# SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

PER SIDÉN

## 4. ORDER STATISTICS

**Exercise 4.1 (4.1 in Gut's book)**

Suppose that $X,Y$ and $Z$ have a joint density function given by

$$f(x,y,z) = \begin{cases} e^{-(x+y+z)} & , x,y,z > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Compute $P(X < Y < Z)$ and $P(X = Y < Z)$.

**Exercise 4.2 (4.5 in Gut's book)**

Let $X_1, X_2, \ldots, X_n$ be independent, continuous random variables with common distribution function $F(x)$, and consider the order statistic $\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$. Compute $E\left(F\left(X_{(n)}\right) - F\left(X_{(1)}\right)\right)$.

**Exercise 4.3 (4.6 in Gut's book)**

Let $X_1, X_2, X_3$ and $X_4$ be independent, $U(0,1)$-distributed random variables. Compute

(a) $\qquad P\left(X_{(3)} + X_{(4)} \le 1\right)$,

(b) $\qquad P(X_3 + X_4 \le 1)$.

**Exercise 4.4 (4.24 in Gut's book)**

Let $X_1, X_2, \ldots, X_n$ be independent, $Exp(a)$-distributed random variables. Determine the distribution of $\sum_{k=1}^{n} X_{(k)}$.

**Exercise 4.5\* (4.16 in Gut's book)**

Let $X_1$ and $X_2$ be independent, $Exp(a)$-distributed random variables.

(a) $\qquad$ Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent, and determine their distributions.

(b) $\qquad$ Compute $E\left(X_{(2)}|X_{(1)} = y\right)$ and $E\left(X_{(1)}|X_{(2)} = x\right)$.

**Exercise 4.6\* (4.18 in Gut's book)**

Suppose that $X \sim U(0,1)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order variables corresponding to a sample of $n$ independent observations of $X$, and set

$$V_i = \frac{X_{(i)}}{X_{(i+1)}}, \ i = 1, 2, \cdots, n-1, \ \text{and} \ V_n = X_{(n)}.$$

Show that

(a)          $V_1, V_2, \ldots, V_n$ are independent,

(b)          $V_i^i \sim U(0,1)$ for $i = 1, 2, \ldots, n$.