Meeting 9:

Recap of Assignment 1
Problem discussion

Recap of Assignment 1

Assignment 1, Task 1

A bank official is concerned about the rate at which the bank's tellers provide service for their customers.

He feels that all of the tellers work at about the same speed, which is either 30, 40 or 50 customers per hour.

Should be interpreted as same **average** speed

States of the world: speed = $\tilde{\lambda} \in \{30, 40, 50\}$ (customers/hour)

Furthermore, 40 customers per hour is twice as likely as each of the two other values, which are assumed to be equally likely.

Prior probabilities: $P(\tilde{\lambda} = 40) = 2 \cdot P(\tilde{\lambda} = 30) = 2 \cdot P(\tilde{\lambda} = 50)$

$$\Rightarrow P(\tilde{\lambda} = 30) = 0.25 \;;\; P(\tilde{\lambda} = 40) = 0.5 \;; P(\tilde{\lambda} = 50) = 0.25$$

In order to obtain more information, the official observes all five tellers for a two-hour period, noting that 380 customers are served during that period.

380 is the sum of all tellers work for 2 hours

Show that the posterior probabilities for the three possible speeds are approximately 0.000045, 0.99996 and 0.00000012 respectively.

Serving customers can be assumed to be a Poisson process

It is actually a process that takes a break when there are no customers waiting, and part of a *birth-and-death process*. However, in this example we can set the number of customers served for 2 hours as approximately equal to the number of customers arriving during the same time.

Let $\tilde{x}_i$ denote the number of customers served by teller *i* during the two hours.

Then, $\tilde{x}_i$ is Poisson distributed with mean $\tilde{\lambda} \cdot 2$.

Assume further that given $\tilde{\lambda} = \lambda$, the number of customers served by one teller is conditionally independent of the number of customers served by another teller.

Let $\tilde{y}$ denote the total number of customers served by the tellers during the two hours.

Then, $\tilde{y} = \sum_{i=1}^{5} \tilde{x}_i$, which is Poisson distributed with mean $5 \cdot \tilde{\lambda} \cdot 2 = 10\tilde{\lambda}$.

Likelihoods:

$$L(\lambda; y) = P\left(\tilde{y} = y \middle| \tilde{\lambda} = \lambda\right) = \frac{(10\lambda)^y}{y!} \cdot e^{-10\lambda}$$

$$\Rightarrow L(\lambda = 30; y = 380) = P\left(\tilde{y} = 380 \middle| \tilde{\lambda} = 30\right) = \frac{(10 \cdot 30)^{380}}{380!} \cdot e^{-10\cdot 30} = \frac{300^{380}}{380!} \cdot e^{-300}$$

$$L(\lambda = 40; y = 380) = P\left(\tilde{y} = 380 \middle| \tilde{\lambda} = 40\right) = \frac{400^{380}}{380!} \cdot e^{-400}$$

$$L(\lambda = 50; y = 380) = P\left(\tilde{y} = 380 \middle| \tilde{\lambda} = 50\right) = \frac{500^{380}}{380!} \cdot e^{-500}$$

Issue: It is not possible to calculate factorials or powers of these magnitudes using a calculator or manually using mathematical operators in software.

But, using dpois in R works:

```
> dpois(x=380,lambda=300)
[1] 1.103549e-06
> dpois(x=380,lambda=400)
[1] 0.01230448
> dpois(x=380,lambda=500)
[1] 3.064927e-09
```

However, R was not invented this exercise was first published (in the former course book.)

Posterior probabilities:

$$P(\tilde{\lambda} = \lambda_i | \tilde{y} = y) = \frac{P\left(\tilde{y} = y \middle| \tilde{\lambda} = \lambda_i\right) \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} P\left(\tilde{y} = y \middle| \tilde{\lambda} = \lambda_j\right) \cdot P(\tilde{\lambda} = \lambda_j)} =$$

$$\frac{\frac{(10 \cdot \lambda_i)^y}{y!} e^{-10 \cdot \lambda_i} \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} \frac{(10 \cdot \lambda_j)^y}{y!} e^{-10 \cdot \lambda_{ji}} \cdot P(\tilde{\lambda} = \lambda_j)} =$$

The factorials disappear!

$$\frac{(10 \cdot \lambda_i)^y e^{-10 \cdot \lambda_i} \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} (10 \cdot \lambda_j)^y e^{-10 \cdot \lambda_j} \cdot P(\tilde{\lambda} = \lambda_j)} = \left\langle \begin{matrix} 10 \cdot \lambda_1 = 10 \cdot 30 = 300 = 3 \cdot 100 \\ 10 \cdot \lambda_2 = 10 \cdot 40 = 400 = 4 \cdot 100 \\ 10 \cdot \lambda_3 = 10 \cdot 50 = 500 = 5 \cdot 100 \end{matrix} \right\rangle =$$

$$\frac{\left(100 \cdot (\lambda_i/10)\right)^y e^{-10 \cdot \lambda_i} \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} \left(100 \cdot (\lambda_i/10)\right)^y e^{-10 \cdot \lambda_j} \cdot P(\tilde{\lambda} = \lambda_j)} = \frac{100^y \cdot (\lambda_i/10)^y e^{-10 \cdot \lambda_i} \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} 100^y \cdot (\lambda_i/10)^y e^{-10 \cdot \lambda_j} \cdot P(\tilde{\lambda} = \lambda_j)} =$$

$$\frac{(\lambda_i/10)^y e^{-10 \cdot \lambda_i} \cdot P(\tilde{\lambda} = \lambda_i)}{\sum_{j=1}^{3} (\lambda_i/10)^y e^{-10 \cdot \lambda_j} \cdot P(\tilde{\lambda} = \lambda_j)}$$

The intractable powers are replaced by tractable ones !

Thus,

$$P(\tilde{\lambda} = 30|\tilde{y} = 380) = \frac{3^{380}e^{-300} \cdot 0.25}{3^{380}e^{-300} \cdot 0.25 + 4^{380}e^{-400} \cdot 0.5 + 5^{380}e^{-500} \cdot 0.25} \approx 0.0000448$$

approx. $\frac{2.6042 \cdot 10^{50}}{5.8076 \cdot 10^{54}}$

$$P(\tilde{\lambda} = 40|\tilde{y} = 380) = \frac{4^{380}e^{-400} \cdot 0.5}{3^{380}e^{-300} \cdot 0.25 + 4^{380}e^{-400} \cdot 0.5 + 5^{380}e^{-500} \cdot 0.25} \approx 0.999955$$

approx. $\frac{5.8073 \cdot 10^{54}}{5.8076 \cdot 10^{54}}$

$$P(\tilde{\lambda} = 50|\tilde{y} = 380) = \frac{5^{380}e^{-500} \cdot 0.25}{3^{380}e^{-300} \cdot 0.25 + 4^{380}e^{-400} \cdot 0.5 + 5^{380}e^{-500} \cdot 0.25} \approx 0.00000012$$

approx. $\frac{7.2327 \cdot 10^{47}}{5.8076 \cdot 10^{54}}$

Alternatively…

Since the first edition of the textbook (from which this exercise is taken) was published in 1972, it was not expected that one could manage powers with very high (380) or very low (–300, –400, –300) exponents – there were no pocket calculators or PC:s

Use normal approximation:

$$P(\tilde{y} = y \mid \tilde{y} \in \mathrm{Po}(\mu)) \approx P\left(\frac{y - 0.5 - \mu}{\sqrt{\mu}} < \tilde{z} \leq \frac{y + 0.5 - \mu}{\sqrt{\mu}} \,\middle|\, \tilde{z} \in N(0,1) \right)$$

$$\text{if } \mu > 15$$

$$\Rightarrow P(\tilde{y} = y \mid \tilde{y} \in \mathrm{Po}(\mu)) \approx \Phi\left(\frac{y + 0.5 - \mu}{\sqrt{\mu}} \right) - \Phi\left(\frac{y - 0.5 - \mu}{\sqrt{\mu}} \right)$$

The likelihood are then approximated as

$$L(\lambda = 30; y = 380) \approx \Phi\left(\frac{380.5 - 300}{\sqrt{300}}\right) - \Phi\left(\frac{379.5 - 300}{\sqrt{300}}\right) \approx \Phi(4.69) - \Phi(4.65)$$

$$L(\lambda = 40; y = 380) \approx \Phi\left(\frac{380.5 - 400}{\sqrt{400}}\right) - \Phi\left(\frac{379.5 - 400}{\sqrt{400}}\right) \approx \Phi(-0.98) - \Phi(-1.03)$$

$$L(\lambda = 50; y = 380) \approx \Phi\left(\frac{380.5 - 500}{\sqrt{500}}\right) - \Phi\left(\frac{379.5 - 500}{\sqrt{500}}\right) \approx \Phi(-5.34) - \Phi(-5.39)$$

...and the posterior probabilities as

$$P\big(\tilde{\lambda} = 30 | \tilde{y} = 380\big) \approx$$

$$\frac{\big(\Phi(4.69) - \Phi(4.65)\big) \cdot 0.25}{\big(\Phi(4.69) - \Phi(4.65)\big) \cdot 0.25 + \big(\Phi(-0.98) - \Phi(-1.03)\big) \cdot 0.5 + \big(\Phi(-5.34) - \Phi(-5.39)\big) \cdot 0.25}$$

$$\approx \langle \text{with very extended tables} \rangle \approx$$

$$\frac{(0.9999986 - 0{,}9999983) \cdot 0.25}{(0.9999986 - 0.9999983) \cdot 0.25 + (0.1635 - 0.1515) \cdot 0.5 + (0.000000046 - 0.000000035 \cdot 0.25)}$$

$$\approx 0.000031$$

$$P\left(\tilde{\lambda} = 40 | \tilde{y} = 380\right) \approx$$

$$\frac{\left(\Phi(-0.98) - \Phi(-1.03)\right) \cdot 0.5}{\left(\Phi(4.69) - \Phi(4.65)\right) \cdot 0.25 + \left(\Phi(-0.98) - \Phi(-1.03)\right) \cdot 0.5 + \left(\Phi(-5.34) - \Phi(-5.39)\right) \cdot 0.25} \approx$$

$$\frac{(0.1635 - 0.1515) \cdot 0.5}{(0.9999986 - 0.9999983) \cdot 0.25 + (0.1635 - 0.1515) \cdot 0.5 + (0.000000046 - 0.000000035 \cdot 0.25)}$$

$$\approx 0.999968$$

$$P\left(\tilde{\lambda} = 50 | \tilde{y} = 380\right) \approx$$

$$\frac{\left(\Phi(-5.34) - \Phi(-5.39)\right) \cdot 0.25}{\left(\Phi(4.69) - \Phi(4.65)\right) \cdot 0.25 + \left(\Phi(-0.98) - \Phi(-1.03)\right) \cdot 0.5 + \left(\Phi(-5.34) - \Phi(-5.39)\right) \cdot 0.25} \approx$$

$$\frac{(0.000000046 - 0.000000035 \cdot 0.25)}{(0.9999986 - 0.9999983) \cdot 0.25 + (0.1635 - 0.1515) \cdot 0.5 + (0.000000046 - 0.000000035 \cdot 0.25)}$$

$$\approx 0.0000011$$

Hence, with normal approximation:		Compare with ”true” Poisson:
$P\left(\tilde{\lambda} = 30 \vert \tilde{y} = 380\right) \approx 0.000031$	$<$	$P\left(\tilde{\lambda} = 30 \vert \tilde{y} = 380\right) \approx 0.000045$
$P\left(\tilde{\lambda} = 40 \vert \tilde{y} = 380\right) \approx 0.999968$	$\approx$	$P\left(\tilde{\lambda} = 40 \vert \tilde{y} = 380\right) \approx 0.999955$
$P\left(\tilde{\lambda} = 50 \vert \tilde{y} = 380\right) \approx 0.0000011$	$\gg$	$P\left(\tilde{\lambda} = 50 \vert \tilde{y} = 380\right) \approx 0.00000012$

Assignment 1, Task 2

1. Assume you have decided to bet on a horse race, and that you have very little knowledge about the competing horses. You consider betting on Little Joe, and you see that the odds for this horse are 9 to 1 (i.e. odds against that the horse will win). You decide to look up some historical tracks on how Little Joe has performed recently and note that he has won in 2 of the last 10 races he competed in. You can assume that these races are fairly comparable with respect to the levels of his competitors.

a) Using the above as your background information, what are your subjective odds for Little Joe?

a) If you bet, you will obtain 9 times the money you have put in the bet. What is your subjective expected return from betting on Little Joe?

Anders' subjective approach

(a)

The base for the prior odds would for most be the current bookmaker odds, i.e. 9 to 1 against. This is the same as a prior probability of $1/10{=}0.10$.

If I temporarily treat this probability as a proportion in a hyperpopulation of potential outcomes for Little Joe, I can update this proportion using the data, i,e, 2 winnings out of 10 <u>comparable</u> races:

Let 1/10 be the mean of a beta prior distribution, i.e. $a/(a+b)$, with a and b being the shape parameters of that distribution. To reflect the meagre background, I let $a = 1$ and $b = 9$.

The data gives a binomial likelihood for the proportion and since beta and binomial is a conjugate family, the posterior for the proportion is also beta with shape parameters $a + 2$ and $b + 10 - 2$, i.e. 3 and 17.

My updated probability is then the mean of this distribution which is
3/20 = 0.15 .

Hence, my subjective odds are 3 to 17 or 5.67 to 1 against.

(b)

Let x be the money I put in (my stakes). My subjective expected return is then

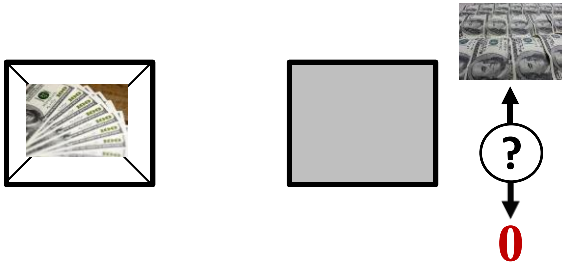
$$9x \cdot \frac{3}{20} + 0 \cdot \frac{17}{20} = \frac{27}{20} \cdot x = 1.35 \cdot x$$

My subjective expected payoff is

$$8x \cdot \frac{3}{20} - x \cdot \frac{17}{20} = \frac{7}{20} \cdot x = 0.35 \cdot x$$

Problem discussion

Newcomb's problem



You are exposed to two "boxes".

In Box 1 you can see that there is an amount of $1000.
You cannot see what is in Box 2, but you are told that it is either nothing or $1 000 000.

You are offered to make one the following two choices:

- A1: Take <u>both</u> boxes
- A2: Take Box 2

<u>Before</u> you make your choice, a prediction expert will predict which choice you will make.
If her prediction is that your choice will be A2, $ 1 000 000 will be put in Box 2
If her prediction is that your choice will be A1, nothing will be put in Box 2.

The prediction expert's accuracy is 99%, i.e. she has been right in 99% of her predictions.

What should you do?

The S:t Petersburg game

Assume a casino is offering you the following game:

Toss a coin until it lands "head up" for the first time. If *n* tosses are required, you will win 2^n dollars.

- How much would you be willing to pay to play that game?
- How much would the casino charge you for playing that game?

The prisoner's dilemma

Assume two perpetrators of two crimes (one serious, one less serious) are arrested.

They are put in different cells and cannot communicate with each other

The prosecutor gives each of the perpetrators the following information:

- "If you both deny, you will each get two years in prison for the less serious crime."
- "If one of you denies and the other confesses, the former will get 20 years in prison, and the latter will get 1 year in prison (*thanks for confessing*)."
- "If you both confess, you will both get 10 years in prison."

How would each perpetrator reason?

Do the rational decisions lead to the best consequence for any of them?