Functions

Synonyms: Mapping, transformation, correspondence.

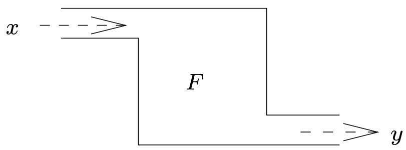
Function/Relation from A to B.
Domain: The elements in A that can be find in the left position in some tuple.
Range: The elements in B that can be found in the right position in some tuple.

A function from A to B is a relation where:

- Every element in the domain is paired with exactly one element from the range.
- The domain is equivalent to A.

Input/Output Principle: "One value in always yields exactly one value out"

A function can be seen as a process that recieves a value and returns another value.



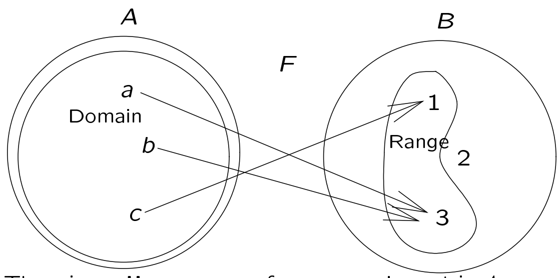
Functions are useful in many situations:
Mathematical functions, e.g. $sin(x)$, $cosin(x)$, $x+y$...
In computer programming, e.g. LISP is entirely built on the notion of functions.
In logic, e.g. functions from logical formulae to their truth values.
In linguistics, e.g. a function that describes how a negation changes the meaning of a statement.
In every-day life, e.g. the hair colour of a peron, or the price of something.

Example (Function):

$$\begin{aligned} F &: A \rightarrow B \\ A &= \{a,b,c\} \\ B &= \{1,2,3\} \\ F &= \{\langle a,3\rangle, \langle b,3\rangle, \langle c,1\rangle\} \end{aligned}$$

The domain of F is $\{a,b,c\} = A$.
The range of F is $\{1,3\} \subseteq B$.
All elements of A can be found in *exactly one* tuple.

Example (Function Cont'd):



There is **exatly one** arrow from every element in A.

Example (Function):

$$\begin{aligned} Age &: M \rightarrow \mathbf{N} \\ M &= \{\text{People in this room}\} \\ \mathbf{N} &= \{0,1,2,3,\ldots\} \\ Age &= \{\langle Marcus, 29\rangle, \ldots\} \end{aligned}$$

Example (Function):

$$\begin{aligned} D &: \mathbf{N} \rightarrow \mathbf{N} \\ \mathbf{N} &= \{0,1,2,3,\ldots\} \\ D &= \{\langle x,y\rangle | y = 2x+1\} \\ D(x) &= 2x+1 \quad \text{Alternative way of writing!} \end{aligned}$$

E.g. we can write $F(1) = 3$ instead of $\langle 1,3\rangle \in F$.

Example (Function):

$$
\begin{aligned}
Morse &: Alfa \to Code \\
Alfa &= \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots\} \\
Code &= \{\cdot, -, \cdot\cdot, \cdot-, -\cdot, --, \ldots\} \\
Morse &= \{\langle \mathrm{a}, \cdot- \rangle, \ldots\}
\end{aligned}
$$

Example (Functions with more than one argument):

$$
\begin{aligned}
+ &: \underbrace{(\mathbf{N} \times \mathbf{N})}_{\text{Domain}} \to \mathbf{N} \\
\mathbf{N} &= \{0, 1, 2, 3, \ldots\} \\
+ &= \{\ \langle\langle 0,0\rangle,0\rangle,\ \langle\langle 0,1\rangle,1\rangle,\ \langle\langle 1,1\rangle,2\rangle, \ldots\ \}
\end{aligned}
$$

Example: $+(1,1) = 2$, or, usually $1 + 1 = 2$.

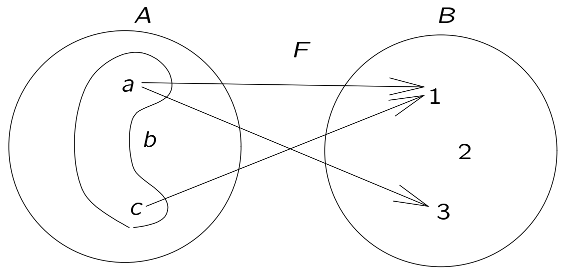
Function *on* one set are sometimes called *operations*.

Example (Not a Function):

$$
\begin{aligned}
F &: A \to B \\
A &= \{a, b, c\} \\
B &= \{1, 2, 3\} \\
F &= \{\langle a, 3\rangle, \langle a, 1\rangle, \langle c, 1\rangle\}
\end{aligned}
$$

a can be found in two tuples, b in none.

Example (Not a Function Cont'd):



Inverse and Complement

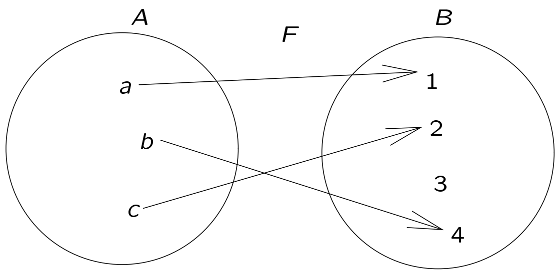
Just as other relations functions have inverse F^{-1} and complement F'.

However, F^{-1} and F' are not always functions.

Example (Inverse and Complement):

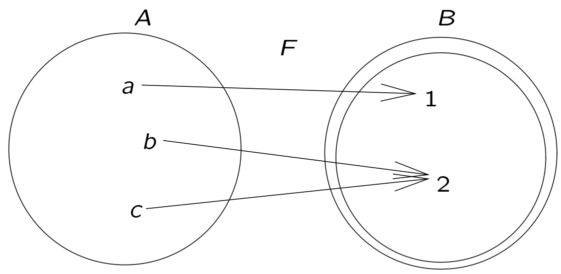
$$
\begin{aligned}
F &: A \to B \\
A &= \{a, b, c\} \\
B &= \{1, 2, 3\} \\
F &= \{\langle a, 3\rangle, \langle b, 3\rangle, \langle c, 1\rangle\} \\
F^{-1} &= \{\langle 3, a\rangle, \langle 3, b\rangle, \langle 1, c\rangle\} \\
F' &= \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}
\end{aligned}
$$

One-To-One Functions



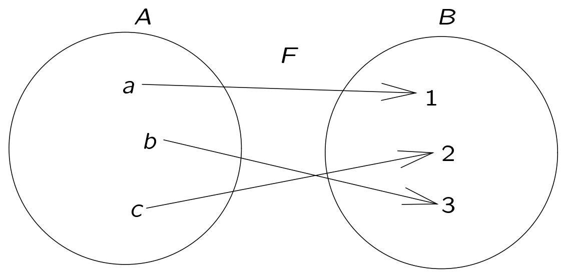
Every element in B is a map of *at most* one element in A, i.e. if $x \neq y$ then $F(x) \neq F(y)$.

Onto Functions



Every element in B is a map of *at least* one element in A, i.e. the range of F is equal to B.

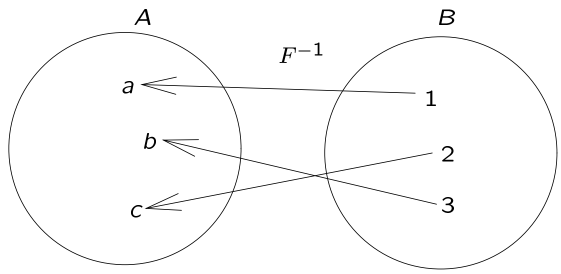
One-To-One and Onto Functions: *One-To-One Correspondences*



Every element in B is a map of *exactly* one element in A.

One-To-One Correspondences Cont'd

The inverse, F^{-1}, of a one-to-one correspondence F, is always a one-to-one correspondence.



Example (Functions): Let $A = \{di, da\}$.
$A \times A = \{\langle di, di\rangle, \langle di, da\rangle, \langle da, di\rangle, \langle da, da\rangle\}$.
Relations on A ($2^4 = 16$ st):

$$\begin{aligned}
R_1 &= \{\ \} \\
R_2 &= \{\langle di, di\rangle\} \\
R_3 &= \{\langle di, da\rangle\} \\
R_4 &= \{\langle da, di\rangle\} \\
R_5 &= \{\langle da, da\rangle\} \\
R_6 &= \{\langle di, di\rangle, \langle di, da\rangle\}
\end{aligned}$$

Example (Functions, Cont'd):

$$\begin{aligned}
R_7 &= \{\langle di, di\rangle, \langle da, di\rangle\} \\
R_8 &= \{\langle di, di\rangle, \langle da, da\rangle\} \\
R_9 &= \{\langle di, da\rangle, \langle da, di\rangle\} \\
R_{10} &= \{\langle di, da\rangle, \langle da, da\rangle\} \\
R_{11} &= \{\langle da, di\rangle, \langle da, da\rangle\} \\
R_{12} &= \{\langle di, di\rangle, \langle di, da\rangle, \langle da, di\rangle\} \\
R_{13} &= \{\langle di, di\rangle, \langle di, da\rangle, \langle da, da\rangle\} \\
R_{14} &= \{\langle di, di\rangle, \langle da, di\rangle, \langle da, da\rangle\}
\end{aligned}$$

Example (Functions Cont'd):

$$\begin{aligned}
R_{15} &= \{\langle di, da\rangle, \langle da, di\rangle, \langle da, da\rangle\} \\
R_{16} &= \{\langle di, di\rangle, \langle di, da\rangle, \langle da, di\rangle, \langle da, da\rangle\}
\end{aligned}$$

Example (Functions Cont'd):

Functions on A ($2^2 = 4$ st):

$$\begin{aligned}
R_7 &= \{\langle di, di\rangle, \langle da, di\rangle\} \\
R_8 &= \{\langle di, di\rangle, \langle da, da\rangle\} \\
R_9 &= \{\langle di, da\rangle, \langle da, di\rangle\} \\
R_{10} &= \{\langle di, da\rangle, \langle da, da\rangle\}
\end{aligned}$$

Example (Functions Cont'd):

One-to-one functions on A?

$$\begin{aligned} R_7 &= \{\langle di, di\rangle, \langle da, di\rangle\} \text{ Nej!}\\ R_8 &= \{\langle di, di\rangle, \langle da, da\rangle\} \text{ Ja!}\\ R_9 &= \{\langle di, da\rangle, \langle da, di\rangle\} \text{ Ja!}\\ R_{10} &= \{\langle di, da\rangle, \langle da, da\rangle\} \text{ Nej!} \end{aligned}$$

Every element should be found at most once to the right in a pair.

Example (Functions Cont'd):

Onto functions on A (at least once to the right)?

One-to-one correspondences A? (exactly once to the right)?

$$\begin{aligned} R_6 &= \{\langle di, di\rangle, \langle da, di\rangle\} \text{ Nej!/Nej!}\\ R_7 &= \{\langle di, di\rangle, \langle da, da\rangle\} \text{ Ja!/Ja!}\\ R_{10} &= \{\langle di, da\rangle, \langle da, di\rangle\} \text{ Ja!/Ja!}\\ R_{11} &= \{\langle di, da\rangle, \langle da, da\rangle\} \text{ Nej!/Nej!} \end{aligned}$$

Example (One-To-One, Onto):

Age: Many people can be of the same age — Not one-to-one. No one in here is two years old — Not onto.

$Morse$: Every word has a unique code — one-to-one. Every possible code cannot be used— not onto.

$Neg : \mathbf{Z} \to \mathbf{Z}$ where $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $Neg(x) = -x$: One-to-one and onto — One-to-one correspondence.

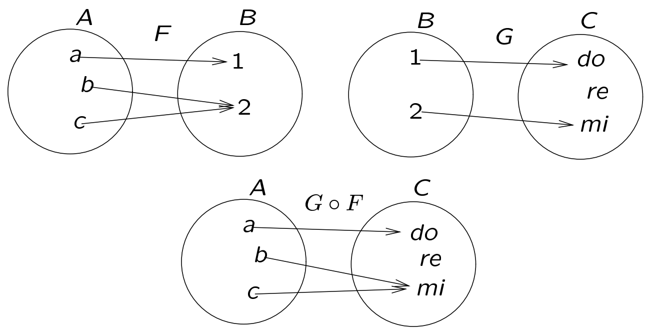
Compositions of functions

Written as $G \circ F$.

Definition:

$G \circ F = \{\langle x, z\rangle |$ *for some* $y, \langle x, y\rangle \in F$ *and* $\langle y, z\rangle \in G\}$.

$(G \circ F)(x)$ is the same as $G(F(x))$.

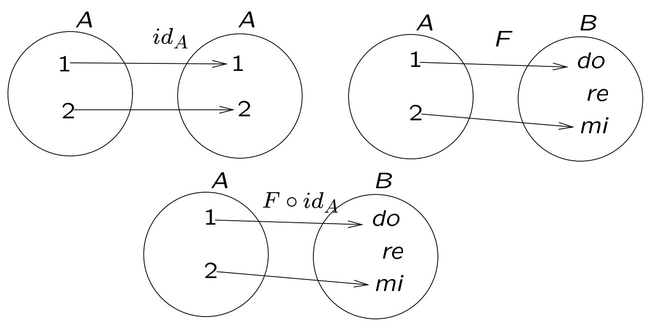


Identity Function

An identity function $id_A : A \to A$, is a function with the property $id_A(x) = x$, i.e. $id_A = \{\langle x, x\rangle | x \in A\}$.

If $F : A \to B$ then

$$F \circ id_A = F \quad (1)$$
$$id_B \circ F = F \quad (2)$$
$$F^{-1} \circ F = id_A \; F \text{ one-to-one correspondence.} \quad (3)$$
$$F \circ F^{-1} = id_B \; F \text{ one-to-one correspondence.} \quad (4)$$



Functions: Summary

- $F : A \to B$. The domain $= A$, the range $\subseteq B$.
- Inverse $F^{-}1$ and complement F'.
- One-to-one, onto, on-to-one correspondence,
- Composition $G \circ F$, identity function id_A.