Ex. 3.3 p.42

Show that if X has a normal distribution such that the mean is zero and the inverse of the variance is Γ-distributed

$$X \mid \Sigma^2 = \lambda \sim N\left(0, \frac{1}{\lambda}\right) \quad \text{with} \quad \Sigma^2 \sim \Gamma\left(\frac{n}{2}, \frac{2}{n}\right)$$

then $X \sim t(n)$.

Solution:

We use
$$f_X(x) = \int_{\mathbb{R}} \boxed{f_{X|\Theta=\theta}(x)}\, \boxed{f_\Theta(\theta)}\, d\theta$$

hence

$$f_X(x) = \int_0^\infty \boxed{\frac{1}{\sqrt{2\pi \frac{1}{\lambda}}}\, e^{-\frac{x^2}{2\cdot\frac{1}{\lambda}}}} \cdot \boxed{\frac{1}{\Gamma(\frac{n}{2})}\, \lambda^{\frac{n}{2}-1} \frac{1}{(\frac{2}{n})^{n/2}}\, e^{-\frac{\lambda}{2/n}}}\, d\lambda$$

$$= \frac{1}{\sqrt{\pi}\,\Gamma(\frac{n}{2})} \int_0^\infty \frac{\sqrt{\lambda}}{\sqrt{2}} \left(\frac{\sqrt{n}}{\sqrt{2}}\right)^n \lambda^{\frac{n}{2}-1}\, e^{-\frac{\lambda x^2}{2} - \frac{n\lambda}{2}}\, d\lambda$$

$$= \frac{1}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \left(\frac{\sqrt{n}}{\sqrt{2}}\right)^{n+1} \int_0^\infty \lambda^{\frac{n-1}{2}}\, e^{-\frac{\lambda}{2}(x^2+n)}\, d\lambda = *$$

Observe that
$$\int_0^\infty t^{k-1} e^{-\frac{t}{\tau}}\, dt = \tau^k\, \Gamma(k)$$

so for $k = \frac{n+1}{2}$ and $\tau = \frac{2}{x^2+n}$ we get

$$\int_0^\infty \lambda^{\frac{n-1}{2}}\, e^{-\lambda\cdot\left(\frac{x^2+n}{2}\right)}\, d\lambda = \left(\frac{2}{x^2+n}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)$$

Finaly

$$* = \frac{1}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \frac{n^{\frac{n+1}{2}}}{\cancel{2^{\frac{n+1}{2}}}} \frac{\cancel{2^{\frac{n+1}{2}}}}{(x^2+n)^{\frac{n+1}{2}}}\, \Gamma\left(\frac{n+1}{2}\right)$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \frac{1}{\left(\frac{x^2}{n}+1\right)^{\frac{n+1}{2}}}$$

that is density function for t-Student distribution $t(n)$

$$X \sim t(n).$$