Ex. (DISTRIBUTIONS WITH RANDOM PARAMETERS)

Let $X \mid Y=y \sim N(0,y)$ and $Y \sim Exp(1)$. Show that $X \sim L\left(\frac{1}{\sqrt{2}}\right)$ (using characteristic functions).

Solution

We know that $Y \sim Exp(1)$ so $\varphi_Y(t) = \frac{1}{1-ait} = \frac{1}{1-it}$

Moreover $\varphi_X(t) = Ee^{itX} = E\left[E\left(e^{itX} \mid Y\right)\right]$ by law of iter. expect.

and $\varphi_{X|N}(t) = e^{i\mu t - \frac{1}{2}t^2\sigma^2} = e^{-\frac{1}{2}t^2 y}$ as $X \mid Y=y \sim N(0,y)$.

$$\varphi_X(t) = E\left[\varphi_{X|N}(t)\right] = E\left[e^{-\frac{1}{2}t^2 Y}\right] = \varphi_Y\left(-\tfrac{1}{2}t^2\right)$$

Moment generating function of $Exp(1)$ is $\frac{1}{1-at} = \frac{1}{1-t}$ so

$$\varphi_X(t) = \frac{1}{1-\left(-\frac{1}{2}t^2\right)} = \frac{1}{1+\frac{1}{2}t^2} \Longrightarrow X \sim Laplace\left(\tfrac{1}{\sqrt{2}}\right)$$