THEOREM

Let $S_n = X_1 + X_2 + \dots + X_n$, X_i iid random variables and N be non-negative integer valued random variable then

$$g_{S_N}(t) = g_N(g_X(t))$$

$$\psi_{S_N}(t) = g_N(\psi_X(t))$$

$$\varphi_{S_N}(t) = g_N(\varphi_X(t)).$$

Example

Let $X_1, X_2, \dots \sim Exp(1)$ iid and $N \sim Fs(p)$. What is a distr. of S_N?

Solution

$\psi_{S_N}(t) = g_N(\psi_X(t))$ by theorem above

$\psi_X(t) = \frac{1}{1-t}$ for $t<1$ as $X_i \sim Exp(1)$

$N \sim Fs(p)$ so $P(N=k) = (1-p)^{k-1} p$, $k=1,2,\dots$

and

$$g_N(t) = Et^X = \sum_{k=1}^{\infty} t^k P(X=k) = \sum_{k=1}^{\infty} t^k q^{k-1} p =$$

$$= tp \sum_{k=1}^{\infty} (tq)^{k-1} = tp \sum_{k=0}^{\infty} (tq)^k = tp \frac{1}{1-tq}$$

Finally

$$\psi_{S_N}(t) = g_N\left(\frac{1}{1-t}\right) = \frac{1}{1-t} \cdot p \frac{1}{1 - \frac{1}{1-t}(1-p)} =$$

$$= \frac{p}{1-t-(1-p)} = \frac{p}{p-t} = \frac{1}{1-\frac{t}{p}}$$

so $S_N \sim Exp\left(\frac{1}{p}\right)$