Exercises: Functions
Discrete Mathematics (HKGAB4) 1998
(not to be handed in)

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1. Let $M = \{m, n, v, w\}$, $B = \{0, 1\}$ and $N4 = \{0, 1, 2, 3, 4\}$, and $\mathbf{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. Which of the following relations are functions? What is the domain and range of every function?

   (a) $R_1 = \{\langle m, 0\rangle, \langle m, 1\rangle, \langle n, 1\rangle, \langle v, 1\rangle, \langle v, 0\rangle, \langle w, 0\rangle\} \subseteq M \times B$.

   (b) $R_2 = \{\langle 0, 1\rangle, \langle 1, 0\rangle\} \subseteq B \times B$.

   (c) $R_3 = \{\langle 0, m\rangle, \langle 1, n\rangle, \langle 2, n\rangle, \langle 3, w\rangle, \langle 4, v\rangle\} \subseteq N4 \times M$.

   (d) $R_4 = \{\langle 0, n\rangle\} \subseteq B \times N4$.

   (e) $R_5 = \{\ \} \subseteq M \times \emptyset$.

   (f) $R_6 = \{\langle w, 0\rangle, \langle v, 1\rangle, \langle n, 0\rangle, \langle m, 0\rangle\} \subseteq M \times B$.

   (g) $R_7 = \{\langle w, v\rangle, \langle v, v\rangle, \langle m, v\rangle, \langle n, v\rangle\} \subseteq M \times M$.

   (h) $f_1 \subseteq \mathbf{N} \times \mathbf{N}$ such that $f1 = \{\langle x, y\rangle \mid y = 2x + 1\}$, i.e. $f(x) = 2x + 1$.

   (i) $f_2 \subseteq \mathbf{N} \times \mathbf{N}$ such that $f2 = \{\langle x, y\rangle \mid y = 2x + n,$ some $n\ \}$.

   (j) $Father \subseteq Humans \times Humans$ such that
   $Father = \{\langle x, y\rangle \mid y$ is the biological father of $x\ \}$.

   (k) $Son \subseteq Humans \times Humans$ such that
   $Son = \{\langle x, y\rangle \mid y$ is the biological son of $x\}$.

   (l) $NumberOfSons \subseteq Humans \times \mathbf{N}$ such that
   $NumberOfSons = \{\langle x, n\rangle \mid x$ has n biolgical sons $\}$.

2. What is the inverse and complement of the following functions? Is the inverse or the complement also functions? The sets M, B and $N4$ are defined as above.

   (a) $F_1 : M \to M = \{\langle m, w\rangle, \langle n, w\rangle, \langle v, v\rangle, \langle w, m\rangle\}$.

(b) $F_2 : M \to B = \{\langle m, 0\rangle, \langle n, 1\rangle, \langle v, 0\rangle, \langle w, 1\rangle\}$.

(c) $F_3 : B \to B = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$.

(d) $F_4 : N4 \to N4 = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle, \langle 4, 0\rangle\}$.

(e) $F_5 : B \to M = \{\langle 0, m\rangle, \langle 1, n\rangle\}$.

3. Which of the functions F_1, F_2, F_3, F_4 and F_5 above are one-to-one, onto, and are one-to-one correspondences?

4. Describe the function $F_2 \circ F_1$ as a set of tuples.

5. Describe the function $F_3 \circ F_2$ as a set of tuples. What special property does F_3 have?

Solutions

1. Functions, domain and range?

   (a) R_1 is not a function (v is mapped to two values).

   (b) R_2 is a function. The domain and range is B.

   (c) R_3 is a function. The domain is $N4$ and the range is M.

   (d) R_4 is not a function (1 is not mapped to a value).

   (e) R_5 is not a function. None of the elements in M are mapped to a value.

   (f) R_6 is a function. The domain is M and the range is B.

   (g) R_7 is a function. The domain is M and the range is $\{v\} \subset M$.

   (h) f_1 is a function. The domain is $\mathbf{N}$ and the range is $\{1, 3, 5, 7, \ldots\}$.

   (i) f_2 is not a function. Every element is mapped to an infinite number of values.

   (j) $Father$ is a function. Every human being has a unique biological father.

   (k) Son is not a function. Every human can have no, one or more biolgical sons.

   (l) $NumberOfSons$ is a function. The domain is the set of all humans, and the range is a subset of $\mathbf{N}$.

2. Inverse and complement.

   (a) $F_1^{-1} = \{\langle w, m\rangle, \langle w, n\rangle, \langle v, v\rangle, \langle m, w\rangle\}$, not a function.
   $F_1' = \{\langle m, m\rangle, \langle m, n\rangle, \langle m, v\rangle, \langle n, m\rangle, \langle n, n\rangle, \langle n, v\rangle, \langle v, m\rangle, \langle v, n\rangle, \langle v, w\rangle,$ $\langle w, n\rangle, \langle w, v\rangle, \langle w, w\rangle\}$, not a function.

   (b) $F_2^{-1} = \{\langle 0, m\rangle, \langle 1, n\rangle, \langle 0, v\rangle, \langle 1, w\rangle\}$, not a function.
   $F_2' = \{\langle m, 1\rangle, \langle n, 0\rangle, \langle v, 1\rangle, \langle w, 0\rangle\}$, function.

   (c) $F_3^{-1} = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$, function.
   $F_3' = \{\langle 0, 1\rangle, \langle 1, 0\rangle\}$, function.

   (d) $F_4^{-1} = \{\langle 1, 0\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle, \langle 4, 3\rangle, \langle 0, 4\rangle\}$, function.
   $F_4' = \{\langle 0, 0\rangle, \langle 0, 2\rangle, \langle 0, 3\rangle, \langle 0, 4\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 4\rangle, \langle 3, 0\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle\}$, not a function.

   (e) $F_5^{-1} = \{\langle m, 0\rangle, \langle n, 1\rangle\}$, not a function.
   $F_5' = \{\langle 0, n\rangle, \langle 0, v\rangle, \langle 0, w\rangle, \langle 1, m\rangle, \langle 1, v\rangle, \langle 1, w\rangle\}$, not a function.

3. One-to-one, onto, or one-to-one correspndences?

(a) F_1 none.

(b) F_2 is onto (the range is all of B).

(c) F_3 is one-to-one, onto, and, therefore a one-to-one correspondence.

(d) F_4 is one-to-one, onto, and, therefore a one-to-one correspondence.

(e) F_5 is one-to-one.

4. $F_2 \circ F_1 = \{\langle m, 1\rangle, \langle n, 1\rangle, \langle v, 0\rangle, \langle w, 0\rangle\}$.

5. $F_3 \circ F_2 = \{\langle m, 0\rangle, \langle n, 1\rangle, \langle v, 0\rangle, \langle w, 1\rangle\}$. F_3 is an identity function on B.