Softmax regression

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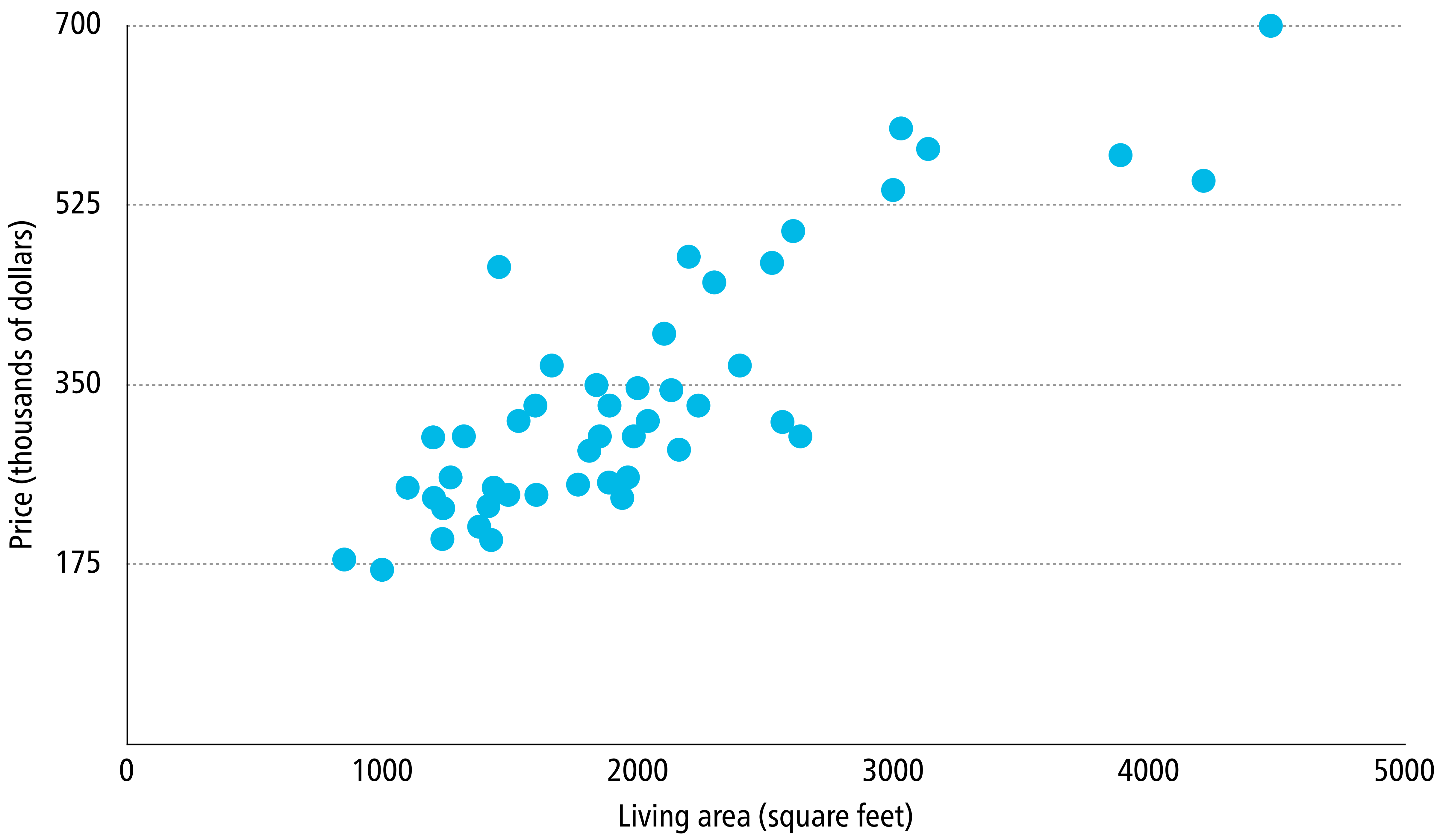
Linear regression with one variable



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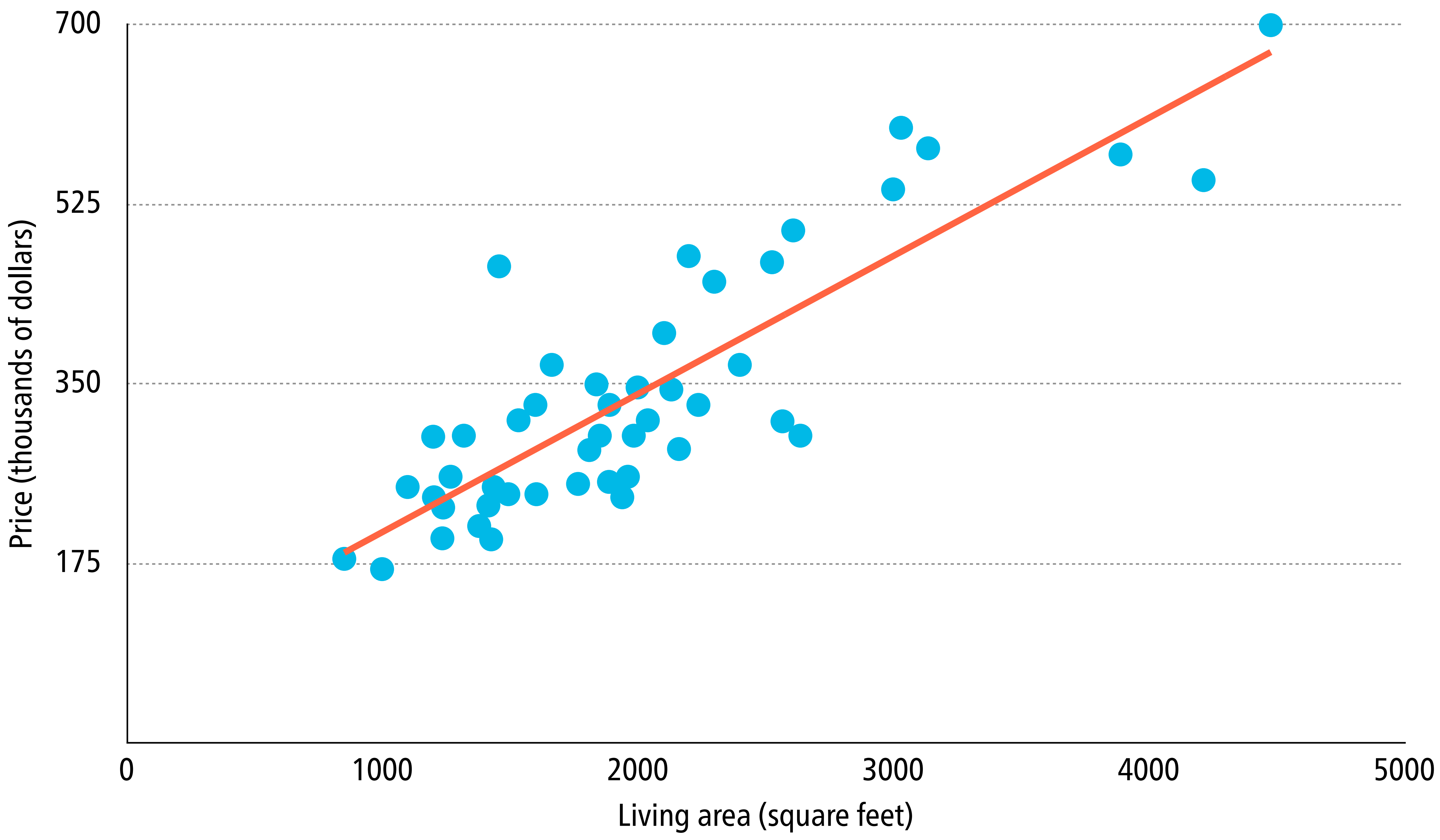


The simple linear model

input vector (1-by-F)

Convention:
Input vectors
are row vectors

output
value

$$\hat{y} = \boldsymbol{x}\boldsymbol{w} + b$$

bias

weight vector (F-by-1)

F = number of features (independent variables)

The general linear model

input vector (1-by-m)

output vector (1-by-n)

$$\hat{\boldsymbol{y}} = \boldsymbol{x}\boldsymbol{W} + \boldsymbol{b}$$

bias vector (1-by-n)

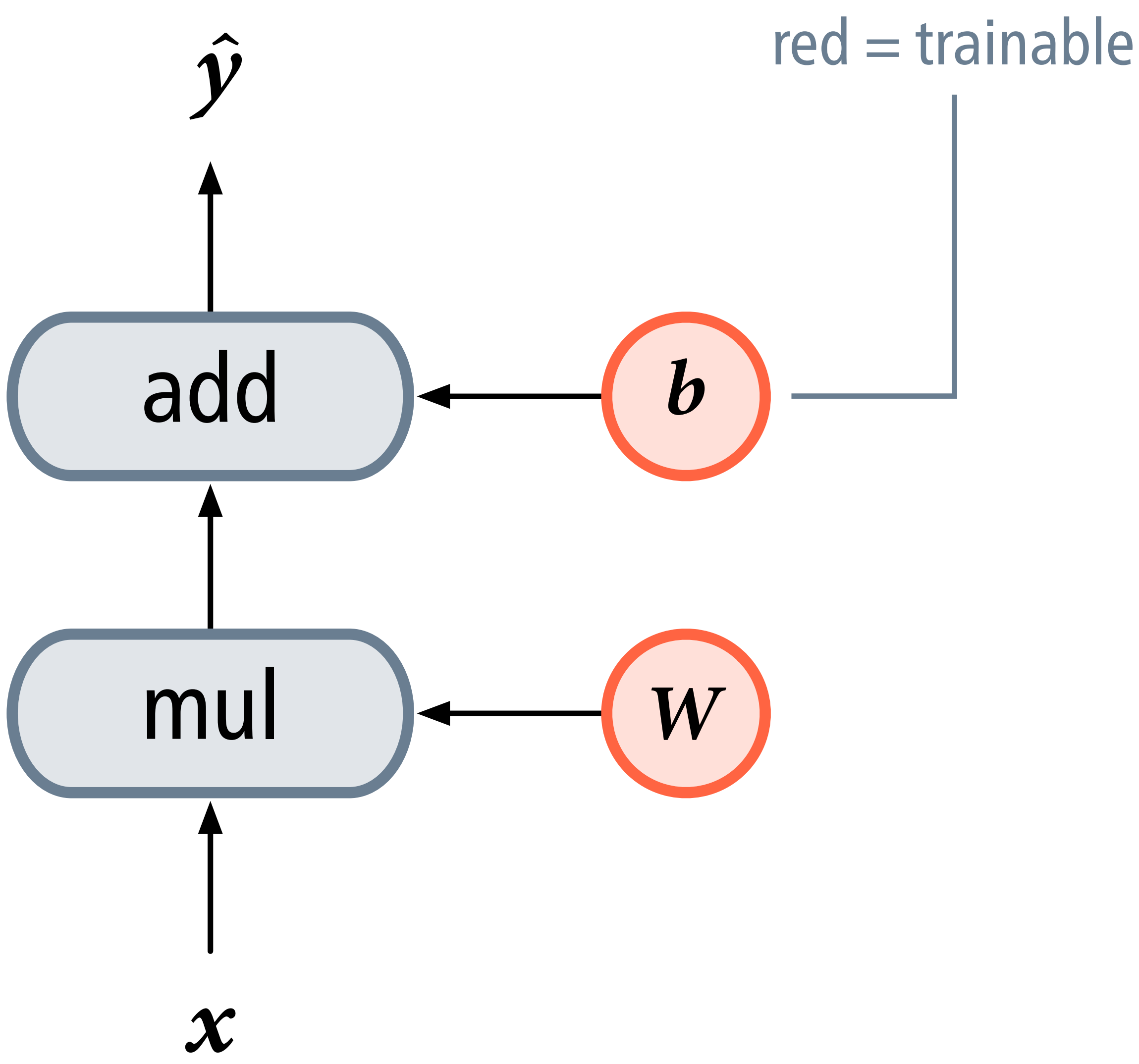
weight matrix (m-by-n)

m = number of input features, n = number of output features

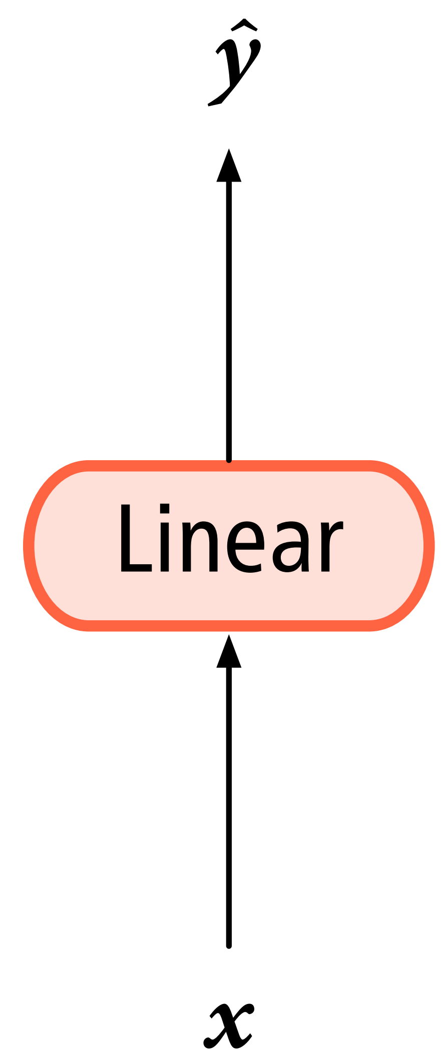
Linear classification

- We think of $\boldsymbol{z} = \boldsymbol{x}W + \boldsymbol{b}$ as an n-dimensional vector of scores that quantify the compatibility of the input $\boldsymbol{x}$ with each class k.
- In **linear classification**, we predict the input $\boldsymbol{x}$ to belong to the highest-scoring class k.
- With linear models, we can only solve a rather restricted class of classification problems (linearly separable).

Graphical notation



computation graph



shorthand notation

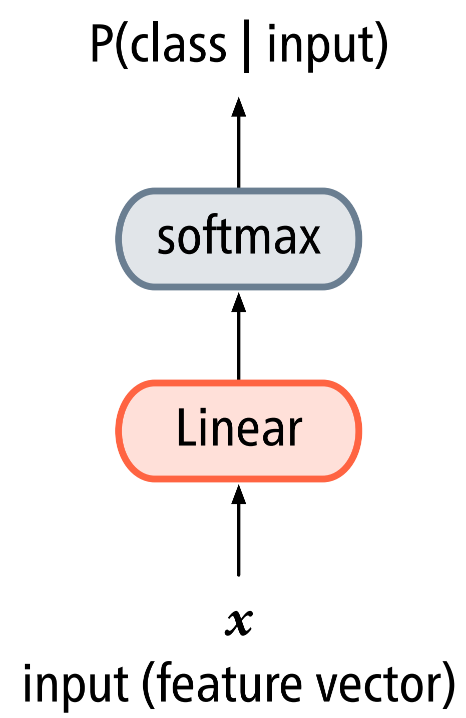
Softmax regression (log-linear classification)

- We convert the scores into a probability distribution $P(k \mid \boldsymbol{x})$ over the classes by feeding them through the **softmax function**:

$$\text{softmax}(\boldsymbol{z})[k] \;=\; \frac{\exp(\boldsymbol{z}[k])}{\sum_i \exp(\boldsymbol{z}[i])}$$

- Similar to the case of linear classification, we now predict the input $\boldsymbol{x}$ to belong to the highest-probability class k.

Softmax regression as a neural network



$$P(k \mid \boldsymbol{x}) \;=\; \text{softmax}(\boldsymbol{x}\boldsymbol{W} + \boldsymbol{b})$$

Training a softmax regression model

- We present the model with training samples of the form $(\boldsymbol{x}, y)$ where $\boldsymbol{x}$ is a feature vector and y is the gold-standard class.
- The output of the model is a vector of conditional probabilities $P(k \mid \boldsymbol{x})$ where k ranges over the possible classes.
- We want to train the model so as to maximize the likelihood of the training data under this probability distribution.

Cross-entropy loss

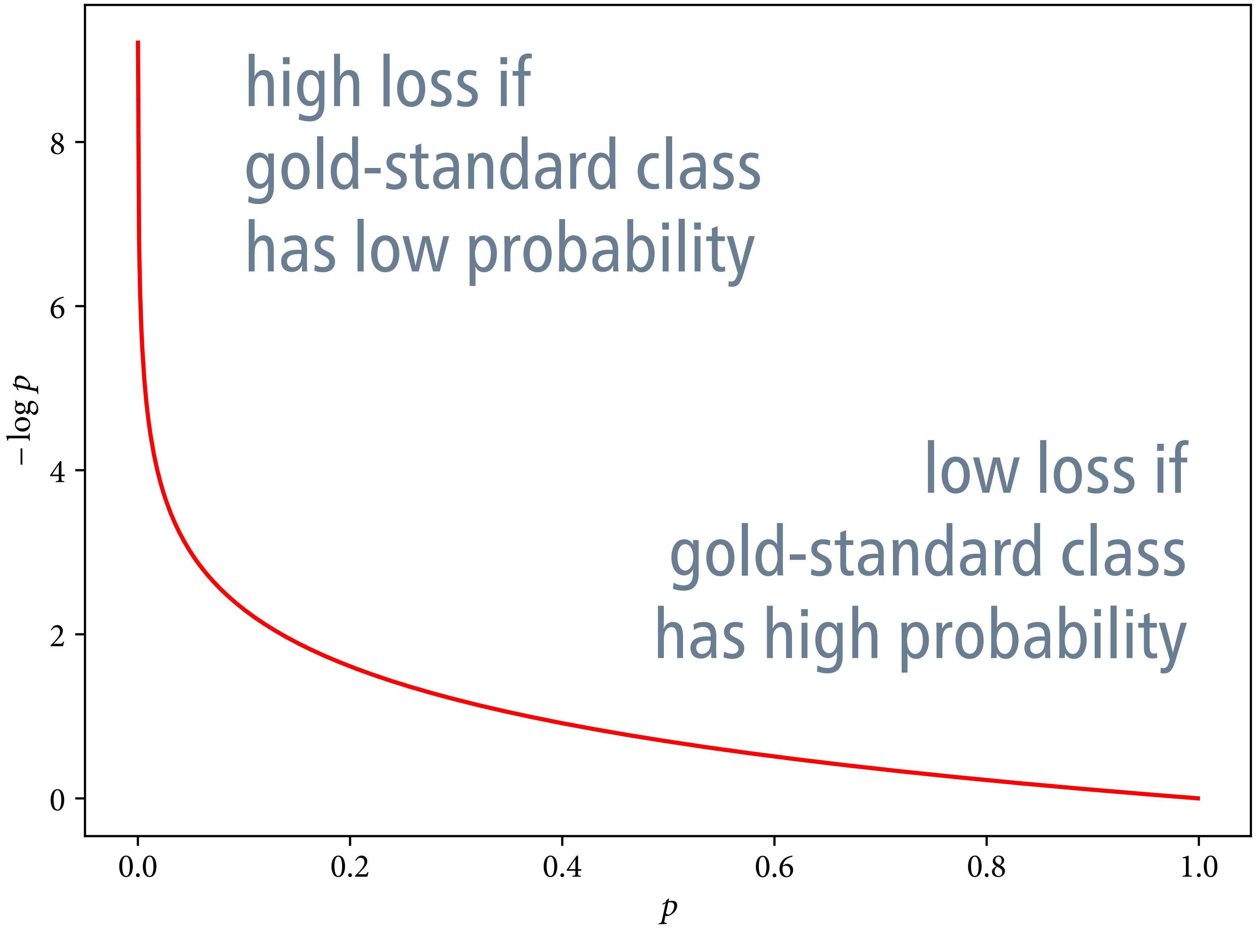
- Instead of maximizing the likelihood of the training data, we minimize the model's **cross-entropy loss**.

- The cross-entropy loss for a specific sample $(\boldsymbol{x}, y)$ is the negative log probability of the gold-standard class y,

$$L(\boldsymbol{\theta}) = -\sum_k [k = y] \log \operatorname{softmax}(\boldsymbol{x}\boldsymbol{W} + \boldsymbol{b})[k]$$

1 if k and y are identical,
0 otherwise

Cross-entropy loss



Gradient descent

'Follow the gradient into the valley of error.'

- **Step 0:** Start with random values for the parameters $\boldsymbol{\theta}$.
- **Step 1:** Compute the gradient of the loss function for the current parameter settings, $\nabla L(\boldsymbol{\theta})$.
- **Step 2:** Update the parameters $\boldsymbol{\theta}$ as follows: $\boldsymbol{\theta} \coloneqq \boldsymbol{\theta} - \alpha \nabla L(\boldsymbol{\theta})$

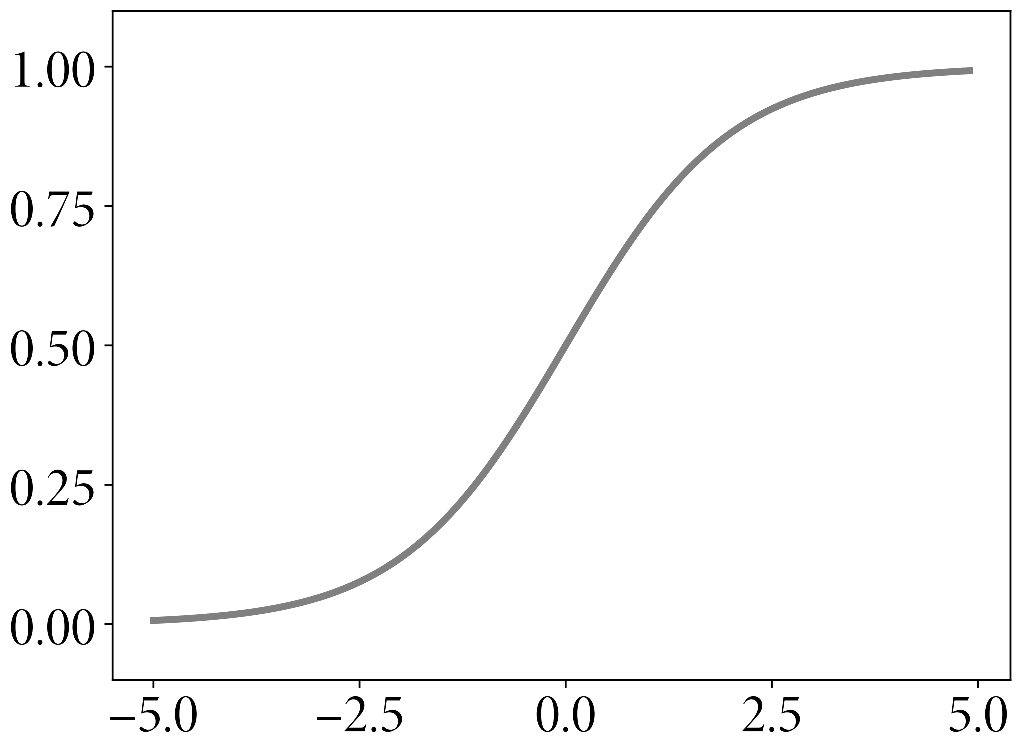
  The parameter α is the learning rate.
- Repeat step 1–2 until the loss is sufficiently low.

Note on terminology

- The softmax function can be viewed as a generalization of the standard logistic function to more than two classes.

- What we call ‘softmax regression’ here is sometimes also called **multinomial logistic regression**.

  or just ‘logistic regression’!



$$y = \frac{1}{1 + \exp(-z)}$$