HKGAB4: Logic and Discrete Mathematics

- Discrete mathematics: v. 4-5, concentrated.
- Logic: v. 6-13.

Administrative Details

- Course Information
- Running Information: http://www.ida.liu.se/labs/kplab/teaching/hkgab4.html
- Examination: problems to solve individually. Finished at the same time as the logic problems (preferrably before).

Discrete Mathematics: Course Form

- Seminars, with handouts. Complements the book, which is (too) compact.
- Lectures: Supervised problem solving
- The Book: Far too expensive (help each other out)

Discrete Mathematics: "Mathematics"?

- Precise reasoning about precise things
- Generalize common phenomena
- Example: natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ are precise generalizations
- "$+$" is a precise generalization

Discrete Mathematics: "Discrete"?

Special flavour of mathematics, developed during the 20th century in connection with developments in Computer Science, Language Theories, Logic and Reasoning.
Means "Separated" in latin

For What, and Why??

Good for *precise* descriptions of common phenomena in

- Computer Programming
- Linguistics (Language Theories)
- Artificial Intelligence
- Logic (precise reasoning and argumentation)

Why Precise Descriptions?

- The only kind of descriptions that a computer can handle.
- You can see that you know what you means!

Challenge:
Describing complex things precisely

Examples of What We Can Describe Precisely

- A sentence as a sequence of words
- Correct, formal reasoning

What Is Needed for Precise Descriptions?

- Useful and precise *concepts*
- Special *notation* for these
  so that everybody know what they mean

Which Concepts Will We Use?

- Sets: Collections of objects without duplicates
- Sequences: Ordered "lists" of objects
- Relations: Relations between objects
- Functions: Mappings from objects to other objects
- Induction, Recursion: *Construct* sets by "adding another object"

How Do We Use the Concepts?

We are interested in *properties* of the concepts which has to be *proven* with precise arguments.
Example: If $\mathrm{N} = \{0, 1, 2, 3, 4, \ldots\}$,
then 0 is a member of N.
Moreover: Whenever n is a member of N, then $n + 1$ is a member of N too.

Implication

A *implies* B, is a true statement if it is the case that whenever A is true, B is true too.
A if, and only if B (A iff B)

Sets

A *set* is a collection of objects, without any order or duplicates.
Thus: Everything we know about a set is whether any other object is a member of it, or not.

Notation for Sets

Let M be the set of cities in Sweden.
Then we know that Ystad is a member of M.
We write Ystad $\in M$,
and say "Ystad belongs to M" or
"Ystad is an *element* of M".
Point: We know what we *mean*!

Example: Sets

Old Example: The natural numbers
$\mathrm{N} = \{0, 1, 2, 3, 4, \ldots\}$.
We know: $1 \in \mathrm{N}$, $17 \in \mathrm{N}$, Motala $\notin \mathrm{N}$.
The last means "Motala does *not* belong to N".

Practical Ways of Writing Sets: Lists

We have already seen $\mathrm{N} = \{0, 1, 2, 3, 4, \ldots\}$.
More: $\{\text{Stockholm}, \text{Motala}, \text{Karesuando}\}$.
Remember: $\{3, 4, 5\} = \{5, 3, 4\} = \{3, 3, 4, 4, 4, 5\}$
Different ways of writing the elements:
$\{1 + 1 + 1, 1 + 3, 2 + 2 + 1\}$ is the same set,
as was $\{III, IV, V\}$ for Romans.

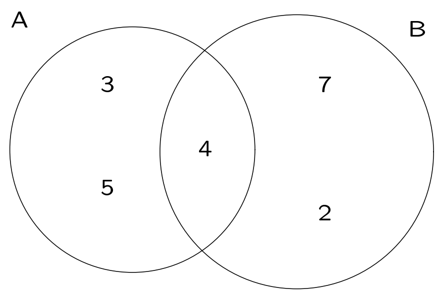
Practical Ways of Writing Sets: With Predicates (1)

The set of elements that satisfy a certain *property*.
Exempel: $\mathrm{N} = \{x | x$ is a natural number$\}$
More: $M = \{x | x$ is a city in Sweden$\}$

Practical ways of of writing sets: with predicates (2)

Compact:
$\{x | Odd(x)\}$ instead of $\{x | x$ is an odd number$\}$
"Reusability":
$\{x | x \in \mathrm{N}$ and $x > 3\} = \{4, 5, 6, 7, \ldots\}$
Generally: $\{x | P(x)\}$ for some P.
P is a *predicate*.

Venn diagrams (John Venn, 1834-1923)



Properties of Sets: Size

$\{$Adam, Bertil$\}$ has two elements.
$\{$Motala, Trelleborg, Halmstad, 3$\}$ has four elements.
But: How many elements does $\mathrm{N} = \{0, 1, 2, 3, 4, \ldots\}$ have?
Not precise: "Infinately many".
We will return to this next week.

Properties of Sets: Equality

When are two sets, A and B equal, i.e. denote the same set?
For every element x,
$x \in A$ exactly when $x \in B$.
We write $A = B$.
Example: $x \in \{1, 2, 3\}$ iff $x \in \{3, 2, 1\}$,
so $\{1, 2, 3\} = \{3, 2, 1\}$.
$\{y | y$ even number$\} = \{y | y + 1$ odd number$\}$.

Equality with Predicate Notation

What do we need to show to get equality between two sets described with predicates?
That is, show $A = \{x | P(x)\} = \{x | Q(x)\} = B$ for any predicates P and Q.
$x \in A$ iff $x \in B$ for every x, yields
$P(x)$ iff $Q(x)$ for every x,
i.e. true for the same elements.

Properties of Sets: Subset (1)

If the members of one set all are members of another:
$\{1, 2, 3\}$ is a *subset* of $\{0, 1, 2, 3, \text{frog}\}$,
written $\{1, 2, 3\} \subseteq \{0, 1, 2, 3, \text{frog}\}$.
We also allow $\{1, 2, 3\} \subseteq \{1, 2, 3\}$.
Proper Subset: $A \subset B$.
Holds when $A \subseteq B$ and $A \neq B$.

Properties of Sets: Subset (2)

The subset property with predicate notation:
$\{x | P(x)\} \subseteq \{x | Q(x)\}$ iff
whenever $P(x)$ is true for some x,
then $Q(x)$ is true.
Example: $\{x | x < 10\} \subseteq \{x | x < 100\}$
"not subset" is written $A \not\subseteq B$
(Note the meaning of this!).

How Small Can Sets Be?

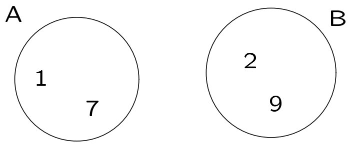
The *empty set*, $\{\}$, or $\emptyset$, as it usually is written, contains no elements.
It is a subset of every set! (Why?)
Example: $\emptyset \subseteq \{0, 1, 2, 3, \ldots\} = \mathbf{N}$,
$\emptyset = \{x | x \text{ is both even and odd}\}$.

The Empty Set with Predicate Notation

How do we show that a set $\{x|P(x)\}$ is even, i.e. that $\{x|P(x)\} = \emptyset$?
Use the definition of equivalence:
$x \in \{x|P(x)\}$ iff $x \in \emptyset$, i.e.
$P(x)$ should never be true!
As before: A is *non-empty* if $A \neq \emptyset$.

Disjunct Sets (1)

Two sets A and B are *disjunct* if they do not share any elements.



Disjunct Sets (2)

Example: $\{1, 2, 3\}$ and $\{95, \text{Solen}, \text{Berlin}\}$,
but not $\{7, 5, 2\}$ and $\{5, 9, \{\text{Apelsin}\}\}$.
Alternatively: The set of elements common to A and B is $\emptyset$.

Operations on Sets: Intersection

The *intersection* of two sets A and B is the set of elements common to A and B, written $A \cap B$.
Example:
$\{x|x < 10\} \cap \{x|x > 2\} = \{3, 4, 5, 6, 7, 8, 9\}$
$A \cap B = \emptyset$ means that A and B are disjunct!
With predicate notation:
$\{x|P(x)\} \cap \{x|Q(x)\} = \{x|P(x) \text{ and } Q(x)\}$

Intersection

Operations on Sets: Union

We want to put together the sets A and B to one set, the *Union* of A and B, written $A \cup B$.
Example: $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
With predicate notation:
$\{x|P(x)\} \cup \{Q(x)\} = \{x|P(x) \text{ or } Q(x)\}$.

Union

Operations on Sets: Difference

Take two sets A and B.
We want to remove everything in A that already is in B.
the resulting set is the *difference* of A and B, written $A - B$.
Example: $\mathbb{N} - \{0,1,2,\text{balloon}\} = \{3,4,5,6,7,\ldots\}$
With predicate notation:
$\{x|P(x)\} - \{x|Q(x)\} = \{x|P(x) \text{ but not } Q(x)\}$.

Difference

Operations on Sets: Complement (1)

Given a set A, we want the set of all elements *not* in A.
We need a *universe* U with elements, that is a set.
We write A', for the *complement* of A, which is $U - A$.
With predicate notation:
$\{x|P(x)\}' = \{x|x \in U \text{ and not } P(x)\}$.

Operations on Sets: Complement (2)

Example: If $U = \mathbb{N}$, we have
$\{1,2,3\}' = \{0,4,5,6,7,\ldots\}$.

Complement

Operations on Sets: Powerset

Given a set A, we want to construct the set of all it's *subsets*, which is called the *powerset* of A,
and written $\mathcal{P}(A)$.
Thus: $\mathcal{P}(A) = \{x|x \subseteq A\}$.
Example: $\mathcal{P}(\{1,2,3\}) =$
$\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

Powerset

Next Seminar

- How can we show the properties of sets?
- What are the relation between the operations and the properties?