Meeting 11:
The value of information

The value of perfect information

Perfect information means that there is no uncertainty left for the decision maker. Hence the loss of the decision must be zero.

The *value of perfect information*, VPI is what this information is worth to the decision maker. This value would be equal in value to the loss of taking an action, a. Since there is more than one action to be taken, VPI can vary with the action.

If we now assume that the decision maker would take the optimal action $a*$ with respect to expected (opportunity) loss the *expected value of perfect information*, EVPI is

$$\text{EVPI} = EL(a^*) - 0 = EL(a^*)$$

Hence, the expected value of perfect information is the expected loss of the optimal action.

The expectation is – if nothing else specified – here assumed to be with respect to the prior distribution of the state of the world, hence using the prior probability density or mass function $f'(\theta)$, Alternatively, we can reason with the payoff instead of the loss.

The value of perfect information will for a specific action vary with the state of the world. For a fix state of the world, θ, the value of perfect information is

$$\text{VPI}(\theta) = R(a_\theta, \theta) - R(a^*, \theta)$$

where a_θ is the optimal action when the state of the world is θ while a^* is the optimal action with respect to expected loss.

Then,

$$\text{EVPI} = \int \text{VPI}(\theta) \cdot f'(\theta)\, d\theta \quad \text{or} \quad \sum \text{VPI}(\theta) \cdot f'(\theta)$$

Exercise 6.5

5. In Exercise 22, Chapter 5, give a general expression for the expected value of perfect information regarding the weather and find the EVPI if
   (a) P(adverse weather) = 0.4, C = 3.5, and L = 10,
   (b) P(adverse weather) = 0.3, C = 3.5, and L = 10,
   (c) P(adverse weather) = 0.4, C = 10, and L = 10,
   (d) P(adverse weather) = 0.3, C = 2, and L = 8.

22. A special type of decision-making problem frequently encountered in meteorology is called the "cost-loss" decision problem. The states of the world are "adverse weather" and "no adverse weather," and the actions are "protect against adverse weather" and "do not protect against adverse weather." C represents the cost of protecting against adverse weather, while L represents the cost that is incurred if you fail to protect against adverse weather and it turns out that the adverse weather occurs. (L is usually referred to as a "loss," but it is not a loss in an opportunity-loss sense.)
    (a) Construct a payoff table and a decision tree for this decision-making problem.
    (b) For what values of P(adverse weather) should you protect against adverse weather?
    (c) Given the result in (b), is it necessary to know the absolute *magnitudes* of C and L?

Payoff table

Action	State of the world	
	adverse weather	no adverse weather
protect	$-C$	$-C$
do not protect	$-L$	0

Optimal action with respect to ER:

$$ER(\text{protect}) = (-C) \cdot P(\text{adverse weather}) + (-C) \cdot P(\text{no adverse weather}) = -C$$

$$ER(\text{do not protect}) = (-L) \cdot P(\text{adverse weather}) + 0 \cdot P(\text{no adverse weather}) =$$

$$= (-L) \cdot P(\text{adverse weather}) = (-L) \cdot P(\text{aw})$$

$$\Rightarrow a^* = \begin{cases} \text{protect} & \text{if} \quad C < L \cdot P(\text{aw}) \\ \text{do not protect} & \text{if} \quad C \geq L \cdot P(\text{aw}) \end{cases}$$

Hence, protect when P(adverse weather) > C/L

$$a^* = \begin{cases} \text{protect} & \text{if} \quad C < L \cdot P(\text{aw}) \\ \text{do not protect} & \text{if} \quad C \geq L \cdot P(\text{aw}) \end{cases}$$

Action	State of the world	
	adverse weather	no adverse weather
protect	–*C*	–*C*
do not protect	–*L*	0

$$\text{VPI(adverse weather)} = \text{VPI(aw)} =$$

$$= \begin{cases} R(\text{protect}, \text{aw}) - R(\text{protect}, \text{aw}) = 0 & \text{if} \quad C < L \cdot P(\text{aw}) \\ R(\text{protect}, \text{aw}) - R(\text{do not protect}, \text{aw}) = L - C & \text{if} \quad L \cdot P(\text{aw}) \leq C < L \\ R(\text{do not protect}, \text{aw}) - R(\text{do not protect}, \text{aw}) = 0 & \text{if} \quad C \geq L \end{cases}$$

$$\text{VPI(no adverse weather)} = \text{VPI(naw)} =$$

$$= \begin{cases} R(\text{do not protect}, \text{naw}) - R(\text{protect}, \text{naw}) = C & \text{if} \quad C < L \cdot P(\text{aw}) \\ R(\text{do not protect}, \text{naw}) - R(\text{do not protect}, \text{naw}) = 0 & \text{if} \quad C \geq L \cdot P(\text{aw}) \end{cases}$$

$$\Rightarrow EVPI =$$

$$= \begin{cases} 0 \cdot P(\text{aw}) + C \cdot P(\text{naw}) = C \cdot P(\text{naw}) & \text{if} \quad C < L \cdot P(\text{aw}) \\ (L - C) \cdot P(\text{aw}) - 0 \cdot P(\text{naw}) = (L - C) \cdot P(\text{aw}) & \text{if} \quad L \cdot P(\text{aw}) \leq C < L \\ 0 \cdot P(\text{aw}) + 0 \cdot P(\text{naw}) = 0 & \text{if} \quad C \geq L \end{cases}$$

(a) $P(\text{aw}) = 0.4$, $C = 3.5$, $L = 10 \Rightarrow C < L \cdot P(\text{aw}) = 4$

$\Rightarrow EVPI = C \cdot P(\text{naw}) = 3.5 \cdot 0.6 = 2.1$

(b) $P(\text{aw}) = 0.3$, $C = 3.5$, $L = 10 \Rightarrow 3 = L \cdot P(\text{aw}) < C < L$

$\Rightarrow EVPI = (L - C) \cdot P(\text{aw}) = (10 - 3.5) \cdot 0.3 = 1.95$

(c) $P(\text{aw}) = 0.4$, $C = 10$, $L = 10 \Rightarrow C = L$

$\Rightarrow EVPI = (L - C) \cdot P(\text{aw}) = (10 - 10) \cdot 0.4 = 0$

(d) $P(\text{aw}) = 0.3$, $C = 2$, $L = 8 \Rightarrow C < L \cdot P(\text{aw}) = 2.4$

$\Rightarrow EVPI = C \cdot P(\text{naw}) = 2 \cdot 0.7 = 1.4$

Exercise 6.8

8. In decision-making problems for which the uncertain quantity of primary interest can be viewed as a future sample outcome $\tilde{y}$, the relevant distribution of interest to the decision maker is the predictive distribution of $\tilde{y}$.
   (a) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{y}$?
   (b) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{\theta}$?
   (c) Explain the difference between your answers to (a) and (b).

(a)

Prior predictive distribution:

$$f(y) = \int f(y|\theta) \cdot f'(\theta) d\theta$$

$$\text{EVPI} = EL_{\tilde{y}}(a^*) = \int L(a^*, y) \cdot f(y) dy = \int_y L(a^*, y) \cdot \left(\int_\theta f(y|\theta) \cdot f'(\theta) d\theta \right) dy$$

$$=$$

$$= \int_y \int_\theta L(a^*, y) \cdot f(y|\theta) \cdot f'(\theta) d\theta dy = \int_\theta \int_y L(a^*, y) \cdot f(y|\theta) \cdot f'(\theta) dy d\theta$$

(b)

$$\text{EVPI} = EL_{\theta|y}(a\,*) = \int L(a^*,\theta)\cdot f''(\theta|y)d\theta =$$

$$= \int_{\theta} L(a^*,\theta)\cdot\left(\frac{f(y|\theta)\cdot f'(\theta)}{\int_{\lambda} f(y|\lambda)\cdot f'(\lambda)d\lambda}\right)d\theta =$$

$$= \int_{\theta} L(a^*,\theta)\cdot\left(\frac{f(y|\theta)\cdot f'(\theta)}{f(y)}\right)d\theta = \frac{1}{f(y)}\int_{\theta} L(a^*,\theta)\cdot f(y|\theta)\cdot f'(\theta)d\theta$$

(c)

$$\text{EVPI}^{(a)} = \int_{\theta}\int_{y} L(a^*,y)\cdot f(y|\theta)\cdot f'(\theta)dyd\theta = \text{constant}$$

$$\text{EVPI}^{(b)} = \frac{1}{f(y)}\int_{\theta} L(a^*,\theta)\cdot f(y|\theta)\cdot f'(\theta)d\theta = g(y)$$

Exercise 6.15

15. A store must decide whether or not to stock a new item. The decision depends on the reaction of consumers to the item, and the payoff table (in dollars) is as follows.

		PROPORTION OF CONSUMERS PURCHASING				
		0.10	0.20	0.30	0.40	0.50
DECISION	*Stock 100*	−10	−2	12	22	40
	Stock 50	−4	6	12	16	16
	Do not stock	0	0	0	0	0

If $P(0.10) = 0.2$, $P(0.20) = 0.3$, $P(0.30) = 0.3$, $P(0.40) = 0.1$, and $P(0.50) = 0.1$, what decision maximizes expected payoff? If perfect information is available, find VPI for each of the five possible states of the world and compute EVPI.

$$\text{VPI}(\theta) = R(a_\theta, \theta) - R(a^*, \theta)$$

$$a^* = \arg\max_a \{ER(a)\}$$

DECISION	PROPORTION OF CUSTOMERS BUYING				
	0.10	0.20	0.30	0.40	0.50
Stock 100	–10	–2	12	22	40
Stock 50	–4	6	12	16	16
Do not stock	0	0	0	0	0

$ER(\text{Stock 100}) = (-10) \cdot 0.2 + (-2) \cdot 0.3 + 12 \cdot 0.3 + 22 \cdot 0.1 + 40 \cdot 0.1 = 7.2$
$ER(\text{Stock 50}) = (-4) \cdot 0.2 + 6 \cdot 0.3 + 12 \cdot 0.3 + 16 \cdot 0.1 + 16 \cdot 0.1 = 7.8$
$ER(\text{Do not stock}) = 0 \cdot 0.2 + 0 \cdot 0.3 + 0 \cdot 0.3 + 0 \cdot 0.1 + 0 \cdot 0.1 = 0$

$\Rightarrow a^* = \text{Stock 50}$

$\Rightarrow$
$\text{VPI}(0.10) = 0 - (-4) = 4$
$\text{VPI}(0.20) = 6 - 6 = 0$
$\text{VPI}(0.30) = 12 - 12 = 0$
$\text{VPI}(0.40) = 22 - 16 = 6$
$\text{VPI}(0.50) = 40 - 16 = 24$

$\Rightarrow$ $\text{EVPI} = 4 \cdot 0.2 + 0 \cdot 0.3 + 0 \cdot 0.3 + 6 \cdot 0.1 + 24 \cdot 0.1 = 3.8$

The value of sample information

Recall the notation

θ	=	State of the world
$f'(\theta)$	=	Prior probability density (or mass) function of θ
y	=	(Potential) sample result
$f(y\|\theta)$	=	Likelihood of θ (from sample result y)
$f''(\theta\|y)$	=	Posterior probability density (or mass) function of θ
$U(a) = U(a, \theta)$	=	Utility of taking action a (with state of world θ)
$R(a) = R(a, \theta)$	=	Payoff from taking action a (with state of world θ)
$L(a) = L(a, \theta)$	=	(Opportunity)loss from taking action a (with state of world θ)

Then, the optimal action a' with respect to the prior distribution of θ satisfies

$$E'U(a') = \int U(a',\theta)\cdot f'(\theta)d\theta \geq \int U(a,\theta)f'(\theta)d\theta = E'U(a) \quad \forall a$$

i.e. $E'U(a') \geq E'U(a) \quad \forall a$

The expected utility with respect to the prior distribution of θ of the optimal action under that prior distribution is larger than or equal to the expected utility with respect to that prior distribution of any action.

and the optimal action a'' with respect to the posterior distribution of θ satisfies

$$E''U(a''|y) = \int U(a'',\theta|y)\cdot f''(\theta|y)d\theta \geq \int U(a,\theta)f''(\theta|y)d\theta = E''U(a|y) \quad \forall a$$

i.e. $E''U(a''|y) \geq E''U(a|y) \quad \forall a$

The expected utility with respect to the posterior distribution of θ of the optimal action under that posterior distribution is larger than or equal to the expected utility with respect to that posterior distribution of any action.

If we now assume that utility is linear in money it also holds that

$$E'R(a') \geq E'R(a) \qquad \forall a$$

$$E''R(a''|y) \geq E''R(a|y) \quad \forall a$$

and that

$$E'L(a') \leq E'L(a) \qquad \forall a$$

$$E''L(a''|y) \leq E''L(a|y) \quad \forall a$$

The *value of sample information* VSI(y) is then defined as

$$\text{VSI}(y) = E''R(a''|y) - E''R(a'|y) = E''L(a'|y) - E''L(a''|y)$$

The posterior expected payoff from taking the posterior optimal action minus the posterior expected payoff from taking the prior optimal action,

or the posterior expected loss from taking the prior optimal action minus the posterior expected loss from taking the posterior optimal action

The *expected* value of sample information

Using the definition of VSI(y) it is

$$\text{EVSI} = \int \text{VSI}(y) \cdot f(y)dy$$

However,

$$\int \text{VSI}(y) \cdot f(y)dy = \int \left(E''R(a''|y) - E''R(a'|y)\right) \cdot f(y)dy =$$

$$= \int E''R(a''|y) \cdot f(y)dy - \int E''R\left(a'^{|y}\right) \cdot f(y)dy =$$

$$= E''R(a'') - E''R(a')$$

"Overall" posterior expected payoff from taking action a'' minus "overall" posterior expected payoff from taking action a' .

Note also that

$$\int E''R(a''|y)\cdot f(y)dy = \int\left(\int R(a'',\theta)\cdot f''(\theta|y)d\theta\right)\cdot f(y)dy =$$

$$= \iint R(a'',\theta)\cdot f''(\theta|y)\cdot f(y)dyd\theta = \int R(a'',\theta)\int f(y|\theta)\cdot f'(\theta)dyd\theta =$$

$$= \int R(a'',\theta)\cdot f'(\theta)\cdot\left(\int f(y|\theta)dy\right)d\theta = \int R(a'',\theta)\cdot f'(\theta)d\theta = E'R(a'')$$

Analogously,

$$\int E''R(a'|y)\cdot f(y)dy = \cdots = E'R(a')$$

$$\Longrightarrow\quad \text{EVSI} = E'R(a'') - E'R(a')$$

Exercise 6.16

16. In Exercise 15, suppose that sample information is available in the form of a random sample of consumers. For a sample of size *one*,
   (a) find the posterior distribution if the one person sampled will purchase the item, and find the value of this sample information;
   (b) find the posterior distribution if the one person sampled will *not* purchase the item, and find the value of this sample information;
   (c) find the expected value of sample information.

Below we have used the word "BUY" instead of "PURCHASE"

(a) Posterior distribution:

$$P(\theta|\text{BUY}) = \frac{P(\text{BUY}|\theta)\cdot P(\theta)}{P(\text{BUY})} = \frac{P(\text{BUY}|\theta)\cdot P(\theta)}{\sum_{\lambda} P(\text{BUY}|\lambda)\cdot P(\lambda)}$$

$P(\mathrm{BUY}|\theta) = \theta$ since θ is the proportion of buying customers

$\Rightarrow$

$$P(\mathrm{BUY}) = 0.10 \cdot 0.2 + 0.20 \cdot 0.3 + 0.30 \cdot 0.3 + 0.40 \cdot 0.1 + 0.50 \cdot 0.1 = 0.26$$

$$P(0.10|\mathrm{BUY}) = 0.10 \cdot 0.2/0.26 \approx 0.0769$$

$$P(0.20|\mathrm{BUY}) = 0.20 \cdot 0.3/0.26 \approx 0.2308$$

$$P(0.30|\mathrm{BUY}) = 0.30 \cdot 0.3/0.26 \approx 0.3462$$

$$P(0.40|\mathrm{BUY}) = 0.40 \cdot 0.1/0.26 \approx 0.1538$$

$$P(0.50|\mathrm{BUY}) = 0.50 \cdot 0.1/0.26 \approx 0.1923$$

$$\mathrm{VSI(BUY)} = E''(a''|\mathrm{BUY}) - E''(a'|\mathrm{BUY})$$

$$a' = \langle = a^* \text{ from Exercise 6.15} \rangle = \text{Stock 50}$$

$$a'' = \arg\max_a \{ER(a|\mathrm{BUY})\}$$

$$E(a|\mathrm{BUY}) = \sum_a R(a,\theta) \cdot P(\theta|\mathrm{BUY})$$

$$\Rightarrow$$

$$ER(\text{Stock 100}|\mathrm{BUY}) = (-10) \cdot 0.0769\ldots + (-2) \cdot 0.2308\ldots + 12 \cdot 0.3462\ldots + 22 \cdot 0.1538\ldots + 40 \cdot 0.1923\ldots = 14.00$$

$$ER(\text{Stock 50}|\mathrm{BUY}) = (-4) \cdot 0.0769\ldots + 6 \cdot 0.2308\ldots + 12 \cdot 0.3462\ldots + 26 \cdot 0.1538\ldots + 16 \cdot 0.1923\ldots \approx 10.77$$

$$ER(\text{Do not stock}|\mathrm{BUY}) = 0 \cdot 0.0769\ldots + 0 \cdot 0.2308\ldots + 0 \cdot 0.3462\ldots + 0 \cdot 0.1538\ldots + 0 \cdot 0.1923\ldots = 0$$

$$\Rightarrow \quad a'' = \text{Stock 100}$$

$$\Rightarrow \mathrm{VSI(BUY)} = E''R(\text{Stock 100}|\mathrm{BUY}) - E''R(\text{Stock 50}|\mathrm{BUY}) =$$

$$= 14.00 - 10.77 = 3.23$$

(b) Posterior distribution:

(b) find the posterior distribution if the one person sampled will *not* purchase the item, and find the value of this sample information;

$$P(\theta|\text{NOT BUY}) = \frac{P(\text{NOT BUY}|\theta)\cdot P(\theta)}{P(\text{NOT BUY})} = \frac{P(\text{NOT BUY}|\theta)\cdot P(\theta)}{\sum_\lambda P(\text{NOT BUY}|\lambda)\cdot P(\lambda)}$$

$P(\text{NOT BUY}|\theta) = 1-\theta$

$\Rightarrow$

$P(\text{NOT BUY}) = 0.90\cdot 0.2 + 0.80\cdot 0.3 + 0.70\cdot 0.3 + 0.60\cdot 0.1 + 0.50\cdot 0.1 = 0.74$

$P(0.10|\text{NOT BUY}) = 0.90\cdot 0.2/0.74 \approx 0.2432$

$P(0.20|\text{NOT BUY}) = 0.80\cdot 0.3/0.74 \approx 0.3243$

$P(0.30|\text{NOT BUY}) = 0.70\cdot 0.3/0.74 \approx 0.2838$

$P(0.40|\text{NOT BUY}) = 0.60\cdot 0.1/0.74 \approx 0.0811$

$P(0.50|\text{NOT BUY}) = 0.50\cdot 0.1/0.74 \approx 0.0676$

$$\text{VSI(NOT BUY)} = E''(a''|\text{NOT BUY}) - E''(a'|\text{NOT BUY})$$

$$a' = \text{Stock 50 (Same as in (a))}$$

$$a'' = \arg\max_{a}\{ER(a|\text{NOT BUY})\}$$

$$E(a|\text{NOT BUY}) = \sum_{a} R(a, \theta) \cdot P(\theta|\text{NOT BUY})$$

$\Rightarrow$

$$ER(\text{Stock 100}|\text{NOT BUY}) = (-10) \cdot 0.2432 \ldots + (-2) \cdot 0.3243 \ldots + 12 \cdot 0.2838 \ldots + 22 \cdot 0.0811 \ldots + 40 \cdot 0.0676 \ldots \approx 4.81$$

$$ER(\text{Stock 50}|\text{NOT BUY}) = (-4) \cdot 0.2432 \ldots + 6 \cdot 0.3243 \ldots + 12 \cdot 0.2838 \ldots + 26 \cdot 0.0811 \ldots + 16 \cdot 0.0676 \ldots \approx 6.76$$

$$ER(\text{Do not stock}|\text{NOT BUY}) = 0 \cdot 0.2432 \ldots + 0 \cdot 0.3243 \ldots + 0 \cdot 0.2838 \ldots + 0 \cdot 0.0811 \ldots + 0 \cdot 0.0676 \ldots = 0$$

$\Rightarrow$ a''=Stock 50

$$\Rightarrow \text{VSI(NOT BUY)} = E''R(\text{Stock 50}|\text{NOT BUY}) - E''R(\text{Stock 50}|\text{NOT BUY}) = 0$$

(c) (c) find the expected value of sample information.

$$\text{EVSI} = \sum_{y} \text{VSI}(y) \cdot P(y) = 3.2 \cdot 0.26 + 0 \cdot 0.74 = 0.832$$

Alternatively:

$$\text{EVSI} = E''R(a'') - E''R(a')$$

$$E''R(a'') = E''R(a''|\text{BUY}) \cdot P(\text{BUY}) + \text{E}''\text{R}(a''|\text{NOT BUY}) \cdot P(\text{NOT BUY}) =$$

$$= E''R(\text{Stock 100}|\text{BUY}) \cdot P(\text{BUY}) + \text{E}''\text{R}(\text{Stock 50}|\text{NOT BUY}) \cdot P(\text{NOT BUY}) =$$

$$= 14.00 \cdot 0.26 + 6.76 \cdot 0.74 = 8.64$$

$$E''R(a') = E''R(a'|\text{BUY}) \cdot P(\text{BUY}) + E''R(a''|\text{NOT BUY}) \cdot P(\text{NOT BUY}) =$$

$$= E''R(\text{Stock 50}|\text{BUY}) \cdot P(\text{BUY}) + \text{E}''\text{R}(\text{Stock 50}|\text{NOT BUY}) \cdot P(\text{NOT BUY}) =$$

$$= 10.77 \cdot 0.26 + 6.76 \cdot 0.74 = 7.80$$

$\Rightarrow$ EVSI = 8.64 – 7.80 = 0.84

Net gain of sampling

Since taking samples (obtaining a sample result) comes with a cost, the value of sample information is of interest when it is compared against the cost of sampling.

Note! This comparison requires that the utility and cost can be expressed in the same unit

Net gain of sampling given sample result y:

$$\mathrm{NGS}(y) = \mathrm{VSI}(y) - \mathrm{CS}$$

where CS is the cost of sampling (not dependent on the sample result)

Expected net gain of sampling, ENGS:

$$\mathrm{ENGS} = \mathrm{EVSI} - \mathrm{CS}$$

…as function of the sample size n :

$$\mathrm{ENGS}(n) = \mathrm{EVSI}(n) - \mathrm{CS}(n)$$

Exercise 6.20

20. In the automobile-salesman example discussed in Section 3.4, suppose that the owner of the dealership must decide whether or not to hire a new salesman. The payoff table (in terms of dollars) is as follows.

		STATE OF THE WORLD		
		Great salesman	*Good salesman*	*Poor salesman*
ACTION	*Hire*	60,000	15,000	-30,000
	Do not hire	0	0	0

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda} = 1/2$ per day for a great salesman, $\tilde{\lambda} = 1/4$ per day for a good salesman, and $\tilde{\lambda} = 1/8$ per day for a poor salesman.

(a) Find VPI(great salesman), VPI(good salesman), and VPI(poor salesman).

(b) Find the expected value of perfect information.

(c) Suppose that the owner of the dealership can purchase sample information at the rate of \$10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

	GREAT	GOOD	POOR
Hire	60,000	15,000	–30,000
Do not hire	0	0	0

(a) Since we are given a payoff table we compute VPI using the result

$$VPI(\theta) = R(a_\theta, \theta) - R(a^*, \theta)$$

where a_θ is the optimal action when θ is the state of the world and a^* is the optimal action with respect to maximised expected payoff.

$\tilde{\theta}$ can here assume the states "GREAT", "GOOD" and "POOR". The (prior) expected payoffs are:

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process ... 1/4 per day for a good salesman, and $\tilde{\lambda}$ = 1/8 per

$$ER(a = \text{"Hire"}) = 60000 \cdot P(\tilde{\theta} = \text{GREAT}) + 15000 \cdot P(\tilde{\theta} = \text{GOOD})$$

$$-30000 \cdot P(\tilde{\theta} = \text{POOR}) =$$

$$= 60000 \cdot 0.10 + 15000 \cdot 0.50 - 30000 \cdot 0.40 = 1500$$

$$ER(a = \text{"Do not hire"}) = 0 \cdot 0.10 + 0 \cdot 0.50 + 0 \cdot 0.40 = 0$$

and hence a_* is "Hire".

If $\tilde{\theta}$ = GREAT or $\tilde{\theta}$ = GOOD action "Hire" is optimal, if $\tilde{\theta}$ = POOR the action "Do not hire" is optimal.

$$\text{VPI(GREAT)} = \text{VPI(GOOD)} = 0$$

$$\text{VPI(POOR)} = 0 - (-30000) = 30000$$

	GREAT	GOOD	POOR
Hire	60,000	15,000	–30,000
Do not hire	0	0	0

(b) The expected value of perfect information is

$$\text{EVPI} = \text{VPI(GREAT)} \cdot P\left(\tilde{\theta} = \text{GREAT}\right) + \text{VPI(GOOD)} \cdot P\left(\tilde{\theta} = \text{GOOD}\right)$$
$$+\text{VPI(POOR)} \cdot P\left(\tilde{\theta} = \text{POOR}\right) = 0 \cdot 0.10 + 0 \cdot 0.50 + 30000 \cdot 0.40 = 12000$$

(c) Let $\tilde{y}$ = Number of automobiles sold during one week. Then

The prior probabilities for the three states of the world are ... P(poor) = 0.40. The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda}$ = 1/2 per day for a great salesman, $\tilde{\lambda}$ = 1/4 per day for a good salesman, and $\tilde{\lambda}$ = 1/8 per day for a poor salesman.

... VPI(good salesman), and VPI(poor salesman).

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GREAT}\right) = \frac{\left(4 \cdot (1/2)\right)^y}{y!} \cdot e^{-4\cdot(1/2)} = \frac{2^y}{y!} \cdot e^{-2}$$

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) = \frac{\left(4 \cdot (1/4)\right)^y}{y!} \cdot e^{-4\cdot(1/4)} = \frac{1^y}{y!} \cdot e^{-1}$$

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) = \frac{\left(4 \cdot (1/8)\right)^y}{y!} \cdot e^{-4\cdot(1/8)} = \frac{0.5^y}{y!} \cdot e^{-0.5}$$

Now,

$$\text{EVSI} = \sum_y \text{VSI}(y) \cdot P(y)$$

$$\text{ENGS} = \text{EVSI} - \text{CS}$$

where $\text{VSI}(y) = E''R(a''|y) - E''R(a'|y)$

In subtask (a) we obtained a_* =“Hire” and a' is thus equal to “Hire”.

The expected posterior payoffs are

$$\begin{aligned} E''\big(R(\text{"Hire"}|y)\big) &= R\big(\text{"Hire"}, \tilde{\theta} = \text{GREAT}\big) \cdot P\big(\tilde{\theta} = \text{GREAT}\big|\tilde{y} = y\big) \\ &+R\big(\text{"Hire"}, \tilde{\theta} = \text{GOOD}\big) \cdot P\big(\tilde{\theta} = \text{GOOD}\big|\tilde{y} = y\big) \\ &+R\big(\text{"Hire"}, \tilde{\theta} = \text{POOR}\big) \cdot P\big(\tilde{\theta} = \text{POOR}\big|\tilde{y} = y\big) \\ &= 60000 \cdot P\big(\tilde{\theta} = \text{GREAT}\big|\tilde{y} = y\big) + 15000 \cdot P\big(\tilde{\theta} = \text{GOOD}\big|\tilde{y} = y\big) \\ &-30000 \cdot P\big(\tilde{\theta} = \text{POOR}\big|\tilde{y} = y\big) \end{aligned}$$

$$\begin{aligned} E''\big(R(\text{"Do not hire"}|y)\big) &= \cdots = 0 \cdot P\big(\tilde{\theta} = \text{GREAT}\big|\tilde{y} = y\big) \\ &+ 0 \cdot P\big(\tilde{\theta} = \text{GOOD}\big|\tilde{y} = y\big) - 0 \cdot P\big(\tilde{\theta} = \text{POOR}\big|\tilde{y} = y\big) = 0 \end{aligned}$$

Now,

$$P\left(\tilde{\theta} = \text{GREAT} | \tilde{y} = y\right) = \frac{P\left(Y = y \middle| \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right)}{P(\tilde{y} = y)}$$

$$P\left(\tilde{\theta} = \text{GOOD} | \tilde{y} = y\right) = \frac{P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)}{P(\tilde{y} = y)}$$

$$P\left(\tilde{\theta} = \text{POOR} | \tilde{y} = y\right) = \frac{P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)}{P(\tilde{y} = y)}$$

where $P(\tilde{y} = y) = P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right) + P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)$
$+ P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)$

This gives

$$E"\left(R(\text{"Hire"}|y)\right) =$$
$$= 60000 \cdot \frac{(2^y/y!)e^{-2} \cdot 0.10}{P(\tilde{y} = y)} + 15000 \cdot \frac{(1^y/y!)e^{-1} \cdot 0.50}{P(\tilde{y} = y)} - 30000 \cdot \frac{(0.5^y/y!)e^{-0.5} \cdot 0.40}{P(\tilde{y} = y)}$$
$$= \frac{6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5}}{P(\tilde{y} = y)}$$

We now must investigate for which values of y the optimal action a" is equal to "Hire" and for which y this action is equal to "Do not hire".

The optimal action is "Hire" when $E''(R(\text{"Hire"}|y)) > E''(R(\text{"Do not hire"}|y)) = 0$

$$\Leftrightarrow$$

$$\frac{6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5}}{P(\tilde{y} = y)} > 0$$

$$\Leftrightarrow \langle \text{since } P(\tilde{y} = y) > 0 \rangle$$

$$6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5} > 0$$

$$\Leftrightarrow \langle \text{since } y! > 0 \rangle$$

$$6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot 0.5^y \cdot e^{-0.5} > 0$$

$$\Leftrightarrow 6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot \frac{1}{2^y} \cdot e^{-0.5} > 0$$

$$\Leftrightarrow \langle \text{since } 2^y > 0 \rangle$$

$$6000 \cdot e^{-2} \cdot 2^{2y} + 7500 \cdot e^{-1} \cdot 2^y - 12000 \cdot e^{-0.5} > 0$$

$$\Leftrightarrow 2^{2y} + \frac{6000}{7500} e^1 \cdot 2^y - \frac{12000}{7500} e^{1.5} > 0$$

$$\Rightarrow \langle \text{since } 2^y > 0 \rangle$$

$$2^y > -\frac{6000}{2 \cdot 7500} e^1 + \sqrt{\left(-\frac{6000}{2 \cdot 7500} e^1\right)^2 + \frac{12000}{7500} e^{1.5}}$$

$$\Leftrightarrow 2^y > 1.802835 \Leftrightarrow y > \frac{\log(1.802835)}{\log(2)} \approx 0.85$$

Hence,

$$E''\left(R(a''|y)\right) = \begin{cases} E''\left(R(\text{"Hire"}|y)\right) & y \geq 1 \\ E''\left(R(\text{"Do not hire"}|y)\right) & y = 0 \end{cases}$$

and

$$\text{VSI}(y) = E''\left(R(a''|y)\right) - E''\left(R(a'|y)\right)$$
$$= \begin{cases} E''\left(R(\text{"Hire"}|y)\right) - E''\left(R(\text{"Hire"}|y)\right) = 0 & y \geq 1 \\ E''\left(R(\text{"Do not hire"}|y=0)\right) - E''\left(R(\text{"Hire"}|y=0)\right) & y = 0 \end{cases}$$

$$E''\left(R(\text{"Do not hire"}|y=0)\right) = 0$$

$$E''\left(R(\text{"Hire"}|y=0)\right) = \frac{6000 \cdot (2^0/0!)e^{-2} + 7500 \cdot (1^0/0!)e^{-1} - 12000 \cdot (0.5^0/0!)e^{-0.5}}{P(\tilde{y}=0)}$$
$$= \frac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{P(\tilde{y}=0)}$$

and thus

$$\text{VSI}(y) = \begin{cases} 0 & y \geq 1 \\ -\dfrac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{P(Y=0)} & y = 0 \end{cases}$$

…and finally,

$$EVSI = \sum_{y=0}^{\infty} \mathrm{VSI}(y) \cdot P(\tilde{y} = y) =$$

$$= \mathrm{VSI}(0) \cdot P(\tilde{y} = 0) + \cancel{\sum_{y=1}^{\infty} 0 \cdot P(\tilde{y} = y)} =$$

$$= \mathrm{VSI}(0) \cdot P(\tilde{y} = 0) =$$

$$= -\frac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{\cancel{P(\tilde{y} = 0)}} \times \cancel{P(\tilde{y} = 0)}$$

$$\approx 3707$$

(c) Suppose that the owner of the dealership can purchase sample information at the rate of $10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

$$ENGS = EVSI - CS = 3707 - 4 \cdot 10 = 3667$$