wants to buy more than one shirt after the stock is exhausted.)

5. In Exercise 4(d), attempt to construct a loss table for the owner's problem *without* referring to the payoff table determined in that exercise. That is, using the concept of opportunity loss, try to express the consequences to the clothing-store owner in terms of losses. Then use the payoff table constructed in Exercise 4(d) to determine a loss table and compare the two loss tables, reconciling any differences.

6. A particular product is both manufactured and marketed by two different firms. The total demand for the product is virtually fixed, so neither firm has advertised in the past. However, the owner of Firm A is considering an advertising campaign to woo customers away from Firm

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B. The ad campaign she has in mind will cost $200,000. She is uncertain about the number of customers that will switch to her firm as a result of the advertising, but she is willing to assume that she will gain either 10 percent, 20 percent, or 30 percent of the market. For each 10 percent gain in market share, the firm's profits will increase by $150,000. Construct the payoff table for this problem and find the corresponding loss table.

7. In Exercise 6, the owner of Firm A is worried that if she proceeds with the ad campaign, Firm B will do likewise, in which case the market shares of the two firms will remain constant. How does this affect the set of possible states of the world, S? Construct a modified payoff table to allow for this change.

8. For the payoff and loss tables in Exercise 2, find the actions that are optimal under the following decision-making criteria:
(a) maximin, (b) maximax, (c) minimax loss.

9. For the payoff and loss tables in Exercises 4(d) and 5, find the actions that are optimal under the maximin, maximax, and minimax loss rules.

10. For the payoff table in Exercise 6, find the actions that are optimal under the maximin and maximax rules. Comment on the implications of these particular rules in this example.

11. What is the primary disadvantage of decision-making rules such as maximin, maximax, and minimax loss?

12. One nonprobabilistic decision-making criterion not discussed in the text involves the consideration of a weighted average of the highest and lowest payoffs for each action. The weights, which must sum to 1, can be thought of as an optimism-pessimism index. The action with the highest weighted average of the highest and lowest payoffs is the action chosen by this criterion. Comment on this decision-making criterion and use it for the payoff table in Exercise 2, with the highest payoff in each row receiving a weight of 0.4 and the lowest payoff receiving a weight of 0.6.

13. Use the decision-making criterion discussed in Exercise 12 for the following payoff table, with the highest payoff in each row receiving a weight of 0.8 and the lowest payoff receiving a weight of 0.2.

		STATE OF THE WORLD	
		I	II
ACTION	1	10	4
	2	7	9

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For this payoff table, the *ER* criterion would also involve a weighted average of the two payoffs in each row. Compare the criterion from Exercise 12 with the *ER* criterion.

14. In Exercise 2, find the expected payoff and the expected loss of each action and find the action that has the largest expected payoff and the smallest expected loss, given that the probabilities of the various states of the world are $P(A) = 0.10$, $P(B) = 0.20$, $P(C) = 0.25$, $P(D) = 0.10$, and $P(E) = 0.35$.

15. In Exercise 3, if $P(\text{I}) = 0.25$, $P(\text{II}) = 0.45$, and $P(\text{III}) = 0.30$, find the expected payoff and the expected loss of each of the three actions. Using *ER* (or *EL*) as a decision-making criterion, which action is optimal?

16. In Exercise 4(d), suppose that the owner decides that in comparing possible demands of 200 and 100, the demand is twice as likely to be 200 as it is to be 100. Similarly, 200 is twice as likely as 300, one and one-half times as likely as 150, and twice as likely as 250. Find the expected payoff of each action. How many shirts should the owner of the store order if he uses the *ER* criterion?

17. In Exercise 6, the owner of Firm A feels that if the ad campaign is initiated, the events "gain 20 percent of the market" and "gain 30 percent of the market" are equally likely and each of these events is three times as likely as the event "gain 10 percent of the market." Using the *ER* criterion, what should she do?

18. In Exercise 17, suppose that the probabilities given are conditional on the rival firm not advertising. If Firm B also advertises, then the owner of Firm A is certain (for all practical purposes) that there will be no change in the market share of either firm. She thinks that the chances are 2 in 3 that Firm B will advertise if Firm A does. What should the owner of Firm A do?

19. In Exercises 15, 16, and 17, determine the relationship between the expected payoffs of the various actions and the corresponding expected losses.

20. Explain how game-theory problems (that is, decision-making problems in which there is some "opponent") can be analyzed using the same techniques that are used in decision making under uncertainty. Instead of "states of the world" one must consider "actions of the opponent." Might this make it more difficult to determine the probabilities necessary to calculate expected payoffs and losses? Explain.

21. Consider a situation in which you and a friend both must choose a number from the two numbers 1 and 2. You must make your choices without communicating with each other. The relevant payoffs are as follows.

    If you choose 1 and he chooses 1, you both win \$10.
    If you choose 1 and he chooses 2, he wins \$15 and you win \$0.
    If you choose 2 and he chooses 1, you win \$15 and he wins \$0.
    If you choose 2 and he chooses 2, you both win \$5.

    (a) How would you go about assigning probabilities to his two actions?
    (b) On the basis of the probabilities assigned in (a), what is your optimal action?

(c) Could you have determined that this was your optimal action without using any probabilities? Why?
(d) Would you expect your action to be different if you could get together with your friend and make a bargain with him before the game is played? Explain.

22. A special type of decision-making problem frequently encountered in meteorology is called the "cost-loss" decision problem. The states of the world are "adverse weather" and "no adverse weather," and the actions are "protect against adverse weather" and "do not protect against adverse weather." C represents the cost of protecting against adverse weather, while L represents the cost that is incurred if you fail to protect against adverse weather and it turns out that the adverse weather occurs. (L is usually referred to as a "loss," but it is not a loss in an opportunity-loss sense.)
(a) Construct a payoff table and a decision tree for this decision-making problem.
(b) For what values of P(adverse weather) should you protect against adverse weather?
(c) Given the result in (b), is it necessary to know the absolute *magnitudes* of C and L?

23. A probabilistic decision-making criterion not discussed in the text is the maximum-likelihood criterion, which states that for each action, you find the most likely payoff, and then you choose the action with the largest most likely payoff. Use this criterion for the decision-making problems in Exercises 15, 16, and 17. Do you think that it is a good decision-making criterion? Explain.

24. In the automobile-salesman example presented in Section 3.4, the owner is particularly concerned with $\tilde{r}$, the number of cars the salesman will sell in the *next* six months (after he sells 10 cars in his first 24 days on the job). Hiring the salesman for that six-month period will cost the owner \$25,000 in fixed salary (salary that will be paid to the salesman regardless of how many cars he sells), and the net profit to the owner is \$500 for each car that is sold. Assuming that the six-month period contains 130 working days for the salesman, should the owner hire him for this period?

25. In Exercise 24, suppose that the salesman is offered a job for the next six months and that he is given a choice of three salary plans: (1) a salary of \$25,000 plus a commission of \$200 for each car that he sells, (2) a salary of \$35,000 with no commission, or (3) a commission of \$600 for each car that he sells, with no fixed salary. At the time he is offered this choice, the salesman feels that the probability that he is a "great" salesman is 0.7, the probability that he is a "good" salesman is 0.29, and the probability that he is a "poor" salesman is 0.01. Which salary plan should he choose?

26. Suppose that a production manager is concerned about a particular run of five items from a certain manufacturing process. Her prior distribution for $\tilde{p}$, the proportion of defective items produced by this process, is $P(\tilde{p} = 0.05) = 0.10$, $P(\tilde{p} = 0.10) = 0.20$, $P(\tilde{p} = 0.15) = 0.30$, $P(\tilde{p} = 0.20) = 0.30$, and $P(\tilde{p} = 0.25) = 0.10$. Furthermore, she has the option, for a cost of \$100, of adjusting the process and guaranteeing that none of the five items will be defective. The net profit per item is \$500, and it costs \$80 to repair a defective item.
(a) Determine a predictive distribution for $\tilde{r}$, the number of defectives in the run of five items.
(b) Set up a payoff table and a decision tree for this problem.
(c) Should the production manager adjust the process?

27. What feature of Exercises 24, 25, and 26 distinguishes them from most of the decision-making problems considered in this chapter?

28. Why is the maximization of expected monetary value (that is, *ER* with the payoffs expressed in terms of money) not always a reasonable criterion for decision making? What problems does this create for the decision maker?

29. You must choose between three acts, where the payoff matrix is as follows (in terms of dollars).

		STATE OF THE WORLD		
		A	*B*	*C*
ACTION	*1*	100	70	20
	2	10	50	120
	3	50	80	30

What is the optimal act according to the expected utility criterion if $P(\text{A}) = 0.3$, $P(\text{B}) = 0.3$, $P(\text{C}) = 0.4$, and the utility function in the relevant range is:
(a) $U(R) = 50 + 2R$,
(b) $U(R) = 50 + 2R^2$,
(c) $U(R) = R$,
(d) $U(R) = R^2 + 5R + 6$.

30. Suppose that you are offered a choice between bets A and B.

Bet A: You win \$1,000,000 with certainty (that is, with probability 1).

Bet B: You win \$5,000,000 with probability 0.10.
You win \$1,000,000 with probability 0.89.
You win \$0 with probability 0.01.

Which bet would you choose? Similarly, choose between bets C and D.

Bet C: You win \$1,000,000 with probability 0.11.
You win \$0 with probability 0.89.

Bet D: You win \$5,000,000 with probability 0.10.
You win \$0 with probability 0.90.

Prove that if you chose bet A, then you should have chosen bet C, and if you chose bet B, then you should have chosen bet D. If you selected A and D or B and C, explain your choices. In light of the proof, would you change your choices?

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31. Suppose that you are contemplating drilling an oil well, with the following payoff table (in terms of thousands of dollars):

		STATE OF THE WORLD	
		Oil	*No oil*
ACTION	*Drill*	100	-40
	Do not drill	0	0

If after consulting a geologist you decide that $P(\text{oil}) = 0.30$, find the optimal action according to the
(a) maximin criterion,
(b) maximax criterion,
(c) minimax loss criterion,
(d) *ER* criterion,
(e) *EL* criterion,
(f) *EU* criterion, where $U(0) = 0.40$, $U(100) = +1$, and $U(-40) = 0$.
Explain the differences among the results in parts (a) through (f).

32. Comment on the following statement: "Some individuals appear to be risk-takers for some decisions (such as gambling) and risk-avoiders for other decisions (such as purchasing insurance)." Can you explain why this phenomenon occurs? Call you draw a utility function that would explain it?

33. Attempt to determine your own utility function for money in the range from –$10,000 to +$10,000. If you were given actual decision-making situations, would you act in accordance with this utility function?

34. If someone gave you $100,000 with "no strings attached," how would this affect your attitude toward risk? Assuming a gift of $100,000, attempt to determine your utility function for money in the range from –$10,000 to +$10,000. Comment on any differences between this utility function and the one assessed in Exercise 33.

35. Suppose that a person's utility function for *total* assets (*not* changes in assets) is

$$U(A) = 200A - A^2 \quad \text{for } 0 \le A \le 100,$$

where A represents total assets in thousands of dollars.
(a) Graph this utility function. How would you classify this person with regard to his attitude toward risk?
(b) If the person's total assets are currently $10,000, should he take a bet in which he will win $10,000 with probability 0.6 and lose $10,000 with probability 0.4?
(c) If the person's total assets are currently $90,000, should he take the bet given in (b)?

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(d) Compare your answers to (b) and (c). Does the person's betting behavior seem reasonable to you? How could you intuitively explain such behavior?

36. For each of the following utility functions for changes in assets (monetary payoffs), graph the function and comment on the attitude toward risk that is implied by the function. All of the functions are defined for $-1000 < R < 1000$.
(a) $U(R) = (R + 1000)^2$.
(b) $U(R) = -(1000 - R)^2$.
(c) $U(R) = 1000R + 2000$.
(d) $U(R) = \log(R + 1000)$.
(e) $U(R) = R^3$.
(f) $U(R) = 1 - e^{-R/100}$.

37. For each of the utility functions in Exercise 36, find out if the decision maker should take a bet in which he will win \$100 with probability p and lose \$50 with probability $1 - p$,
(a) if $p = 1/2$,
(b) if $p = 1/3$,
(c) if $p = 1/4$.

38. A function that is often used to measure the degree of risk aversion in a given utility function is the Pratt-Arrow risk-aversion function. This function is of the form $r(A) = -U''(A)/U'(A)$, where the primes denote differentiation and $U(A)$ represents a utility function for *total* assets. Find $r(A)$ for the utility function in Exercise 35.

39. Find the Pratt-Arrow risk-aversion functions for $0 < A < 100$ for the following utility functions, where A represents total assets in thousands of dollars:
(a) $U(A) = 1 - e^{-0.05A}$.
(b) $U(A) = \log A$.
Graph these risk-aversion functions and the risk-aversion function from Exercise 38 and compare them in terms of how the risk aversion changes as A increases.

40. Two persons, A and B, make the following bet: A wins \$40 if it rains tomorrow and B wins \$10 if it does not rain tomorrow.
(a) If they both agree that the probability of rain tomorrow is 0.10, what can you say about their utility functions?
(b) If they both agree that the probability of rain tomorrow is 0.30, what can you say about their utility functions?
(c) Given no information about their probabilities, is it possible that their utility functions could be identical? Explain.
(d) If they both agree that the probability of rain tomorrow is 0.20, is it possible that their utility functions could be identical? Explain.

41. Suppose that an investor is considering two investments. With each investment, he will double his money with probability 0.4 and lose half of his money with probability 0.6. He has \$1000 to invest, and he can either put \$500 in each investment, put the entire \$1000 in one of the two investments, put \$500 in one investment and not invest the remaining \$500, or not invest at all. Assume that the outcomes of the two investments are independent and that the investor's utili-

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ties for *changes* in assets are $U(\$1000) = 1$, $U(\$500) = 0.85$, $U(\$250) = 0.75$, $U(\$0) = 0.5$, $U(-\$250) = 0.25$, and $U(-\$500) = 0$. What should the investor do? What commonly encountered financial concept does this illustrate?

42. In Exercise 16, suppose that $U(\$6{,}000) = 1$, $U(-\$3{,}000) = 0$, and the owner of the clothing store is indifferent between the following lotteries:

    Lottery A: Receive $3,000 for certain.

    Lottery B: Receive $6,000 with probability 0.87 and receive –$3,000 with probability 0.13.

    What is $U(\$3000)$? By considering lotteries such as these, the owner assesses $U(-\$1{,}500) = 0.42$, $U(\$0) = 0.62$, $U(\$1{,}500) = 0.77$, and $U(\$4{,}500) = 0.95$. Plot these points on a graph and attempt to fit a smooth curve to them. Using this curve, find the action that maximizes the owner's expected utility.

43. In Exercise 18, suppose that the owner of Firm A assesses $U(\$500{,}000) = 1$, $U(\$400{,}000) = 0.70$, $U(\$300{,}000) = 0.43$, $U(\$200{,}000) = 0.28$, $U(\$100{,}000) = 0.18$, $U(\$0) = 0.12$, $U(-\$100{,}000) = 0.06$, and $U(-\$200{,}000) = 0$. Graph these points, fit a smooth utility function to them, and use this utility function to find EU(ad campaign) and EU(no ad campaign).

44. For the utility function represented in Figure 5.6.5, the point (to the right of $0) at which the utility function shifts from that of a risk-taker to that of a risk-avoider is sometimes called a "level of aspiration." Explain this terminology.

45. For each of the utility functions in Exercise 36, find the risk premiums for the following gambles.
    (a) You win $100 with probability 0.5 and you lose $100 with probability 0.5.
    (b) You win $100 with probability 0.4 and you lose $50 with probability 0.6.
    (c) You win $70 with probability 0.3 and you lose $30 with probability 0.7.
    (d) You win $200 with probability 0.5 and you win $50 with probability 0.5.

46. If your utility function for monetary payoffs is $U(R) = 40{,}000 - (200 - R)^2$ for $-200 \le R \le 200$, show the risk premium graphically for each of the gambles in Exercise 45.

47. One counterargument to the transitivity of indifference goes something like this. You are probably indifferent between a cup of black coffee and a cup of coffee with one grain of sugar. Similarly, you are indifferent between a cup of coffee with one grain of sugar and one with two grains of sugar. Therefore, if indifference is transitive, you should be indifferent between a cup of black coffee and one with two grains of sugar. By adding a grain of sugar at a time, you can arrive at the conclusion that you should be indifferent between a cup of black coffee and one with a million grains of sugar. This seems to be an unreasonable conclusion; does this mean that transitivity of indifference is an unrealistic assumption? Discuss the issues raised by this example, both with regard to this specific assumption and with regard to the "axioms of coherence" in general.

48. How can the consideration of utility functions and subjective probabilities and the maximization of expected utility be justified formally in problems of decision making under uncertainty?

49. What is the multiattribute utility problem? Illustrate this problem with respect to a high-school senior trying to decide which college to attend.

50. How might a firm develop a utility function that would take into account profits *and* other factors such as pollution control? For a related problem, how might a governmental agency develop a utility function to decide among various potential courses of action?

51. Discuss the role of mathematical functions in providing convenient models for utility functions. Compare the linear, the exponential, and the logarithmic models presented in Section 5.8.

52. One complication that has been ignored in this chapter is the problem of how to handle cash flows over time. An investment may result in a sequence of payoffs rather than in a single payoff at a particular point in time. One approach to this problem is to choose an appropriate discount rate and to discount all future cash flows so that they can be expressed in present dollars rather than in future dollars. This approach obviously is a simplification, ignoring such factors as tax considerations, an individual's time preferences for payoffs, and so on. Discuss this problem and its relation to utility theory.

53. In Exercise 17, suppose that the owner of Firm A is uncertain about some of the elements in her payoff table. In particular, the ad campaign will cost either \$150,000, \$200,000, or \$250,000, with $P(\$150{,}000) = 0.2$, $P(\$200{,}000) = 0.4$, and $P(\$250{,}000) = 0.4$. In addition, the increase in profits for each 10 percent gain in market share will be either \$100,000, \$140,000, or \$180,000, with $P(\$100{,}000) = 0.3$, $P(\$140{,}000) = 0.5$, and $P(\$180{,}000) = 0.2$.
   (a) Express the problem in terms of a decision tree, including branches for the uncertainties about the payoffs.
   (b) What should the owner of Firm A do?

54. Suppose that one person (A) claims that $U(\$100) = 0.8$ and that $U(\$50) = 0.3$, whereas a second person (B) claims that $U(\$100) = 0.8$ and that $U(\$50) = 0.6$. What, if anything, can you say about the relative preferences of A and B? In general, is it meaningful to make interpersonal comparisons of utility functions? Explain.

55. Suppose that you could obtain an interest-free \$50,000 loan with the restriction that it has to be invested in stocks, bonds, or savings accounts (or a combination of these types of investments) and that the loan has to be repaid in exactly one year. How could you set up this problem in terms of the framework presented in Section 5.9? First of all, how would you reduce the number of potential actions to a manageable number? Second, how would you define S, the set of states of the world? Third, how would you determine the possible payoffs or losses? Fourth, how could you introduce a utility function? Fifth, how could you quantify your judgements to assess probabilities for the various states of nature?

56. The problem in Exercise 55 is quite complex and must be simplified (for example, by eliminating actions) if it is to be expressed in terms of the formal decision-theoretic model. For an even more ill-structured problem, consider the high-school senior who must decide among numerous colleges, jobs, military service, and so on. Take a hypothetical high-school senior and try to express his problem in terms of the formal decision-theoretic framework. Which inputs might be most difficult to determine in this situation?

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57. In the oil-drilling venture presented in Section 5.10, suppose that the decision maker finds three other persons who would be willing to take shares in the venture, thereby leaving the decision maker with a 25 percent share. Compute the appropriate payoffs and determine the expected utility of this new action, and compare this expected utility with the *EU*s computed in Section 5.10 for the other actions.

58. In Exercise 16, suppose that one of the employees in the clothing store assesses thc following probabilities for the demand for shirts: $P(50) = 0.05$, $P(100)= 0.15$, $P(150)= 0.25$, $P(200)= 0.35$, $P(250)= 0.10$, $P(300)= 0.05$, and $P(350) = 0.05$. Note that two additional values for demand, 50 and 350, are being considered, so the payoff table must be expanded. Using the expanded payoff table and the employee's probabilities, which action has the largest expected payoff? Comment on the implications of this result with regard to the sensitivity of the shirt-ordering problem to changes in the set of states of the world, S, or to changes in the probabilities.

59. Discuss the importance of sensitivity analysis in problems of decision making under uncertainty. When might the results of a sensitivity analysis greatly simplify the decision maker's problem? When might the results of a sensitivity analysis make the decision maker's problem extremely difficult?

60. In the oil-drilling example presented in Section 5.10, the *EU* of the optimal action is approximately equal to $U(\$15,000)$. Thus, if the decision maker contemplates selling the drilling rights, the minimum selling price should be \$15,000. If the decision maker does not own the drilling rights, is it necessarily true that his maximum buying price for the rights should be \$15,000? In general, are the buying and selling prices determined from a given individual's utility function for, say, a lottery, always equal? Explain.

CH 6

Exercises

1. What is the difference between a terminal decision and a preposterior decision? Are they at all related?

2. In Exercises 14 and 15 in Chapter 5, find the EVPI.

3. In Exercise 16, Chapter 5, how much should the store owner be willing to pay for perfect information about the demand for the type of shirts he is about to order?

4. In Exercise 18, Chapter 5, what is the value of perfect information that the rival firm, Firm B, will advertise if Firm A advertises? What is the value of perfect information that Firm B will not advertise and the increase in Firm A's market share will be 10 percent? What is the value of perfect information that Firm B will not advertise and the increase in Firm A's market share will be 20 percent? Finally, what is the value of perfect information that Firm B will not advertise and the increase in Firm A's market share will be 30 percent? From these VPIs, calculate the EVPI.

5. In Exercise 22, Chapter 5, give a general expression for the expected value of perfect information regarding the weather and find the EVPI if
   (a) $P(\text{adverse weather}) = 0.4$, $C = 3.5$, and $L = 10$,
   (b) $P(\text{adverse weather}) = 0.3$, $C = 3.5$, and $L = 10$,
   (c) $P(\text{adverse weather}) = 0.4$, $C = 10$, and $L = 10$,
   (d) $P(\text{adverse weather}) = 0.3$, $C = 2$, and $L = 8$.

6. In Exercise 5, if $P(\text{adverse weather}) = 0.3$ and $C/L = 1/2$, how much should you be willing to pay for perfect information concerning the absolute magnitudes of C and L? Explain.

7. In Exercise 24, Chapter 5, how much should the owner be willing to pay for perfect information concerning $\tilde{r}$? If he cannot obtain perfect information concerning $\tilde{r}$, how much should he be willing to pay for perfect information concerning $\tilde{\lambda}$?

8. In decision-making problems for which the uncertain quantity of primary interest can be viewed as a future sample outcome $\tilde{y}$, the relevant distribution of interest to the decision maker is the predictive distribution of $\tilde{y}$.
   (a) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{y}$?
   (b) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{\theta}$?
   (c) Explain the difference between your answers to (a) and (b).

9. In Exercise 26, Chapter 5, the uncertain quantity of interest is $\tilde{r}$, the number of defective items in a sample of size five from a production process.
   (a) How much should the production manager be willing to pay for perfect information about $\tilde{r}$?
   (b) How much should she be willing to pay for perfect information about $\tilde{p}$?
   (c) If she obtains perfect information that $\tilde{p} = 0.15$, how much should she be willing to pay for perfect information about $\tilde{r}$?

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(d) If she obtains perfect information that $\tilde{r} = 1$, how much should she be willing to pay for perfect information about $\tilde{p}$?

10. In Exercise 8, is one type of perfect information (that is, perfect information about $\tilde{y}$ or perfect information about $\tilde{\theta}$) more "valuable" to the decision maker than the other? Explain your answer and illustrate with respect to Exercise 9.

11. In Exercise 25, Chapter 5, find the expected value of perfect information (a) about $\tilde{r}$, (b) about $\tilde{\lambda}$.

12. In Exercise 53, Chapter 5, find the expected value of perfect information about
    (a) the cost of the ad campaign,
    (b) the increase in profits for each 10 percent gain in market share, and
    (c) both the cost of the ad campaign *and* the increase in profits for each 10 percent gain in market share.

    Does the answer to part (c) equal the sum of the answers to parts (a) and (b)? Explain.

13. In Exercise 31, Chapter 5, assuming that your utility function for money is linear, find the value of perfect information that there is oil and the value of perfect information that there is no oil. From these results, how much would you be willing to pay for a geological test that will tell you for certain whether or not there is oil?

14. In Exercise 4, show that the EVPI is equal to the expected loss of the action that is optimal under the decision maker's probability distribution.

15. A store must decide whether or not to stock a new item. The decision depends on the reaction of consumers to the item, and the payoff table (in dollars) is as follows.

		PROPORTION OF CONSUMERS PURCHASING				
		0.10	0.20	0.30	0.40	0.50
DECISION	*Stock 100*	-10	-2	12	22	40
	Stock 50	-4	6	12	16	16
	Do not stock	0	0	0	0	0

If $P(0.10) = 0.2$, $P(0.20) = 0.3$, $P(0.30) = 0.3$, $P(0.40) = 0.1$, and $P(0.50) = 0.1$, what decision maximizes expected payoff? If perfect information is available, find VPI for each of the five possible states of the world and compute EVPI.

16. In Exercise 15, suppose that sample information is available in the form of a random sample of consumers. For a sample of size *one*,
    (a) find the posterior distribution if the one person sampled will purchase the item, and find the value of this sample information;
    (b) find the posterior distribution if the one person sampled will *not* purchase the item, and find the value of this sample information;
    (c) find the expected value of sample information.

17. In Exercise 16, suppose that you also want to consider other sample sizes.
    (a) Find EVSI for a sample of size 2.
    (b) Find EVSI for a sample of size 5.
    (c) Find EVSI for a sample of size 10.
    (d) If the cost of sampling is \$0.50 per unit sampled, find the expected net gain of sampling (ENGS) for samples of sizes 1, 2, 5, and 10.

18. Consider a bookbag filled with 100 poker chips. You know that either 70 of the chips are red and the remainder blue, or 70 are blue and the remainder red. You must guess whether the bookbag has 70R – 30B or 70B – 30R. If you guess correctly, you win \$5. If you guess incorrectly, you lose \$3. Your prior probability that the bookbag contains 70R – 30B is 0.40, and you are risk neutral.
    (a) If you had to make your guess on the basis of the prior information, what would you guess?
    (b) If you could purchase perfect information, what is the most that you should be willing to pay for it?
    (c) If you could purchase sample information in the form of one draw from the bookbag, how much should you be willing to pay for it?
    (d) If you could purchase sample information in the form of five draws (with replacement) from the bookbag, how much should you be willing to pay for it?

19. Do Exercise 18 with the following payoff table (in dollars).

		STATE OF THE WORLD	
		70R–30B	*70B–30R*
YOUR GUESS	*70R–30B*	6	−2
	70B–30R	−6	10

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20. In the automobile-salesman example discussed in Section 3.4, suppose that the owner of the dealership must decide whether or not to hire a new salesman. The payoff table (in terms of dollars) is as follows.

		STATE OF THE WORLD		
		Great salesman	*Good salesman*	*Poor salesman*
ACTION	*Hire*	60,000	15,000	-30,000
	Do not hire	0	0	0

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda} = 1/2$ per day for a great salesman, $\tilde{\lambda} = 1/4$ per day for a good salesman, and $\tilde{\lambda} = 1/8$ per day for a poor salesman.

(a) Find VPI(great salesman), VPI(good salesman), and VPI(poor salesman).

(b) Find the expected value of perfect information.

(c) Suppose that the owner of the dealership can purchase sample information at the rate of \$10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

21. In Exercise 20, suppose that the owner hires the salesman for one week and that he sells two cars. At the end of the week, the owner has three choices: hire the salesman permanently, fire the salesman, or hire the salesman temporarily for another week (at the extra cost of \$10 per day). What should he do?

22. In Exercise 18, Chapter 5, suppose that the owner of Firm A can obtain information about Firm B's reaction to advertising by Firm A. In particular, she can ask a contact she has in Firm B. However, the information from the contact cannot be regarded as perfect information. If Firm B would really advertise if Firm A does, the chances are 4-in-5 that the contact would report this correctly. However, if Firm B would not advertise, the chances are only 2-in-3 that the contact would report this correctly.

(a) If the contact reports that Firm B will advertise if Firm A does, what is the value of this sample information?

(b) If the contact reports that Firm B will not advertise even if Firm A does, what is the value of this sample information?

(c) What is the expected value of sample information?

23. In the oil-drilling example in Section 6.4, what is the expected value of the seismic information if the decision maker's utility function is linear with respect to money?

24. A firm is considering the marketing of a new product. For convenience, suppose that the events of interest are simply θ_1 = "new product is a success" and θ_2 = "new product is a failure." The

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prior probabilities are $P(\theta_1) = 0.3$ and $P(\theta_2) = 0.7$. If the product is marketed and is a failure, the firm suffers a loss of $300,000. If the product is not marketed and it would be a success, the firm suffers an opportunity loss of $500,000. The firm is considering two separate surveys, A and B, and the results from each survey can be classified as favorable, neutral, and unfavorable. The conditional probabilities for survey A are

$$P(\text{favorable}|\theta_1) = 0.6,\ P(\text{neutral}|\theta_1) = 0.3,\ P(\text{unfavorable}|\theta_1) = 0.1,$$
$$P(\text{favorable}|\theta_2) = 0.1,\ P(\text{neutral}|\theta_2) = 0.2,\ \text{and}\ P(\text{unfavorable}|\theta_2) = 0.7.$$

The conditional probabilities for survey B are

$$P(\text{favorable}|\theta_1) = 0.8,\ P(\text{neutral}|\theta_1) = 0.1,\ P(\text{unfavorable}|\theta_1) = 0.1,$$
$$P(\text{favorable}|\theta_2) = 0.1,\ P(\text{neutral}|\theta_2) = 0.4,\ \text{and}\ P(\text{unfavorable}|\theta_2) = 0.5.$$

Survey A costs $20,000 and survey B costs $30,000.

(a) Find the expected value of perfect information about $\tilde{\theta}$.
(b) Find the expected net gain from survey A.
(c) Find the expected net gain from survey B.
(d) Suppose that the firm has a choice. It can use no survey, survey A, or survey B, but not both surveys. What is the optimal course of action?

25. In Exercise 24, suppose that the firm also has the option of using *both* surveys. Furthermore, since both surveys would be conducted by the same marketing research firm, the total cost of the two surveys is only $40,000, provided that a decision is made in advance to use both surveys (that is, the firm cannot use one survey and then decide whether or not to use the other). Given any one of the three events θ_1, θ_2, and θ_3, the results of survey B are considered to be independent of the results of survey A. What should the firm do?

26. In Exercise 25, suppose that the firm has the additional option of using survey A and, after seeing the results of survey A, deciding whether or not to use survey B. The reverse procedure, using survey B first and then considering survey A, is not possible. Of course, if the sequential plan is used and both surveys are taken, the total cost of the surveys is $50,000 rather than $40,000. Find the expected net gain of the sequential plan and compare this with the expected net gains of the sampling plans considered in Exercises 24 and 25.

27. In Exercise 17, consider a sequential sampling plan with a maximum total sample size of two and analyze the problem as follows.
(a) Represent the situation in terms of a tree diagram.
(b) Using backward induction, find the ENGS for the sequential plan.
(c) Compare the sequential plan with a single-stage plan having $n = 2$.

28. Repeat Exercise 27 for sequential samples with maximum total sample sizes of three, four, and five. Discuss your results.

29. In Exercise 18, assuming that the cost of sampling is 25 cents per draw, find the ENGS for
(a) a sequential sampling plan with a maximum total sample size of two,
(b) a sequential sampling plan with a maximum total sample size of four,
(c) a single-stage sampling plan with a sample size of two,
(d) a single-stage sampling plan with a sample size of four.

30. Repeat Exercise 29 under the assumption that the cost of sampling is 15 cents per draw plus a fixed cost of 50 cents for each separate sample. Comment on any differences between the results of Exercise 29 and this exercise.

31. Repeat Exercise 29 under the assumption that the sampling from the bookbag is done *without* replacement. Comment on any differences between the results of Exercise 29 and this exercise. In particular, for each sampling plan, is there any difference between the ENGS for sampling with replacement and the ENGS for sampling without replacement? Explain.

32. The preceding exercises illustrate the idea of sequential analysis, or sequential decision making. In a sequential procedure, how does the decision maker know when to stop sampling and make her terminal decision? Is it possible for a sequential sampling plan to have a positive ENGS even though no sampling plan with a fixed sample size has a positive ENGS in the same situation? Explain.

33. Comment on the statement, "Linear payoff and loss functions have wide applicability in real-world decision-making problems." Give some examples of problems in which the payoff and loss functions are linear or approximately linear.

34. Suppose that the payoff functions (in dollars) of two actions are

$$R(a_1, \mu) = 70 - 0.5\mu$$

and

$$R(a_2, \mu) = 50 + 0.5\mu,$$

where $\tilde{\mu}$ is the mean of a normal process with variance 1200.
(a) Find the breakeven value, μ_b.
(b) If the prior distribution is a normal distribution with mean $m' = 25$ and variance $\sigma'^2 = 400$, which action should you choose?
(c) What is the value of perfect information that $\tilde{\mu} = 15$?
(d) What is the value of perfect information that $\tilde{\mu} = 21$?
(e) What is the expected value of perfect information?
(f) Graph the payoff functions and the associated loss functions.
(g) What would the expected value of perfect information be if the prior mean $m' = 30$? Explain the difference between this answer and the answer to (e).

35. The payoff in a certain decision-making problem depends on $\tilde{p}$, the parameter of a Bernoulli process. The payoff functions are

$$R(a_1, p) = 50p$$

and

$$R(a_2, p) = -10 + 100p$$

(a) What is the breakeven value of $\tilde{p}$?
(b) Graph the payoff functions and the associated loss functions.
(c) If the prior distribution is a beta distribution with $r' = 4$ and $n' = 24$, which action should be chosen?
(d) If a sample of size 15 is taken, with 4 successes, find the posterior distribution and the optimal action under this distribution.

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36. In Exercise 34, find the EVSI and ENGS for samples of size 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, assuming that the cost of sampling is \$0.15 per trial. On a graph, draw curves representing EVSI, ENGS, and CS.

37. In Exercise 35(c), find the expected value of perfect information about $\tilde{p}$.

38. A firm is contemplating the purchase of 500 printer cartridges. One supplier, supplier A, offers the cartridges at \$15 each, guarantees each cartridge, and will replace all defective cartridges free. A second supplier, supplier B, offers the cartridges at \$14 each with no guarantee. However, supplier B will replace defective cartridges with good cartridges for \$10 per cartridge. Suppose that the proportion of defective cartridges produced by supplier B is denoted by $\tilde{p}$, and suppose that the prior distribution for $\tilde{p}$ is a beta distribution with parameters $r' = 2$ and $n' = 50$.
   (a) What should the firm do on the basis of the prior distribution?
   (b) How much is it worth to the firm to know the proportion of defective cartridges for certain?
   (c) Suppose that supplier B can provide a randomly chosen sample of 10 cartridges. What is the expected value of this sample?
   (d) In the sample of 10 cartridges, 1 is defective. Find the posterior distribution of $\tilde{p}$ and use this distribution to determine which supplier the firm should deal with.

39. Suppose that an investor has a choice of four investments (assume that he must choose one, and only one, of the four—he cannot divide his money among them):
   A: a savings account in a bank;
   B: a stock that moves counter to the general stock market;
   C: a growth stock that moves at a faster pace than the market;
   D: a second growth stock that is similar to C but not quite as speculative.
   The investor has decided that his payoff depends on his choice of investment and on the *next year's change* in the Dow-Jones Industrials Average (DJIA), which we shall call $\tilde{\theta}$. (Ignore the fact that he should be interested in the change in price of the investments themselves—for example, by assuming perfect correlations between the DJIA and the investments.) His payoff functions are as follows:

$$R(A, \theta) = 0.05,$$
$$R(B, \theta) = 0.05 - \theta,$$
$$R(C, \theta) = -0.15 + 2\theta,$$

   and
$$R(D, \theta) = -0.05 + \theta.$$

   (a) Draw the graphs of these payoff functions and determine a decision rule for the choice of an investment under certainty about $\tilde{\theta}$.
   (b) Suppose that the investor's prior distribution for $\tilde{\theta}$ is normal with mean 0.08 and standard deviation 0.05. Which investment should he choose?
   (c) Show graphically and algebraically the loss function. If the investor obtains perfect information and finds that $\tilde{\theta} = 0.02$, what is the VPI?
   (d) Determine and graph the loss functions for the four actions A, B, C, and D.

40. In the oil-drilling venture example presented in Section 5.10, suppose that $\tilde{\theta}$ is assumed to be continuous. Given the information in Section 5.10, find the payoff functions for the four actions and graph these functions. Furthermore, assume that the mean of the decision maker's prior

distribution for $\tilde{\theta}$ is 125,000. If the decision maker's utility function is taken to be linear with respect to money, find the optimal action under the prior distribution. Why is it necessary to know only the mean of the prior distribution in order to find the optimal action?

41. What are certainty equivalents and how can they help simplify problems of decision making under uncertainty?

42. Give some examples of decision-making situations in which the payoff or loss functions might be nonlinear.

43. Discuss the importance of sensitivity analysis in relation to both terminal and preposterior decisions.

44. In the example presented in Section 6.8, find the optimal sample size
(a) if $CS(n) = 0.10 + 0.02n$,
(b) if $CS(n) = 0.20 + 0.01n$.

45. The cost of sampling is often taken to be a fixed multiple of the sample size, so that the sample costs, say, \$5 per item sampled. The cost functions in Exercise 44, however, include both a cost per item and a fixed cost. Under what circumstances might such cost functions arise, and what are the implications for sequential sampling?

46. In Exercise 35, assume that the payoff functions are given in terms of dollars and that the decision maker's utility function for money is of the form

$$U(M) = (M + 100)^2 \quad \text{for } -100 \le M \le 100,$$

where M represents dollars. Graph the payoff functions and the associated loss functions in terms of utility, determine which of the two actions has the higher expected utility under the prior distribution, and determine which of the two actions has the higher expected utility under the posterior distribution. [*Hint*: The "laws of expectation" presented in Section 3.1 are useful here.]

47. In Exercise 35, suppose that a third action is considered, with payoff function

$$R(a_3, p) = -20 + 140p.$$

(a) Graph the three payoff functions and determine a decision rule for choosing the optimal action under uncertainty about $\tilde{p}$.
(b) Given the posterior distribution from Exercise 35, what action should be chosen?
(c) If the decision maker's utility function for money is as given in Exercise 46, graph the three payoff functions in terms of utility and determine the optimal action under the posterior distribution.

48. In Exercise 34, suppose that the decision maker's utility function for money is of the form

$$U(M) = M^3,$$

where M represents dollars. Graph the payoff functions in terms of utility and determine the optimal action under the prior distribution given in Exercise 34. [*Hint*: For a normal

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distribution with mean μ and variance σ^2, the third moment about the origin is $E(\tilde{M}^3) = \mu(\mu^2 + 3\mu^2)$.]

49. The payoff functions for three actions are

$$R(a_1, p) = -16 + 100p,$$
$$R(a_2, p) = 1000(p - 0.20)^2,$$
and
$$R(a_3, p) = 600(p - 0.15)^2,$$

where $\tilde{p}$ is the parameter of a Bernoulli process. It the prior distribution of $\tilde{p}$ is a beta distribution with $r' = 10$ and $n' = 50$, find the expected payoffs of the three actions.

CH 7

Exercises

1. Comment on the statement, "If the classical statistician uses prior information at all, he uses it in an informal manner, whereas the Bayesian formally incorporates it into his inferential and decision-making framework."
2. Discuss the relationship between classical statistical procedures and Bayesian procedures under a diffuse prior distribution, both with regard to numerical results and with regard to the interpretation of results.
3. The conditional probability $P(y \mid \theta)$ or the conditional density $f(y \mid \theta)$ can be interpreted as a sampling distribution or as a likelihood function; explain the difference between these two interpretations.
4. Discuss the importance of the likelihood principle and the concept of sufficient statistics with regard to problems of statistical inference and decision.
5. In Exercise 23, Chapter 3, find a point estimate for $\tilde{p}$, the proportion of consumers who will purchase the product, based
   (a) on the prior distribution alone,
   (b) on the sample information alone, and
   (c) on the posterior distribution.
6. Do Exercise 5 under the assumption that the prior distribution of $\tilde{p}$ is a beta distribution with $r' = 4$ and $n' = 10$.
7. In Exercise 33, Chapter 3, find a point estimate for $\tilde{\lambda}$, the intensity of occurrence of accidents along a particular stretch of highway,
   (a) based on the prior distribution alone,
   (b) based on the sample information alone,
   (c) based on the posterior distribution.
8. In Exercise 43, Chapter 3, find a point estimate for $\tilde{\theta}$
   (a) given that $\tilde{\phi} = 1$,
   (b) given that $\tilde{\phi} = 0$,
   (c) given only that $P(\tilde{\phi} = 1) = 0.3$.
9. Suppose that a sample from a given population with known variance σ^2 consists of four independent trials and that the sample information on the ith trial is represented by the random variable $\tilde{x}_i$, i=1, 2, 3, 4. Consider the following estimators of the population mean, $\tilde{\mu}$:

   $\tilde{y}_1 = \tilde{x}_1$, $\tilde{y}_2 = (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_4)/4$,

   $\tilde{y}_3 = (4\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4)/10$, and $\tilde{y}_4 = (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_4)/3$.

   (a) Find $E(\tilde{y}_i \mid \tilde{\mu} = \mu)$ for $i = 1, 2, 3, 4$.
   (b) Find $V(\tilde{y}_i \mid \tilde{\mu} = \mu)$ for $i = 1, 2, 3, 4$.
   (c) Find $B(\tilde{y}_i \mid \tilde{\mu} = \mu)$ for $i = 1, 2, 3, 4$.
   (d) Find MSE$(\tilde{y}_i \mid \tilde{\mu} = \mu)$ for $i = 1, 2, 3, 4$.
   (e) Comment on the merits of $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$, and $\tilde{y}_4$ as classical estimators with respect to unbiasedness and efficiency.

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10. In Exercise 28, Chapter 4, find
    (a) a point estimate for $\tilde{\mu}$ based on the prior distribution alone,
    (b) a point estimate for $\tilde{\mu}$ based on the sample information alone,
    (c) a point estimate for $\tilde{\mu}$ based on the posterior distribution,
    (d) a 90 percent credible interval for $\tilde{\mu}$ based on the prior distribution alone,
    (e) a 90 percent confidence interval for $\tilde{\mu}$ based on the sample information alone,
    (f) a 90 percent credible interval for $\tilde{\mu}$ based on the posterior distribution.
    (g) Comment on any differences in your answers to (a) through (c) and to (d) through (f).

11. In Exercise 33, Chapter 4, find
    (a) an 86.6 percent confidence interval for $\tilde{\mu}$,
    (b) a point estimate for $\tilde{\mu}$ based on the posterior distribution,
    (c) a 38.3 percent credible interval for $\tilde{\mu}$.

12. In Exercise 6, find
    (a) a 90 percent credible interval for $\tilde{p}$ based on the prior distribution,
    (b) a 90 percent credible interval for $\tilde{p}$ based on the posterior distribution,
    (c) a 98 percent credible interval for $\tilde{p}$ based on the prior distribution,
    (d) a 50 percent credible interval for $\tilde{p}$ based on the prior distribution.

13. *Carefully* distinguish between a classical confidence interval and a Bayesian credible interval and also between the classical and Bayesian approach to point estimation.

14. Suppose that a contractor must decide whether or not to build any speculative houses (houses for which he would have to find a buyer), and if so, how many. The houses that this contractor builds are sold for a price of $30,000, and they cost him $26,000 to build. Since the contractor cannot afford to have too much cash tied up at once, any houses that remain unsold three months after they are completed will have to be sold to a realtor for $25,000. The contractor's prior distribution for $\tilde{\theta}$, the number of houses that will be sold within three months of completion, is:

θ	$P(\tilde{\theta} = \theta)$
0	0.05
1	0.10
2	0.10
3	0.20
4	0.25
5	0.20
6	0.10

    If the contractor's utility function is linear with respect to money, how many houses should he build? How much should he be willing to pay to find out for certain how many houses will be sold within three months?

15. A hot-dog vendor at a football game must decide in advance how many hot dogs to order. He makes a profit of $0.10 on each hot dog that is sold, and he suffers a $0.20 loss on hot dogs that are unsold. If his distribution of the number of hot dogs that will be demanded at the football game is a normal distribution with mean 10,000 and standard deviation 2000, how many hot dogs should he order? How much is it worth to the vendor to know in advance exactly how many hot dogs will be demanded?

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16. A sales manager is asked to forecast the total sales of his division for a forthcoming period of time. He feels that his loss function is linear as a function of the difference between his estimate and the true value, but he also feels that an error of overestimation is three times as serious as an error of underestimation (given that the magnitudes of the errors are equal). He feels this way because his superiors will criticize him if the division does worse than he predicts, but they will be happy if the division does better than predicted and will be less likely to be concerned about an error in predicting. If his actual judgments can be represented by a normal distribution with mean 50,000 and standard deviation 10,000, what value should he report as his forecast of sales?

17. Suppose that you want to estimate $\tilde{p}$, the proportion of the market for a particular product that will be obtained by a new brand of the product. Your prior distribution for $\tilde{p}$ is a diffuse beta distribution with $r' = 0$ and $n' = 0$, and you are willing to assume that the process behaves like a Bernoulli process. You take a random sample of 200 purchasers of the product and find that 24 of them buy the new brand. If the loss function for the estimation problem is linear and if the per unit cost of overestimation is three times the per unit cost of underestimation, find the optimal estimate of $\tilde{p}$.

18. Prove that if a decision maker's loss function in a point estimation problem is given by Equation 7.4.1, then a $k_u/(k_u + k_o)$ fractile of the decision maker's distribution of $\tilde{\theta}$ is an optimal point estimate.

19. An economist is asked to predict a future value of a particular economic indicator, and she thinks that her loss function is linear with $k_u = 4k_o$. If her distribution is a uniform distribution on the interval from 650 to 680, what should her estimate be?

20. In Exercises 14, 15, 16, and 19, what are the optimal estimates if the loss functions are quadratic instead of linear (that is, if the loss functions are of the form of Equation 7.4.5)? In each case, if $k = 1$, what is the expected value of perfect information?

21. In Exercise 17, what is the optimal estimate if the loss function is quadratic instead of linear? Find the EVPI and, assuming that a second sample of purchasers of the product is being considered, find the EVSI for samples of size 50, 100, 200, and 500.

22. Comment on the following statement: "In taking the sample mean as an estimator of the population mean, the classical statistician is acting essentially as though he had a quadratic loss function."

23. If a person faces a point estimation problem with a linear loss function with $k_u = 4$ and $k_o = 3$, does he need to assess an entire probability distribution or can he determine a certainty equivalent? Explain.

24. If a statistician wishes to estimate an uncertain quantity $\tilde{\theta}$ subject to a loss function that is linear with $k_o = 2k_u$ and if his distribution of $\tilde{\theta}$ is an exponential distribution,

$$f(\theta \mid \lambda) = \begin{cases} \lambda e^{-\lambda\theta} & \text{if } \theta > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

with $\lambda = 4$, what is his optimal estimate of $\tilde{\theta}$?

25. If the loss function in a point estimation problem is of the form

$$L(a, \theta) = \begin{cases} 0 & \text{if } |a - \theta| < k, \\ 1 & \text{otherwise,} \end{cases}$$

where k is some very small positive number, what is the optimal estimate? Can you think of any realistic situations in which the loss function might be of this form?

26. In Exercise 6, determine an estimate of $\tilde{p}$ from the posterior distribution if the loss function is

$$L(a,\theta) = \begin{cases} 19(a-\theta) & \text{if } \theta \le a, \\ (\theta - a) & \text{if } \theta \ge a. \end{cases}$$

How does this differ from the estimate obtained from the *prior* distribution using the same loss function?

27. In Exercise 10, the production manager must make an estimate of the mean weight of items turned out by the process in question. His loss function is linear with $k_o = k_u$. What should his estimate be?

28. In Exercise 10, suppose that instead of estimating $\tilde{\mu}$, the production manager wants to predict $\tilde{m}$, the sample mean from a sample of size 10. He is making this prediction after having seen the first sample, and his loss function for the prediction problem is linear with $k_u = 6$ and $k_o = 2$. What is his optimal prediction of $\tilde{m}$? If the sample size is 100 rather than 10, what is the optimal prediction of $\tilde{m}$?

29. In Exercise 27, suppose that the production manager wants an interval estimate for $\tilde{\mu}$, given the following loss function: the interval must be of length 0.5, and the loss is 1 if the interval includes $\tilde{\mu}$ and 0 if it does not include $\tilde{\mu}$. In general, what type of interval is optimal for this "all-or-nothing" type of loss function?

30. Suppose that a marketing manager is interested in $\tilde{p}$, the proportion of consumers that will buy a particular new product. He considers the following two hypotheses:

$$H_1\colon \tilde{p} = 0.10$$

and

$$H_2\colon \tilde{p} = 0.20.$$

His prior probabilities are $P(\tilde{p} = 0.10) = 0.85$ and $P(\tilde{p} = 0.20) = 0.15$, and a random sample of eight consumers results in three consumers who state that they will buy the product if it is marketed.

(a) What is the prior odds ratio?
(b) What is the likelihood ratio?
(c) What is the posterior odds ratio?
(d) The manager decides that the posterior probability of H_1 must be no larger than 0.40 to make it worthwhile to market the product. Should the product be marketed?
(e) On the basis of the sample information alone, should the product be marketed? If the decision is made on this basis, what is implied about the prior distribution?

31. Suppose that you are uncertain about which of two authors, A or B, wrote a particular essay, and you feel that you would be indifferent between the following lotteries.

Lottery I: You win \$10 if A wrote the essay, \$0 otherwise.

Lottery II: You win \$10 with probability 0.3, \$0 with probability 0.7.

You know from past analyses of the writings of A and B that A uses a certain key word with an average frequency of four times in 100 words, and B uses this word with an average frequency of twice in 100 words (assume that the process is stationary and independent). In the particular essay of interest there are 500 words and the key word appears eight times. You are interested in finding out which author wrote the essay.

(a) What statistical model would you use to represent the data-generating process?
(b) Express the "real-world" hypotheses in terms of the statistical model chosen in (a).
(c) Find the prior odds ratio.
(d) Find the likelihood ratio.
(e) Find the posterior odds ratio.
(f) What would you conclude about the authorship of the essay?

32. Comment on the statement "Hypothesis testing can sometimes be thought of as a problem of classification or as a problem of discrimination."

33. In Exercise 10, assume that the prior distribution is discrete with $P(\tilde{\mu} = 109) = 0.2$ and $P(\tilde{\mu} = 110) = 0.8$. For H_1: $\tilde{\mu} = 109$ and H_2: $\tilde{\mu} = 110$, find

(a) the prior odds ratio
(b) the likelihood ratio
(c) the posterior odds ratio.

34. In Exercise 30, it would be more realistic to consider hypotheses such as the following:

$$H_1: \tilde{p} \leq 0.15$$

and

$$H_2: \tilde{p} > 0.15 .$$

If the prior distribution is a beta distribution with $r' = 2$ and $n' = 18$, find the posterior distribution and use this posterior distribution to find the posterior odds ratio of H_1 to H_2.

35. In Exercise 10, suppose that the production manager is interested in the hypotheses H_1: $\tilde{\mu} \leq 110$ and H_2: $\tilde{\mu} > 110$. From the prior distribution, find the prior odds ratio; also, from the posterior distribution, find the posterior odds ratio. If the losses involved in a decision-making problem involving the production process are such that H_1 should be accepted only if the posterior odds ratio is greater than three, what decision should be made?

36. A statistician is interested in the mean $\tilde{\mu}$ of a normal population, and his prior distribution for $\tilde{\mu}$ is normal with mean 800 and variance 12. The variance of the population is known to be 72. How large a sample is needed to guarantee that the variance of the posterior distribution will be no larger than 1? How large a sample is needed to guarantee that the variance of the posterior distribution will be no larger than 0.1?

37. In testing a hypothesis concerning the mean of a normal process with known variance against a one-tailed alternative, discuss the relationship between the classical significance level and the posterior probability $P''(H_1)$

(a) if the prior distribution is diffuse,
(b) if the prior distribution is not diffuse.

38. Comment on the following statement: "A hypothesis such as $\tilde{\mu} = 100$ is not realistic and should be modified somewhat; otherwise, the Bayesian approach to hypothesis testing may not be applicable."

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39. Suppose that you are sampling from a normal data-generating process with unknown mean $\tilde{\mu}$ and known variance $\sigma^2 = 25$. You are interested in the hypotheses H_1: $\tilde{\mu} = 50$ versus H_2: $\tilde{\mu} \neq 50$. Find the classical two-tailed level of significance if

(a) $n = 1$ and $m = 51$,
(b) $n = 25$ and $m = 51$,
(c) $n = 100$ and $m = 51$,
(d) $n = 10{,}000$ and $m = 51$,
(e) $n = 1$ and $m = 50.1$,
(f) $n = 25$ and $m = 50.1$,
(g) $n = 100$ and $m = 50.1$,
(h) $n = 10{,}000$ and $m = 50.1$.

Comment on these results, especially with respect to Lindley's paradox.

40. An automobile manufacturer claims that the average mileage per gallon of gas for a particular model is normally distributed with $m' = 20$ and $\sigma' = 4$, provided that the car is driven on a level road at a constant speed of 30 miles per hour. A rival manufacturer decides to use this as a prior distribution and to obtain additional information by conducting an experiment. In the experiment, it is assumed that the variance of mileage is $\sigma^2 = 96$ and that the mileage is normally distributed. The experiment is conducted on 10 randomly chosen cars, with the sample mean (the average mileage in the sample) equaling 18. Find the posterior distribution for $\tilde{\mu}$, the average mileage per gallon of gas. On the basis of this posterior distribution, what is the probability that $\tilde{\mu}$ is greater than or equal to 20? What is the probability that $\tilde{\mu}$ is between 19 and 21?

41. In Exercise 35, suppose that the manager is interested in the hypotheses H_1: $\tilde{\mu} = 110$ and H_2: $\tilde{\mu} \neq 110$. From the posterior distribution, what is the posterior odds ratio of H_1 to H_2? What is the posterior odds ratio if the hypotheses are H_1: $109 < \tilde{\mu} < 111$ and H_2: $\tilde{\mu} \leq 109$ or $\tilde{\mu} \geq 111$?

42. Suppose that a statistician is interested in H_1: $\tilde{\mu} = 50$ and H_2: $\tilde{\mu} \neq 50$. His prior distribution consists of a mass of probability of 0.25 at $\tilde{\mu} = 50$, with the remaining 0.75 of probability distributed uniformly over the interval from $\tilde{\mu} = 40$ to $\tilde{\mu} = 60$. Find the prior probability that

(a) $45 < \tilde{\mu} < 55$,
(b) $47 \leq \tilde{\mu} \leq 50$,
(c) H_1 is exactly true,
(d) the hypothesis $49 < \tilde{\mu} < 51$ is true.

43. Suppose that $\tilde{x}$ is normally distributed with mean $\tilde{\mu}$ and variance $\sigma^2 = 400$. The prior distribution of $\tilde{\mu}$ is normally distributed with mean $m' = -60$ and variance $\sigma'^2 = 40$. Furthermore, a sample of size 20 is taken, with sample mean $m = -69$.

(a) Find the posterior distribution of $\tilde{\mu}$.
(b) Find a 63 percent credible interval for $\tilde{\mu}$.
(c) Find $P(H_1)$ from the posterior distribution, where H_1 is the hypothesis that $\tilde{\mu}$ is greater than or equal to -70.

44. In Exercise 43, suppose that three hypotheses are under consideration:

H_1: $\tilde{\mu} \leq -70$,
H_2: $-70 < \tilde{\mu} < -60$,
and H_3: $\tilde{\mu} \geq -60$.

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(a) Find the posterior probabilities of these three hypotheses.
(b) If the decision problem at hand involves three actions (corresponding to the three hypotheses) and if the hypothesis to be selected is the one that is most probable, which hypothesis would be selected under the posterior distribution?
(c) In part (b), which hypothesis would be selected on the basis of the *prior* distribution?

45. Suppose that a statistician is interested in the difference in the means of two normal populations, $\tilde{\mu}_1$ and $\tilde{\mu}_2$. Let $\tilde{\mu} = \tilde{\mu}_1 - \tilde{\mu}_2$. Each population has variance 100. A random sample of size 25 is taken from the first population, with sample mean $m_1 = 80$, and a random sample of size 25 is taken from the second population, with sample mean $m_2 = 60$. The two samples are independent.

(a) Suppose that the prior distribution of $\tilde{\mu} = \tilde{\mu}_1 - \tilde{\mu}_2$ is normally distributed with mean $m' = 10$ and variance $\sigma'^2 = 50$. Find $P'(H_1)$ and $P'(H_2)$, where the hypotheses are H_1: $\tilde{\mu} \leq 0$ and H_2: $\tilde{\mu} > 0$.
(b) Find the posterior distribution of $\tilde{\mu} = \tilde{\mu}_1 - \tilde{\mu}_2$.
(c) From the posterior distribution, find the posterior odds ratio of H_1 to H_2.

46. Exercise 45 suggests a Bayesian approach to inferences regarding the difference between two means under the conditions that the two populations of interest are normally distributed and the variances are known. Using the notation of Exercise 45, determine a general formula for finding the posterior distribution of the difference between two means if the prior distribution of the difference in means has mean m' and variance σ'^2 and independent samples are taken from the two populations.

47. In Exercise 30, the marketing manager decides that the loss that will be suffered if the company markets the product and $\tilde{p}$ is in fact only 0.10 is three times as great as the loss that will be suffered if the company fails to market the product and $\tilde{p}$ is actually 0.20. Should the product be marketed?

48. In Exercise 31, the loss due to claiming incorrectly that A authored the essay is considered to be three times as great as the loss due to claiming incorrectly that B authored the essay. If you must make a claim concerning the authorship, which one should you claim as the author?

49. In Exercise 33, if $L_I = 7$ and $L_{II} = 5$, should you accept H_1 or H_2?

50. A statistician is interested in testing the hypotheses

$$H_1\text{: } \tilde{\mu} \geq 120$$

and

$$H_2\text{: } \tilde{\mu} < 120,$$

where $\tilde{\mu}$ is the mean of a normally distributed population with variance 144. The prior distribution for $\tilde{\mu}$ is a normal distribution with mean $m' = 115$ and variance $\sigma'^2 = 36$. A sample of size eight is taken, with sample mean $m = 121$.

(a) Find the prior odds ratio of H_1 to H_2.
(b) Find the posterior odds ratio of H_1 to H_2.
(c) If $L_I = 4$ and $L_{II} = 6$, which hypothesis should be accepted according to the posterior distribution?

51. For a sample of size one from a normal population with known variance 25, show that the classical test of

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$$H_1: \tilde{\mu} = 50$$

versus

$$H_2: \tilde{\mu} = 60$$

is a likelihood ratio test. That is, show that the rejection region can be expressed in the form $LR \leq c$, where LR is the likelihood ratio. [*Hint*: Consider the equation $LR \leq c$ and attempt to manipulate it algebraically to get the result $x \geq k$, where x is the sample outcome and k is some constant, since you know that the rejection region must be of this form.]

52. Consider the hypotheses H_1: $\tilde{\mu} = 10$ and H_2: $\tilde{\mu} = 12$, where $\tilde{\mu}$ is the mean of a normally distributed population with variance 1. The prior distribution is diffuse (take a normal distribution with $n' = 0$), and a sample of size 10 is to be taken.
    (a) If $L_{\text{I}} = 100$ and $L_{\text{II}} = 50$, find the region of rejection (the sample results for which you would reject H_1 in favor of H_2), using the decision-theoretic approach.
    (b) From the region of rejection determined in (a), find P(Type I error) and P(Type II error).
    (c) Do (a) and (b) if $L_{\text{I}} = 50$ and $L_{\text{II}} = 50$.
53. In classical hypothesis testing, the rejection region is often determined simply by choosing an arbitrarily small value, such as 0.05, and requiring that P(Type I error) equal this value. What advantages or disadvantages does this procedure have in comparison with the decision-theoretic approach, as illustrated in Exercise 52?
54. Comment on the statement, "The entire posterior distribution constitutes an inferential statement, and for many (perhaps most) purposes, the entire distribution is much more informative and useful than any summarizations in the form of estimates or tests of hypotheses."
55. Carefully distinguish among classical inferential statistics, Bayesian inferential statistics, and decision theory.

Answers to Selected Exercises

Chapter 2

3. 2^n.
6. (b) 1/6 (e) 13/18 (g) 1/12.
8. 25/52, 1/2.
9. 125/216.
17. (a) 2/3 (c) 3/10.
18. (b) 1 to 4 (c) 7 to 1.
41. (b) 3/7, 12/13, 4/7, 1/13 (e) No.
43. $P(E_1, E_2, E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1,E_2)$.
44. 2/3.
45. 0, 0.24, 0.4.
48. 8/23, 5/23, 10/23.
50. 343/370.
51. 5/29.

Chapter 3

1. (c) 0.5 (e) 0.9 (g) 0.
2. (b) 0 (c) $E(\tilde{x}) = 7/4$, $V(\tilde{x}) = 3/16$.
4. (b) 9/13 (d) 8/13.
5. (a) 1.8 (c) 5.56 (f) 0, 1.
6. (a) $E(\tilde{x}) = 8205$, $V(\tilde{x}) = 445{,}475$ (b) $E(\tilde{z}) = 3{,}025$, $V(\tilde{z}) = 11{,}136{,}875$.
11. The values of $\tilde{z}$ are –1.81, –1.06, –0.31, 0.44, 1.19, 1.94, and 2.69.
12. \$5.18.
14. (b) Yes (e) 1.5, 2.4, 3.9, 1.2 (f) 0.45, 0.84, 1.29, 10.56.

17. (a) $P(\tilde{x} = 1 \mid \tilde{y} = 1) = 0.83$, $P(\tilde{x} = 2 \mid \tilde{y} = 1) = 0.17$
    (d) $P(\tilde{y} = 1 \mid \tilde{x} = 2) = 0.042$, $P(\tilde{y} = 2 \mid \tilde{x} = 2) = 0.333$,
    $P(\tilde{y} = 3 \mid \tilde{x} = 2) = 0.500$, $P(\tilde{y} = 4 \mid \tilde{x} = 2) = 0.125$
    (e) 1.17, 1.40, 2.862, 2.708.
21. 0.0584, 0.0781.
23. 0.1933, 0.3844, 0.3097, 0.1126.
25. $P(\tilde{p} = 0.4) = 2/15$, $P(\tilde{p} = 0.5) = 10/15$, $P(\tilde{p} = 0.6) = 3/15$.
26. The posterior probabilities are 0.06990, 0.10744, 0.15713, 0.31012, 0.28831, and 0.06710, respectively, and the posterior mean and variance are 0.1568 and 0.000699.
31. 0.0067, 0.0681, 0.1251, 10.
33. 0.109, 0.561, 0.330.
34. The posterior probabilities are $P(\tilde{\lambda} = 10) = 0.071$, $P(\tilde{\lambda} = 11) = 0.210$, $P(\tilde{\lambda} = 12) = 0.470$, and $P(\tilde{\lambda} = 13) = 0.249$.
38. $P(\tilde{\lambda} = 50) = 0.00$, $P(\tilde{\lambda} = 75) = 0.14$, $P(\tilde{\lambda} = 100) = 0.86$.
41. $P(\tilde{\lambda} = 4) = 0.3244$, $P(\tilde{\lambda} = 5) = 0.5027$, $P(\tilde{\lambda} = 6) = 0.1729$.
47. From the hypergeometric, 0.881; from the binomial approximation, 0.821.
48. 0.0324, 0.0220.
49. 0.0353, 0.0014.
50. 0.000685, 0.0154.
53. $P(\tilde{\theta} = 2) = 1/19$, $P(\tilde{\theta} = 1) = 6/19$, $P(\tilde{\theta} = 0) = 12/19$.
57. $P(\tilde{r} = 0) = 0.03194$, $P(\tilde{r} = 1) = 0.10406$, $P(\tilde{r} = 2) = 0.17550$,
    $P(\tilde{r} = 3) = 0.20824$, $P(\tilde{r} = 4) = 0.19391$, $P(\tilde{r} = 5) = 0.14558$,
    $P(\tilde{r} = 6) = 0.08661$, $P(\tilde{r} = 7) = 0.03905$, $P(\tilde{r} = 8) = 0.01240$,
    $P(\tilde{r} = 9) = 0.00247$, $P(\tilde{r} = 10) = 0.00023$.
59. Some predictive probabilities are $P(\tilde{r} = 0) = 0.0487$, $P(\tilde{r} = 5) = 0.1121$, and $P(\tilde{r} = 9) = 0.0059$.

Chapter 4

1. (a) 6 (c) 5/32 (d) 1 if $x \geq 1$, $3x^2 - 2x^3$ if $0 < x < 1$, 0 if $x \leq 0$ (e) 1/2, 3/10, 1/20.
2. (a) $x/2$ if $0 \leq x < 2$, 0 elsewhere (d) −1, 2.
3. 7/36, 1/9.
6. (a) 1/8 (c) $(x + 1)/4$ if $0 \leq x \leq 2$, 0 elsewhere (d) $(2x + 1)/6$ if $0 \leq x \leq 2$, 0 elsewhere (f) 7/6, 7/6.
9. (b) $f(\theta \mid y) = 229(\theta + 0.10)^2$ if $-0.10 \leq \theta \leq 0.10$, $459(\theta + 0.10)(0.20 - \theta)$ if $0.10 \leq \theta \leq 0.12$, $1261(0.20 - \theta)^2$ if $0.12 \leq \theta \leq 0.20$, 0 elsewhere.
13. $r' = 10$, $n' = 15$.
14. $r' = 15$, $n' = 75$.
16. $r' = 2$, $n' = 4$, $r'' = 4$, $n'' = 10$.
20. (a) 0.5328, 0.6915, 0.248 (b) $c = 7.56$ (c) −3.56, 0.32, 4.76, 12.32.
21. 109.1, 10.5.
24. 0.969, 0.881.
27. The posterior probabilities are $P(\tilde{\mu} = 109.4) = 0.0734$, $P(\tilde{\mu} = 109.7) = 0.2572$, $P(\tilde{\mu} = 110.0) = 0.5063$, $P(\tilde{\mu} = 110.3) = 0.1412$, and $P(\tilde{\mu} = 110.6) = 0.0219$.
28. normal, $m'' = 109.73$, $\sigma''^2 = 0.267$.
30. normal, $m' = 50$, $\sigma'^2 = 16.67$.
33. normal, $m'' = 51.4$, $\sigma''^2 = 8.97$; $P(\tilde{\mu} \geq 50) = 0.681$.
34. normal, $m'' = 967$, $\sigma''^2 = 1667$.

36. (c) normal-gamma, $m'' = 967$, $n'' = 30$, $v'' = 55{,}111$, $d'' = 21$.
43. beta, $r' = 450$, $n' = 1000$.
44. $\lambda = 0.55$.
45. gamma, $r'' = 34.5$, $t'' = 10$, $E(\tilde{\lambda}) = 3.45$, $V(\tilde{\lambda}) = 0.345$.
55. normal, $E(\tilde{m}) = 50$, $V(\tilde{m}) = 39$; $P(\tilde{m} \geq 55) = 0.212$.
56. beta-binomial, $n = 6$, $r' = 4$, $n' = 10$; $E(\tilde{r}) = 2.4$, $V(\tilde{r}) = 2.09$.
57. $P(\tilde{r} = r) = 1/(n + 1)$ for $r = 0, 1, \ldots, n$.

Chapter 5

2. (a) Action 3
4. (d)

	DEMAND				
	100	*150*	*200*	*250*	*300*
Order 100	600	525	450	375	300
Order 200	0	900	1800	1725	1650
Order 300	-450	450	1350	2250	3150

7.

	PERCENTAGE CHANGE IN MARKET SHARE			
	0	*10%*	*20%*	*30%*
Advertise	-200,000	-50,000	100,000	250,000
Don't Advertise	0	0	0	0

8. (a) Action 2 (b) Action 4 (c) Action 4.
9. Order 100, order 300, order 300.
12. Action 2.
14. $ER(1) = 26.0$, $ER(2) = 48.0$, $ER(4) = 39.5$, $EL(1) = 90.5$, $EL(2) = 68.5$, $EL(4) = 77.0$.
15. Action 2.
16. $ER(\text{Order } 100) = 8{,}625/19$, $ER(\text{Order } 200) = 24{,}525/19$, $ER(\text{Order } 300) = 24{,}750/19$.

18. *ER*(advertise) = –1,800,000/21, *ER*(Don't advertise) = 0.
22. (b) *P*(Adverse weather) > C/L.
24. *ER*(Hire) = –657.5, *ER*(Don't hire) = 0; don't hire.
25. *ER*(Plan 1) = 36,017.5, *ER*(Plan 2) = 35,000, ER(Plan 3) = 33,052.5.
26. (c) *ER*(Adjust) = 2400, *ER*(Don't adjust) = 2438.
29. (a) Action 2 (d) Action 2.
31. (a) Don't drill (c) Drill (e) Drill (f) Don't drill.
35. (b) *EU*(bet) = 2160, *EU*(No bet) = 1900 (c) *EU*(bet) = 9840, *EU*(No bet) = 9900.
37. (b) For the 6 utility functions, t.he optimal actions are bet, don't bet, either, don't bet, bet, and don't bet, respectively.
40. (d) Yes.
41. *EU*(Don't invest) = 0.50, *EU*($1000 in one investment) = 0.40, *EU*($500 in one investment) = 0.49, *EU*($500 in each investment) = 0.52.
43. Approximate expected utilities: *EU*(Advertise) = 0.08, *EU*(Don't advertise) = 0.12.
45. (b) For the 6 utility functions, the risk premiums are –2.67, 2.72, 0, 2.62, –59, and 22.75, respectively.
53. *ER*(Advertise) = 100,857, *ER*(Don't advertise) = 0.
58. *ER*(Order 0) = –273.75, *ER*(Order 100) = 427.50, *ER*(Order 200) = 1,143.75, *ER*(Order 300) = 986.25, *ER*(Order 400) = 285.00.

Chapter 6

2. In Exercise 14, EVPI = 68.5; in Exercise 15, EVPI = 0.70.
3. $414.47.
4. EVPI = 50,000.
5. (b) 1.95 (d) 1.40.
7. For $\tilde{\lambda}$, EVPI = 3,757.5.
9. (a) 14.38 (b) 0 (c) 12.22 (d) 0.
12. (a) 0 (b) 0 (c) 2571.
13. VPI (Oil) = 0, VPl (No Oil) = 40,000, EVP1 = 28,000.
15. EVPI = 3.8.
16. (a) VSI = 3.231 (b) VSI = 0 (c) EVSI = 0.84.
17. (a) 1.09 (c) 2.05.
18. (b) $3.20 (c) $0.80 (d) $1.90.
19. (b) $4.80 (c) $1.20 (d) $2.84.
20. (b) 12,000 (c) ENGS = 3,667.
22. (a) 0 (b) 14,286 (c) 5,079.
23. 27,466.
24. (a) $150,000 (b) $52,000 (c) $69,000.
25. ENGS (Use A *and* B) = $62,300.
26. ENGS (Sequential Plan) = $73,900.
27. (b) 0.34.
34. (a) 20 (c) 5 (d) 0 (e) 5.726 (g) 3.956.
35. (c) a_1 (d) beta distribution, $r'' = 8$ and $n'' = 39$, a_2.
36. ENGS(1) = 1.83, ENGS(5) = 3.35, ENGS(10) = 3.32.
39 (b) A (c) VPl = 0.

40. Drill with 100% interest.
44. (a) $n = 6$ (b) $n = 10$.
46. $E'U(a_1) = 11{,}750$, $E'U(a_2) = 11{,}433$, $E''U(a_1) = 12{,}167$, $E''U(a_2) = 12{,}254$.
47. (b) a_2 (c) $E''U(a_3) = 11{,}899$.
48. $E'U(a_1) = 207{,}359$, $E'U(a_2) = 262{,}891$.
49. $E'R(a_1) = 4$, $E'R(a_2) = 3.14$, $E'R(a_3) = 3.38$.

Chapter 7

9. (b) σ^2, $\sigma^2/4$, $3\sigma^2/10$, $4\sigma^2/9$ (c) 0, 0, 0, $\mu/3$.
10. (d) (108.96, 111.04) (e) (107.73, 110.67) (f) (108.88, 110.58).
11. (a) (46.5, 61.5) (c) (49.9, 52.9).
12. (b) (0.1875, 0.5300) (d) (0.2910, 0.5020).
14. 5 houses; EVPI = $2,000.
15. 9,140; EVPI = $218.
16. 43,300.
17. 0.1032.
19. 674.
24. 0.10.
26. 0.1875 for the posterior distribution, 0.1688 for the prior distribution.
27. 109.73.
29. (109.48,109.98).
30. (a) 5.667 (b) 0.225 (c) 1.275 (d) $P''(H_1) = 0.56$.
31. (a) Poisson (d) 13/1126 (e) 0.005.
33. (a) 0.25 (b) 1.45 (c) 0.36.
35. $\Omega' = 1$, $\Omega'' = 2.33$.
36. $n \geq 66$, $n \geq 714$.
39. (a) 0.842 (c) 0.046 (e) 0.984 (g) 0.842.
40. $P''(\tilde{\mu} \geq 20) = 0.305$, $P''(19 < \tilde{\mu} < 21) = 0.281$.
41. 0; 10.6.
42. (b) 0.3625 (d) 0.325.
43. (a) normal, $m'' = -66$ and $\sigma''^2 = 13.33$ (b) (−69.65, −62.35) (c) 0.864.
44. (a) 0.136, 0.813, 0.051.
47. $\Omega'' \times L_{I/II} = 3.825$; do not market.
48. B.
49. $\Omega'' \times L_{I/II} = 0.50$; accept H_2.
50. (a) 0.2552 (b) 0.6284 (c) $\Omega'' \times L_{I/II} = 0.4189$.