SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

JOLANTA PIELASZKIEWICZ
(EXERCISES BY PER SIDÉN)
28 SEPTEMBER 2020

4. Order Statistics

Exercise 4.1 (4.1 in Gut's book)
Suppose that X,Y and Z have a joint density function given by

$$f(x,y,z) = \begin{cases} e^{-(x+y+z)} & , x,y,z > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Compute $P(X < Y < Z)$ and $P(X = Y < Z)$.

Exercise 4.2 (4.5 in Gut's book)
Let $X_1, X_2, \ldots, X_n$ be independent, continuous random variables with common distribution function $F(x)$, and consider the order statistic $\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$. Compute $E\left(F\left(X_{(n)}\right) - F\left(X_{(1)}\right)\right)$.

Exercise 4.3 (4.6 in Gut's book)
Let X_1, X_2, X_3 and X_4 be independent, $U(0,1)$-distributed random variables. Compute

(a) $P\left(X_{(3)} + X_{(4)} \leq 1\right)$,
(b) $P(X_3 + X_4 \leq 1)$.

Exercise 4.4 (4.24 in Gut's book)
Let $X_1, X_2, \ldots, X_n$ be independent, $Exp(a)$-distributed random variables. Determine the distribution of $\sum_{k=1}^{n} X_{(k)}$.

Exercise 4.5* (4.16 in Gut's book)
Let X_1 and X_2 be independent, $Exp(a)$-distributed random variables.

(a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent, and determine their distributions.
(b) Compute $E\left(X_{(2)} | X_{(1)} = y\right)$ and $E\left(X_{(1)} | X_{(2)} = x\right)$.

Exercise 4.6* (4.18 in Gut's book)

Suppose that $X \sim U(0,1)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order variables corresponding to a sample of n independent observations of X, and set

$$V_i = \frac{X_{(i)}}{X_{(i+1)}},\ i = 1, 2, \cdots, n-1,\ \text{and } V_n = X_{(n)}.$$

Show that

(a) $V_1, V_2, \ldots, V_n$ are independent,

(b) $V_i^i \sim U(0,1)$ for $i = 1, 2, \ldots, n$.