Ex. 4.4 (4.24 in Gut's book)

Let $X_1, \dots, X_n$ be independent. $Exp(a)$-distributed random variables. Determine the distribution of $\sum_{k=1}^{n} X_{(k)}$.

Solution:

Observe that $\sum_{k=1}^{n} X_{(k)} = \sum_{i=1}^{n} X_i$ so

moment generating function

$$\psi_{X_{(1)}+\dots+X_{(n)}}(t) = \psi_{X_1+\dots+X_n}(t) \overset{\text{because of independence}}{=} \prod_{i=1}^{n} \psi_{X_i}(t) = (\psi_{X}(t))^n =$$

$$= \left(\frac{1}{1-at}\right)^n = \frac{1}{(1-at)^n}$$

We know that $Y \sim \Gamma(p, a)$

$$\psi_Y(t) = \frac{1}{(1-at)^p}$$

so

$X_{(1)} + \dots + X_{(n)} \sim \Gamma(n, a)$ distributed.