Exercises
Discrete Mathematics 1998:
Boolean Algebra, Induction and Recursion
(not to be handed in)

February 1997

Exercises

1. Give a solution for each of the following boolean equations:

   (a) $((a \wedge b \wedge c^*) \vee (a^* \wedge b^* \wedge c)) \vee (a^* \wedge b \wedge c)$.

   (b) $((a \vee b \vee c^*) \wedge (a^* \vee b^* \vee c)) \wedge (a^* \vee b \vee c)$.

2. Simplify the boolean expression

   $$((a \vee b) \wedge (b \vee c)^*) \wedge (b^* \wedge c)^*$$

   as much as possible.

3. Define the set of strings with an *even number of letters* $EStr(A)$ from an alphabeth A.

4. Show that $S \in EStr(A)$ has an even number of letters with an inductive proof. Use induction over the length of the strings. Hint: If n is even number, then $n = 2k$ for some k.

5. Define the a recursive function $EvenStr : Str(A) \to B$ that yields the value 1 if a string has an even number of letters and 0 else. Compute $EvenStr(\mathsf{telefon})$.

6. Define the recursive function $OddSum(n)$ that yields the sum of the first n odd numbers, for all $n \geq 1$. Hint: The nth odd number can be written $2 \cdot n - 1$. Compute $OddSum(3)$.

7. Prove $\sum_{i=1}^{n}(i \cdot i!) = (n+1)! - 1$ with an induction proof. Remember that $(k+1)(k!) = (k+1)!$.

8. Construct a recursive function $f : \mathbf{N} \to \mathbf{N}$ where $f(n) = \sum_{i=1}^{n}(i \cdot i!)$. You can assume that the faculty function ! already is defined and can be used freely, but not $\sum$. Compute $f(3)$.

Solutions

1. (a) Disjunctive normal form. We can easily find three solutions: $a = 1, b = 1, c = 0$ by looking at the left hand side, $a = 0, b = 0, c = 1$ by looking in the middle, and $a = 0, b = 1, c = 1$ by looking at the right hand side.

   (b) Conjunctive normal form. We pick one literal from every subexpression. Following solutions exist: $a = 1, b = 0, c = 1$ (a frome the left subexpression, b^* from the middle and c from the right), $b = 1, a = 0, c = 1$ and, finally, $c = 0, a = 0, b = 0/1$ (in the last solution we have choosen a^* in both the right and middle subexpression, and, thus, we can arbitrarily choose between 1 and 0 for b).

2. Remark: We start by "distributing" the "not:s" as close to the variables as possible, with De Morgan's laws (2). The next step is to get b and b^* as close together as possible on the left side (3-10). The we try to get the remaining b together with b^* on the right side, and the two c^* together (11-17), which gives us a possibility to simplify (18-23).

1)	$((a \vee b) \wedge (b \vee c)^*) \wedge (b^* \wedge c)^*$	
2)	$((a \vee b) \wedge (b^* \wedge c^*)) \wedge (b \vee c^*)$	De Morgan x 2
3)	$((b^* \wedge c^*) \wedge (a \vee b)) \wedge (b \vee c^*)$	Comm.
4)	$(((b^* \wedge c^*) \wedge a) \vee ((b^* \wedge c^*) \wedge b)) \wedge (b \vee c^*)$	Distr.
5)	$(((b^* \wedge c^*) \wedge a) \vee ((c^* \wedge b^*) \wedge b)) \wedge (b \vee c^*)$	Comm.
6)	$(((b^* \wedge c^*) \wedge a) \vee (c^* \wedge (b^* \wedge b))) \wedge (b \vee c^*)$	Ass.
7)	$(((b^* \wedge c^*) \wedge a) \vee (c^* \wedge (b \wedge b^*))) \wedge (b \vee c^*)$	Comm.
8)	$(((b^* \wedge c^*) \wedge a) \vee (c^* \wedge 0)) \wedge (b \vee c^*)$	Kompl.
9)	$(((b^* \wedge c^*) \wedge a) \vee 0) \wedge (b \vee c^*)$	Ident.
10)	$((b^* \wedge c^*) \wedge a) \wedge (b \vee c^*)$	Ident.
11)	$(((b^* \wedge c^*) \wedge a) \wedge b) \vee (((b^* \wedge c^*) \wedge a) \wedge c^*)$	Distr.
12)	$(((c^* \wedge b^*) \wedge a) \wedge b) \vee (((b^* \wedge c^*) \wedge a) \wedge c^*)$	Comm.
13)	$((c^* \wedge (b^* \wedge a)) \wedge b) \vee ((b^* \wedge (c^* \wedge a)) \wedge c^*)$	Ass. x 2
14)	$((c^* \wedge (a \wedge b^*)) \wedge b) \vee ((b^* \wedge (a \wedge c^*)) \wedge c^*)$	Comm. x 2
15)	$(c^* \wedge ((a \wedge b^*) \wedge b)) \vee (b^* \wedge ((a \wedge c^*) \wedge c^*))$	Ass. x 2
16)	$(c^* \wedge (a \wedge (b^* \wedge b))) \vee (b^* \wedge (a \wedge (c^* \wedge c^*)))$	Ass. x 2
17)	$(c^* \wedge (a \wedge (b \wedge b^*))) \vee (b^* \wedge (a \wedge (c^* \wedge c^*)))$	Comm.
18)	$(c^* \wedge (a \wedge (b \wedge b^*))) \vee (b^* \wedge (a \wedge c^*))$	Idem.
19)	$(c^* \wedge (a \wedge 0)) \vee (b^* \wedge (a \wedge c^*)$	Comp.
20)	$(c^* \wedge 0) \vee (b^* \wedge (a \wedge c^*)$	Ident.
21)	$0 \vee (b^* \wedge (a \wedge c^*)$	Ident.
22)	$(b^* \wedge (a \wedge c^*)) \vee 0$	Comm.
23)	$(b^* \wedge (a \wedge c^*))$	Ident.

3. We start by defining a base. The empty string ϵ is the shortest string with an even number of letters (i.e. 0 elements).

   Base: $\epsilon \in EStr(A)$.

   If we have an even string, and add two elements we get a new even string.

   Induction: If $x, y \in A$ and $\alpha \in EStr(A)$, then $xy\alpha \in EStr(A)$.

   Closure: Nothing else is in $EStr(A)$.

   Example: Let $A = \{\mathsf{a}, \mathsf{o}\}$. According to the base ϵ belongs to $EStr(A)$. Since $\epsilon \in EStr(A)$, we also know that aa, ao, oa and oo belong to $EStr(A)$ (we use the induction part on ϵ). Then also aaaa, aoaa, oaaa and ooaa on $EStr(A)$ (the induction part again, this time on aa), and so on, an infinite number of times.

4. We start with the base:

   Base: The length of ϵ is 0, and $0 = 2 \cdot 0$, i.e. the length is an even number.

   Next we have to construct an induction hypothesis:

   Induction Hypothesis: Assume that all strings of length $x \leq n$ for some $n \geq 1$, has an even number of letters, i.e. $x = 2k$ for some k.

   The we will do the induction step, where the even-number-of-letters property is extended to hold for strings of length $n + 1$.

   Induction Step: Let $S \in EStr(A)$ have the length $n + 1$. According to the (inductive) definition of $EStr(A)$, the string S must be on the form $xy\alpha$, where $x,\ y \in A$ and $\alpha \in EStr(A)$. According the induction hypothesis the length of α is equal to $2k$, for some k, and, thus, the length of $S = xy\alpha$ must be $2 + 2k = 2(k + 1)$, which also is an even number.

   Remark: The crux with this proof is that we are doing induction over the length of the strings, which also is the property we want to prove something about.

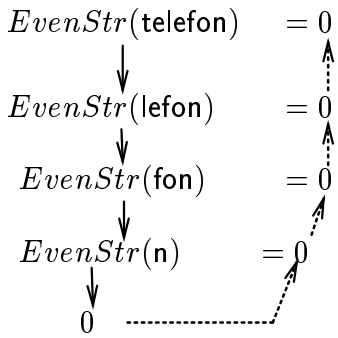
5. In this case we need two bases, one even and one odd.

$$\begin{aligned} \textbf{Base:} EvenStr(\epsilon) &= 1 \\ EvenStr(x) &= 0,\ x \in A \end{aligned}$$

Induction: $EvenStr(xy\alpha) \;=\; EvenStr(\alpha),$
$x, y \in A,\ \alpha \in EStr(A)$

I.e. a string $xy\alpha$ has even length if the string α (without the two first elements xy) has even length.

The computation of $EvenStr(\mathsf{telefon})$ is shown below.



First we use the fact that $EvenStr(xy\alpha) = EvenStr(\alpha),\ x, y \in A,\ \alpha \in EStr(A)$ where $x = \mathsf{t}$, $y = \mathsf{e}$ and $\alpha = \mathsf{lefon}$ and we get $EvenStr(\mathsf{lefon})$. Then we set $x = \mathsf{l}$, $y = \mathsf{e}$ and $\alpha = \mathsf{fon}$ and get $EvenStr(\mathsf{fon})$, then we set $x = \mathsf{f}$, $y = \mathsf{o}$ and $\alpha = \mathsf{n}$, so we have $EvenStr(\mathsf{n})$. According to the rule $EvenStr(x) = 0,\ x \in A$ the result is 0.

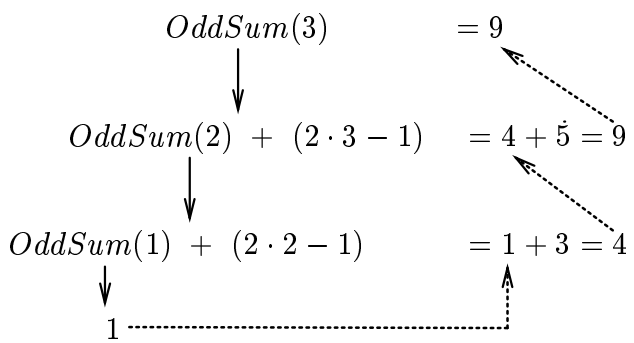
6. The base case is $n = 1$. The sum of the first odd number 1 is 1.

   Base: $OddSum(1) \;=\; 1$

   The sum of the n first odd numbers is computed as the sum of the first $n-1$ first numbers plus the nth odd number $(2 \cdot n - 1)$.

   Induction: $OddSum(n) \;=\; OddSum(n-1) + (2 \cdot n - 1),\ n > 1$

   The computation of $OddSum(3)$ is done as follows.



7. The base is straightforward: Let $n = 1$.

   Base: $\sum_{i=1}^{1}(i \cdot i!) = (1 \cdot 1!) = 1$ and $(1+1)! - 1 = 2! - 1 = 1$.

   The we make the assumption for n.

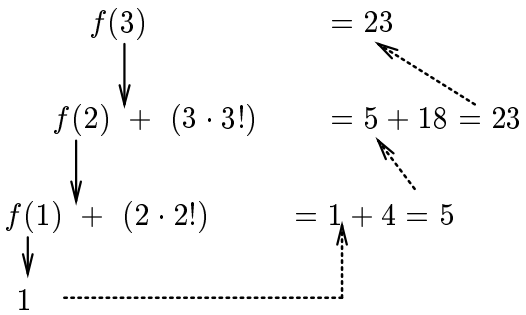
   Induction Hypothesis: $\sum_{i=1}^{n}(i \cdot i!) = (n+1)! - 1$.

   Finally we have the induction step. We will now show that $\sum_{i=1}^{(n+1)}(i \cdot i!) = ((n+1)+1)! - 1$.

   Induction Step:

$$\begin{aligned}
& \sum_{i=1}^{(n+1)} (i \cdot i!) \\
= & \sum_{i=1}^{n}(i \cdot i!) + (n+1)((n+1)!) \\
= & ((n+1)!) - 1 + (n+1)((n+1)!) \text{ according to the hypothesis} \\
= & (n+1)((n+1)!) + 1 \cdot ((n+1)!) - 1 \\
= & (n+2)((n+1)!) - 1 \\
= & (n+2)! - 1 \\
= & ((n+1)+1)! - 1
\end{aligned}$$

8. We let $f(0) = 0$.

$$\begin{aligned}
\textbf{Base: } f(0) &= 0 \\
f(1) &= 1 \\
\textbf{Induction: } f(n) &= f(n-1) + (n \cdot n!),\ n > 1
\end{aligned}$$

   The computation of $f(3)$ looks like this.