

## Exam in Probability Theory, 6 credits

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Exam time: 8-12

Allowed: Pocket calculator.  
Table with common formulas and moment generating functions (distributed with the exam).  
Table of integrals (distributed with the exam).  
Table with distributions from Appendix B in the course book (distributed with the exam).

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Grades: Grades: Maximum is 20 points.  
A=19-20 points  
B=17-18 points  
C=12-16 points  
D=10-11 points  
E=8-9 points  
F=0-7 points

- Write clear and concise answers to the questions.
  - Remember that you may get points from explaining how you would solve the problem, even if you don't fully solve it.
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1. The random variables  $X$  and  $Y$  have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} \exp[-(x + \frac{y}{x})] & \text{if } 0 \leq x \leq \infty \text{ and } 0 \leq y \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

- (a) Compute the marginal density function for  $X$ . Does it belong to any of the known distributions?  
1.5p.

**Solution:** The marginal density of  $X$  is

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y) dy = \exp(-x) \int_0^\infty \exp\left(-\frac{y}{x}\right) dy$$

where

$$\int_0^\infty \exp\left(-\frac{y}{x}\right) dy = \left[ \exp\left(-\frac{y}{x}\right) (-x) \right]_0^\infty = [0 - 1 \cdot (-x)] = x.$$

So

$$f_X(x) = x \cdot \exp(-x)$$

for  $0 \leq x \leq \infty$ . This can be recognized as a  $X \sim \Gamma(2, 1)$  density.

- (b) Compute the conditional density of  $Y|X = x$ . Does it belong to any of the known distributions?  
1.5p.

**Solution:** The conditional density of  $Y|X = x$  is

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\exp\left[-\left(x + \frac{y}{x}\right)\right]}{x \cdot \exp(-x)} = \frac{1}{x} \exp\left(-\frac{y}{x}\right),$$

which means that  $Y|X = x \sim Exp(x)$ .

(c) Compute  $Cov(X, Y)$  2p.

**Solution:** We know that

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$$

where  $E(X) = 2 \cdot 1 = 2$  (since  $X \sim \Gamma(2, 1)$ ). To compute  $E(Y)$  we use the law of iterated expectation

$$E(Y) = E[E(Y|X)] = E(X) = 2$$

where we have used that  $Y|X = x \sim Exp(x)$ . Now

$$E(XY) = E[E(XY|X)] = E[X \cdot E(Y|X)] = E[X \cdot X] = E(X^2) = V(X) + [E(X)]^2 = 2 \cdot 1^2 + 2^2 = 6.$$

Hence

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y) = 6 - 2 \cdot 2 = 2.$$

2. Let  $Y|(\Lambda = \lambda) \sim Po(\lambda)$  and  $\Lambda \sim Exp(a)$ .

(a) Calculate the expected value and variance of  $Y$ . 2p.

**Solution:** By the law of iterated expectation we have

$$E[E(Y|\Lambda)] = E(\Lambda) = a$$

and by the law of total variance

$$V(Y) = E[V(Y|\Lambda)] + V[E(Y|\Lambda)] = E(\Lambda) + V(\Lambda) = a + a^2 = a(1 + a).$$

(b) Calculate the moment generating function  $Y$ . 2p.

**Solution:** The moment generating function of  $Y$  is most easily computed by the law of iterated expectation as follows

$$\psi_Y(t) = E(e^{tY}) = E[E(e^{tY}|\Lambda = \lambda)] = E[e^{\lambda(e^t - 1)}] = \psi_\Lambda(e^t - 1)$$

where

$$\psi_\Lambda(t) = \frac{1}{1 - at}$$

is the moment generating function for the  $Exp(a)$  distribution, which exists if  $t < 1/a$ . Hence

$$\psi_Y(t) = \psi_\Lambda(e^t - 1) = \frac{1}{1 - a(e^t - 1)},$$

which exists if  $e^t - 1 < 1/a$ , that is if  $e^t < 1 + \frac{1}{a}$  or  $t < \ln(1 + \frac{1}{a})$ .

(c) Calculate the moment generating function of  $Z = nY$ , where  $n$  is a positive integer. 1p.

**Solution:** The moment generating function of  $Z = nY$  is by definition

$$E(e^{tZ}) = E(e^{(tn)Y}) = \psi_Y(tn) = \frac{1}{1 - a(e^{tn} - 1)}.$$

3. Let  $X_k \sim \text{Exp}(a)$ ,  $k = 1, 2, \dots$  be independent random variables.

- (a) Derive the density of  $Z_n = \min(X_1, X_2, \dots, X_n)$ . [for full points it is not enough to just write down the formula, you have to derive it.] 2p.

**Solution:** The distribution function of  $Z_n$  is

$$F_{Z_n}(z) = P(Z_n \leq z) = 1 - P(Z_n > z) = 1 - P(X_1 > z, X_2 > z, \dots, X_n > z)$$

since  $Z_n$  is the minimum and  $Z_n > z$  is therefore the same as all  $X_1, \dots, X_n$  being larger than  $z$ . Now, because of independence and that all  $X_k$  have same distribution, we have

$$F_{Z_n}(z) = 1 - (1 - F_X(z))^n = 1 - \left(1 - \left[1 - \exp\left(-\frac{z}{a}\right)\right]\right)^n = 1 - \exp\left(-\frac{nz}{a}\right)$$

and the density is therefore

$$f_Z(z) = \frac{1}{(a/n)} \exp \exp\left(-\frac{z}{a/n}\right)$$

which shows that  $Z \sim \text{Exp}\left(\frac{a}{n}\right)$ .

- (b) Derive the distribution of  $Y_n = \sum_{k=1}^n X_k$  for  $n = 1, 2, \dots$  1p.

**Solution:** Since the  $X_k$  are independent and identically distributed we know that

$$\psi_{Y_n}(t) = \left(\frac{1}{1-at}\right)^n = \frac{1}{(1-at)^n} \sim \Gamma(n, a)$$

for  $t < 1$ .

- (c) Let again  $Y_n = \sum_{k=1}^n X_k$  for  $n = 1, 2, \dots$ . Find  $a$  such that  $Y_n \xrightarrow{P} 1$  as  $n \rightarrow \infty$ . 2p.

**Solution:** The expected value of  $Y_n \sim \Gamma(n, a)$  is  $na$  and the variance is  $na^2$ . For  $a = 1/n$  we therefore have  $E(Y_n) = 1$  and  $V(Y_n) = 1/n$ . We need to prove that for any  $\varepsilon > 0$

$$P(|Y_n - 1| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We can use Chebyshev's inequality

$$P(|Y_n - 1| > \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} = \frac{1/n}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \rightarrow 0$$

as  $n \rightarrow \infty$ .

4. Let  $X_1, X_2$  and  $X_3$  follow a multivariate normal distribution with mean vector  $\mu = (0, 1, 2)'$  and covariance matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

- (a) What is the bivariate distribution of  $X_1$  and  $X_3$ ? 1p.

**Solution:** Since the distribution is normal, the distribution is obtained by just picking out the relevant elements of  $\mu$  and  $\Lambda$ :

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\right)$$

- (b) Compute the correlation between  $X_1$  and  $X_3$ . 1p.

**Solution:** From  $\Lambda$  we see that  $\text{Cov}(X_1, X_3) = 1$ . We know that  $\rho_{X_1, X_3} = \frac{\text{Cov}(X_1, X_3)}{\sigma_{X_1} \sigma_{X_3}} = \frac{1}{\sqrt{1} \sqrt{4}} = \frac{1}{2}$ .

- (c) Compute  $\text{Var}(c_1 \cdot X_1 + c_2 \cdot X_2)$  where  $c_1$  and  $c_2$  are constants. 1p.

**Solution:** From  $\Lambda$  we see that  $Cov(X_1, X_2) = 0$  so  $\rho_{X_1, X_2} = 0$  and, because the variables are normal, this means that  $X_1$  and  $X_2$  are independent. We therefore have

$$Var(c_1 \cdot X_1 + c_2 \cdot X_2) = c_1^2 Var(X_1) + c_2^2 Var(X_2) = c_1^2 + 2c_2^2.$$

(d) What is the conditional distribution of  $X_1 + X_2 + X_3$  given  $X_1 - X_3 = c$  for some constant  $c$ ?  
2p.

**Solution:** Let us first derive joint distribution of  $Y_1 = X_1 + X_2 + X_3$  and  $Y_2 = X_1 - X_2$ . Since these are linear transformations and  $X_1, X_2$  and  $X_3$  are jointly normal, we have

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = BX$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

We then know that

$$Y \sim N(B\mu, BAB')$$

that is

$$Y \sim N\left(\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 11 & -4 \\ -4 & 3 \end{pmatrix}\right)$$

Now,

$$Y_1 | (Y_2 = c) \sim N\left[\mu_1 + \rho_{12} \frac{\sigma_1}{\sigma_2} (c - \mu_2), \sigma_1^2 (1 - \rho_{12}^2)\right]$$

where the indicies are with respect to the random vector  $Y$ . Filling in the right values, we obtain

$$Y_1 | (Y_2 = c) \sim N\left[3 - \frac{4}{3}(c + 2), 11\left(1 - \frac{16}{11 \cdot 3}\right)\right] = N\left[\frac{1 - 4c}{3}, \frac{17}{3}\right]$$

since  $\rho_{12} = \frac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2} = \frac{-4}{\sqrt{11}\sqrt{3}}$ , so  $\rho_{12}^2 = \frac{16}{11 \cdot 3}$  and  $\rho_{12} \frac{\sigma_1}{\sigma_2} = \frac{-4}{\sqrt{11}\sqrt{3}} \frac{\sqrt{11}}{\sqrt{3}} = -\frac{4}{3}$ .

Good luck!

Mattias