Paper I

The Computational Complexity of Relating Points with Intervals on the Real Line

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ABSTRACT

Vilain has suggested the point-interval algebra for reasoning about the relative positions of points and intervals on the real line. The satisfiability problem for the full point-interval algebra is known to be NP-complete but the computational complexity of its $4,294,967,296$ subclasses has not been investigated. We classify these subclasses with respect to whether their corresponding satisfiability problems are polynomial-time or NP-complete. The classification reveals that there are exactly five maximal subclasses having a polynomial-time satisfiability problem.

1 Introduction

Problems concerned with the realizability of a set of points or intervals (along the real line) subject to constraints have been a very active research area for some time. Such problems arise in several application areas including operations research [Papadimitriou and Yannakakis, 1979], combinatorics [Roberts, 1976], planning [Allen, 1991], natural language processing [Song and Cohen, 1988] and temporal reasoning such as time serialization in archeology [Golumbic and Shamir, 1993], The archetypical example is *Allen's interval algebra* [Allen, 1983] which is based on relations between pairs of intervals on the real line. Intervals are related by disjunctions of 13 basic relations which yields 8192 possible relations. Given a set of interval variables related by Allen relations, the satisfiability problem is to decide whether there exists an interval interpretation of the variables satisfying the relations or not. This problem (which is NP-complete [Vilain and Kautz, 1986]) has been studied in a large number of papers [Vilain *et al.*, 1989; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1997c; Drakengren and Jonsson, 1997a; Drakengren and Jonsson, 1997b]. A variant of Allen's algebra where all intervals are required to have unit length is studied in [Pe'er and Shamir, 1997].

Another algebra which has been thoroughly investigated is the *point algebra* [Vilain, 1982]. This algebra considers relations between pairs of points on the real line. In this case, we have three basic relations and eight disjunctive relations. The satisfiability problem for the point-interval algebra is defined analogously to the satisfiability problem for Allen's algebra. However, this problem is solvable in polynomial time [Vilain *et al.*, 1989].

As a natural intermediate problem, the *point-interval algebra* has been suggested [Vilain, 1982]. Here, a point p and an interval I on the real line can be related in five mutually exclusive ways depicted in Table 1. The satisfiability problem for the full point-interval algebra is known to be NP-complete [Meiri, 1996]. One way of dealing with the complexity of the point-interval algebra is to restrict the algebra only to contain subsets of its 32 relations and study the complexity of these subclasses. This approach has been quite successful in the case of Allen's algebra where a large number of polynomial-time solvable subclasses has been reported [van Beek and Cohen, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1997a; Drakengren and Jonsson, 1997c]. The main result of this article is a complete classification of all subclasses of the point-interval algebra with respect to the computational complexity of the corresponding satisfiability problem. Our study reveals that there exist five maximal subclasses whose satisfiability problem is solvable in polynomial time. Furthermore, if a subclass is not a subset of one of these subclasses then the satisfiability problem is NP-complete. Similar dichotomy results have been shown for the generalized satisfiability problem [Schaefer, 1978] and the directed subgraph homeomorphism problem [Fortune *et al.*, 1980], just to mention a few.

A few words on methodology are appropriate at this point. The proof of the main theorem relies on a quite extensive case analysis performed by a computer. The number of cases considered in this analysis was approximately 2.4×10^5. Naturally, such an analysis cannot be reproduced in an article or be verified manually. To allow for the verification of our results, we include a description of the program used in the analysis. The program can be obtained from the authors.

The rest of this article is organized as follows: Section 2 defines the point-interval algebra and some auxiliary concepts. Section 3 contains the classification of subclasses and Section 4 concludes the article. Most of the proofs are postponed to the appendix. This article is an extended and corrected version of an earlier conference paper [Jonsson *et al.*, 1996].

2 Point-Interval Relations

The point-interval algebra is based on the notions of *points*, *intervals* and *binary relations* on these. A point p is a variable interpreted over the set of real numbers $\mathbb{R}$. An interval I is represented by a pair $\langle I^-, I^+\rangle$ satisfying $I^- < I^+$ where I^- and I^+ are interpreted over $\mathbb{R}$. We assume that we have a fixed universe of variable names for points and intervals. Then, a $\mathcal{V}$*-interpretation* is a function $\Im$ that maps point variables to $\mathbb{R}$ and interval variables to $\mathbb{R} \times \mathbb{R}$ and satisfies the previously stated restrictions. We will frequently extend the notation by denoting the first component of $\Im(I)$ by $\Im(I^-)$ and the second by $\Im(I^+)$.

Given an interpreted point and an interpreted interval, their relative positions can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic point-interval relations* where each basic relation can be defined in terms of its endpoint relations (see Table 1). A formula of the form pBI where p is a point, I an interval and $B \in \mathbf{B}$, is said to be satisfied by a $\mathcal{V}$-interpretation iff the interpretation of the points and intervals satisfies the endpoint relations specified in Table 1.

To express indefinite information, unions of the basic relations are used which lead to 2^5 distinct binary *point-interval relations*. Naturally, a set of basic relations is to be interpreted as a disjunction of its member relations. A point-interval relation is written as a list of its members, *e.g.*, $(\mathtt{b}\ \mathtt{d}\ \mathtt{a})$. The set of all point-interval relations $2^{\mathbf{B}}$ is denoted by $\mathcal{V}$. Relations of special interest are the *null* relation $\varnothing$ (also denoted by $\perp$) and the *universal* relation $\mathbf{B}$ (also denoted $\top$).

A formula of the form $p(B_1, \ldots, B_n)I$ is called a *point-interval* formula. Such a formula is satisfied by a $\mathcal{V}$-interpretation $\Im$ iff pB_iI is satisfied by $\Im$ for some i, $1 \leq i \leq n$. A set Θ of point-interval formulae is said to be $\mathcal{V}$-*satisfiable* iff there exists an $\mathcal{V}$-interpretation $\Im$ that satisfies every formula of Θ. Such a satisfying $\mathcal{V}$-interpretation is called a $\mathcal{V}$-*model* of Θ. The reasoning problem we will study is the following:

INSTANCE: A finite set Θ of point-interval formulae.
QUESTION: Does there exist a $\mathcal{V}$-model of Θ?

We denote this problem $\mathcal{V}$-SAT. In the following, we often consider restricted versions of $\mathcal{V}$-SAT where relations used in the formulae in Θ are only from a subset $\mathcal{S}$ of $\mathcal{V}$. In this case we say that Θ is a set of formulae over $\mathcal{S}$ and use a parameter in the problem description to denote the subclass under consideration, *e.g.* $\mathcal{V}$-SAT($\mathcal{S}$).

Meiri's extended definition of the point-interval algebra consists of $\mathcal{V}$ equipped with the two binary operations *intersection* and *composition*. However, such a definition does not constitute an algebra because it is not closed under composition. We replace the composition operation with an operation on $\mathcal{V}$ which we will refer to as *3-composition*.

Definition 2.1 Let $\mathbf{B} = \{\mathtt{b}, \mathtt{s}, \mathtt{d}, \mathtt{f}, \mathtt{a}\}$. The *point-interval algebra* consists of the set $\mathcal{M} = 2^{\mathbf{B}}$ and the operations binary *intersection* (denoted by $\cap$) and ternary *3-composition* (denoted by $\otimes$). They are defined as follows:

$$\forall p, I : p(R \cap S)I \Leftrightarrow pRI \wedge pSI;$$

$$\forall p, I : p(R \otimes S \otimes T)I \Leftrightarrow \exists q, J : (qRI \wedge qSJ \wedge pTJ).$$

It can easily be verified that

$$R \otimes S \otimes T = \bigcup \{B \otimes B' \otimes B'' | B \in R, B' \in S, B'' \in T\},$$

Basic relation	Symbol	Example	Endpoint relation
p before I	`b`	`p   ` ` III`	$p < I^-$
p starts I	`s`	` p  ` ` III`	$p = I^-$
p during I	`d`	`  p ` ` III`	$I^- < p < I^+$
p finishes I	`f`	`   p` ` III`	$p = I^+$
p after I	`a`	`    p` ` III`	$p > I^+$

Table 1: The five basic relations of the $\mathcal{V}$-algebra. The endpoint relation $I^- < I^+$ that is required for all relations has been omitted.

i.e., 3-composition is the union of the component-wise 3-composition of basic relations. In Table 2, we present the tables for 3-composition of basic relations. We can see that, for instance, $(\mathtt{f}) \otimes (\mathtt{b}) \otimes (\mathtt{s}) = (\mathtt{a})$.

Next, we introduce a *closure* operation $\mathcal{C}_{\mathcal{V}}$. This operation will simplify some of the following proofs.

Definition 2.2 Let $\mathcal{S} \subseteq \mathcal{M}$. Then we denote by $\mathcal{C}_{\mathcal{V}}(\mathcal{S})$ the $\mathcal{V}$-*closure* of $\mathcal{S}$, defined as the least subset of $\mathcal{M}$ containing $\mathcal{S}$ which is closed under intersection and 3-composition.

Given a set $\mathcal{S} \subseteq \mathcal{M}$, we can easily compute $\mathcal{C}_{\mathcal{V}}(\mathcal{S})$. Let $\Psi : 2^{\mathcal{M}} \to 2^{\mathcal{M}}$ be defined as

$$\Psi(X) = X \cup \{x \cap y \mid x, y \in X\} \cup \{x \otimes y \otimes z \mid x, y, z \in X\}.$$

Obviously, Ψ is a monotone function on the complete lattice $\langle 2^{\mathcal{M}}, \subseteq \rangle$. Since $\Psi^i(\mathcal{S}) \subseteq \Psi^{i+1}(\mathcal{S})$ for all i and $|\mathcal{M}| = 32$, there exists a $k \leq 32$ such that $\Psi^k(\mathcal{S}) = \Psi^{k+1}(\mathcal{S})$. Clearly, $\Psi^k(\mathcal{S}) = \mathcal{C}_{\mathcal{V}}(\mathcal{S})$. A program for computing $\mathcal{V}$-closures can be obtained from the authors.

The proof of the following result is omitted since proofs of analogous results can be found in [Nebel and Bürckert, 1995].

Lemma 2.3 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT$(\mathcal{S})$ is in P iff $\mathcal{V}$-SAT$(\mathcal{C}_{\mathcal{V}}(\mathcal{S}))$ is in P. $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard iff $\mathcal{V}$-SAT$(\mathcal{C}_{\mathcal{V}}(\mathcal{S}))$ is NP-hard.

We continue by defining a duality operation on $\mathcal{V}$.

$R =$ b					
	b	s	d	f	a
b	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)
s	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
d	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
f	(b)	(b)	(b)	(b)	(b s d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ s					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b)	(s)	(d f a)	(d f a)	(d f a)
d	(b)	(b)	(b s d f a)	(d f a)	(d f a)
f	(b)	(b)	(b)	(s)	(d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ d					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b s d)	(d)	(d f a)	(d f a)	(d f a)
d	(b s d)	(b s d)	(b s d f a)	(d f a)	(d f a)
f	(b s d)	(b s d)	(b s d)	(d)	(d f a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ f					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d)	(f)	(a)	(a)	(a)
d	(b s d)	(b s d)	(b s d f a)	(a)	(a)
f	(b s d)	(b s d)	(b s d)	(f)	(a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ a					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d f a)	(a)	(a)	(a)	(a)
d	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
f	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
a	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)

Table 2: 3-composition in the $\mathcal{V}$-algebra. Each subtable represents one of the five possible choices of the R relation. Within each subtable, the vertical axis represents the S relation and the horizontal axis the T relation.

Definition 2.4 Let $R \in \mathcal{V}$. Define $\mathcal{D}_{\mathcal{V}}(R)$ as the set $\{\beta(r) \mid r \in R\}$ where $\beta(r)$ is defined as follows:

- $\beta(\mathtt{b}) = \mathtt{a}$;
- $\beta(\mathtt{s}) = \mathtt{f}$;
- $\beta(\mathtt{d}) = \mathtt{d}$;
- $\beta(\mathtt{f}) = \mathtt{s}$;
- $\beta(\mathtt{a}) = \mathtt{b}$.

Let $\mathcal{S} \subseteq \mathcal{V}$. Define $\mathcal{D}_{\mathcal{V}}(\mathcal{S})$ as the set $\{\mathcal{D}_{\mathcal{V}}(R) \mid R \in \mathcal{S}\}$.

Lemma 2.5 Let $\mathcal{S} \subseteq \mathcal{V}$. There is a polynomial-time reduction from $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) to to $\mathcal{V}$-SAT($\mathcal{S}$) and vice versa.

Proof: We show the reduction from $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) to $\mathcal{V}$-SAT($\mathcal{S}$); the other reduction is analogous. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}\mathcal{S}$) problem. Let $\Theta' = \{p\mathcal{D}_{\mathcal{V}}(R)I \mid pRI \in \Theta\}$. Obviously, Θ' is an instance of the $\mathcal{V}$-SAT(S) problem. Assume Θ has a $\mathcal{V}$-model $\Im$. Construct a new $\mathcal{V}$-interpretation $\Im'$ as follows:

$\Im'(p) = -\Im(p)$ for each point p appearing in Θ;

$\Im'(I^-) = -\Im(I^+)$ and $\Im'(I^+) = -\Im(I^-)$ for each interval I appearing in Θ.

It is not hard to see that $\Im'$ is a $\mathcal{V}$-model of Θ'. Showing the converse direction is analogous. $\Box$

Corollary 2.6 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT($\mathcal{S}$) is in P iff $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) is in P. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-hard iff $\mathcal{V}$-SAT($\mathcal{D}_{\mathcal{V}}(\mathcal{S})$) is NP-hard.

3 Classification of $\mathcal{V}$

We begin this section by defining five subalgebras of the point-interval algebra having a polynomial-time $\mathcal{V}$-SAT problem. Later on, we show that these algebras are the only maximal subalgebras of $\mathcal{V}$ with this property. Before we can define the algebras, we need a definition concerning the point algebra.

Definition 3.1 A *point algebra (PA) formula* is an expression of the form xry where r is a member of $\{<, \leq, =, \neq, \geq, >, \bot, \top\}$ and x, y denote real-valued variables. The symbols $<, \leq, =, \neq, \geq, >$ denote the relations "strictly less than", "less than" and so on. The symbol $\bot$ denotes the relation $\varnothing$ which is unsatisfiable for every choice of $x, y \in \mathbb{R}$ and $\top$ denotes the relation $\mathbb{R} \times \mathbb{R}$ which is satisfiable for every choice of $x, y \in \mathbb{R}$.

Let Ω be a set of PA formulae and X the set of variables appearing in Ω. An assignment of real values to the variables in X is said to be a *PA-interpretation* of Ω. Furthermore, Ω is *PA-satisfiable* iff there exists a PA-interpretation $\Im$ such that for each formula $xry \in \Omega$, $\Im(x) r \Im(y)$ holds. Such a PA-interpretation $\Im$ is said to be a *PA-model* of Ω.

The first point-interval subalgebra we will consider has a very close connection to PA.

Definition 3.2 The set $\mathcal{V}^{23}$ consists of those relations in $\mathcal{V}$ that can be expressed as one or more PA formulae over points and endpoints of intervals.

Alternatively, $\mathcal{V}^{23}$ can be characterized in the following two ways:

$$\mathcal{V}^{23} = \mathcal{C}_{\mathcal{V}}(\{(\mathtt{s}), (\mathtt{f}\ \mathtt{a}), (\mathtt{b}\ \mathtt{d}\ \mathtt{f})\}); \text{ or}$$

$$\mathcal{V}^{23} = \{r \in \mathcal{V} \mid (\mathtt{d}) \subseteq r \text{ or } r \subseteq (\mathtt{b}\ \mathtt{s}) \text{ or } r \subseteq (\mathtt{f}\ \mathtt{a})\}.$$

The other four subalgebras are defined with the aid of $\mathcal{C}_{\mathcal{V}}$ and $\mathcal{D}_{\mathcal{V}}$.

Definition 3.3

$$v^{20}_{\mathtt{s}} = \{(\mathtt{s}), (\mathtt{b}\ \mathtt{s}), (\mathtt{b}\ \mathtt{a}), (\mathtt{b}\ \mathtt{s}\ \mathtt{f}\ \mathtt{a}), (\mathtt{b}\ \mathtt{s}\ \mathtt{d}\ \mathtt{a})\},$$

$$\mathcal{V}^{20}_{\mathtt{s}} = \mathcal{C}_{\mathcal{V}}(v^{20}_{\mathtt{s}})$$

$$\mathcal{V}^{20}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{20}_{\mathtt{s}})$$

$$\mathcal{V}^{17}_{\mathtt{s}} = \{\bot\} \cup \{r \in \mathcal{V} \mid (\mathtt{s}) \subseteq r\}$$

$$\mathcal{V}^{17}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{17}_{\mathtt{s}})$$

Given a subalgebra $\mathcal{V}^{x}_{y}$, x denotes the number of relations in the algebra. Let $\mathcal{V}_P$ be the set of all subalgebras in Definition 3.3 together with the algebra $\mathcal{V}^{23}$. The relations included in each of these algebras can be found in Table 3. Further, let $\mathcal{V}_{NP}$ denote the set of subalgebras listed in Table 4. We have the following theorem.

Theorem 3.4 If $V \in \mathcal{V}_P$ then $\mathcal{V}$-SAT(V) is in P. If $V \in \mathcal{V}_{NP}$ then $\mathcal{V}$-SAT(V) is NP-hard.

Proof: See Appendices A and B for the results concerning $\mathcal{V}_P$ and $\mathcal{V}_{NP}$, respectively. $\square$

The main theorem can now be stated.

Theorem 3.5 For $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{V}$-SAT$(\mathcal{S})$ is in P iff $\mathcal{S}$ is a subset of some member of $\mathcal{V}_P$. Otherwise, $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-complete.

Proof: *if:* For each $C \in \mathcal{V}_P$, $\mathcal{V}$-SAT(C) is in P by Theorem 3.4.
only-if: Choose $\mathcal{S} \subseteq \mathcal{V}$ such that $\mathcal{S}$ is not a subset of any algebra in $\mathcal{V}_P$. For each subalgebra C in $\mathcal{V}_P$, choose a relation x such that $x \in \mathcal{S}$ and $x \notin C$. This can always be done since $\mathcal{S} \not\subseteq C$. Let X be the set of these relations. We make three observations about X:

1. $|X| \leq 5$;
2. X is not a subset of any algebra in $\mathcal{V}_P$;
3. $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard if $\mathcal{V}$-SAT(X) is NP-hard since $X \subseteq \mathcal{S}$.

To show that $\mathcal{V}$-SAT$(\mathcal{S})$ has to be NP-hard, a machine-assisted case analysis of the following form was performed:

1. Generate all subsets of $\mathcal{V}$ of size ≤ 5. There are
$$\sum_{i=0}^{5} \binom{32}{i} \approx 2.4 \times 10^5$$
such subsets.
2. Let $\mathcal{T}$ be such a set. Test: $\mathcal{T}$ is a subset of some subalgebra in $\mathcal{V}_P$ or $D \subseteq \mathcal{C}_{\mathcal{V}}(\mathcal{T})$ for some $D \in \mathcal{V}_{NP}$.

The test succeeds for all $\mathcal{T}$. Hence, $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard. Since $\mathcal{V}$-SAT is in NP [Meiri, 1996], NP-completeness follows. $\square$

	$\mathcal{V}^{23}$	$\mathcal{V}^{20}_{\mathtt{s}}$	$\mathcal{V}^{20}_{\mathtt{f}}$	$\mathcal{V}^{17}_{\mathtt{s}}$	$\mathcal{V}^{17}_{\mathtt{f}}$
$\perp$	•	•	•	•	•
(b)	•	•	•		
(s)	•	•		•	
(b s)	•	•	•	•	
(d)	•				
(b d)	•		•		
(s d)	•			•	
(b s d)	•		•	•	
(f)	•		•		•
(b f)			•		•
(s f)				•	•
(b s f)			•	•	•
(d f)	•				•
(b d f)	•		•		•
(s d f)	•			•	•
(b s d f)	•		•	•	•
(a)	•	•	•		
(b a)		•	•		
(s a)		•		•	
(b s a)		•	•	•	
(d a)	•	•			
(b d a)	•	•	•		
(s d a)	•	•		•	
(b s d a)	•	•	•	•	
(f a)	•	•	•		•
(b f a)		•	•		•
(s f a)		•		•	•
(b s f a)		•	•	•	•
(d f a)	•	•			•
(b d f a)	•	•	•		•
(s d f a)	•	•		•	•
$\top$	•	•	•	•	•

Table 3: The maximal subalgebras of $\mathcal{V}$ which have a polynomial-time satisfiability problem.

Subclass	Relations	Proof of NP-hardness
A_0	$(\mathtt{d}), (\mathtt{b\ a})$	Lemma 2.6
A_1	$(\mathtt{d}), (\mathtt{b\ s\ a})$	Lemma 2.7
A_2	$(\mathtt{d}), (\mathtt{b\ f\ a})$	Lemma 2.7
A_3	$(\mathtt{s\ d}), (\mathtt{b\ a})$	Lemma 2.7
A_4	$(\mathtt{s\ d}), (\mathtt{b\ f\ a})$	Lemma 2.7
A_5	$(\mathtt{d\ f}), (\mathtt{b\ a})$	Lemma 2.7
A_6	$(\mathtt{d\ f}), (\mathtt{b\ s\ a})$	Lemma 2.7
A_7	$(\mathtt{s\ d\ f}), (\mathtt{b\ a})$	Lemma 2.7
A_8	$(\mathtt{d}), (\mathtt{b\ s\ f\ a})$	Lemma 2.7
B_0	$(\mathtt{d}), (\mathtt{b\ f})$	Lemma 2.8
B_0^D	$(\mathtt{d}), (\mathtt{s\ a})$	$B_0^D = \mathcal{D}_{\mathcal{V}}(B_0)$
B_1	$(\mathtt{d}), (\mathtt{b\ s\ f})$	Lemma 2.9
B_1^D	$(\mathtt{d}), (\mathtt{s\ f\ a})$	$B_1^D = \mathcal{D}_{\mathcal{V}}(B_1)$
B_2	$(\mathtt{s\ d}), (\mathtt{b\ f})$	Lemma 2.9
B_2^D	$(\mathtt{d\ f}), (\mathtt{s\ a})$	$B_2^D = \mathcal{D}_{\mathcal{V}}(B_2)$
B_3	$(\mathtt{d\ a}), (\mathtt{b\ f})$	Lemma 2.10
B_3^D	$(\mathtt{b\ d}), (\mathtt{s\ a})$	$B_3^D = \mathcal{D}_{\mathcal{V}}(B_3)$
B_4	$(\mathtt{s\ d\ a}), (\mathtt{b\ f})$	Lemma 2.11
B_4^D	$(\mathtt{b\ d\ f}), (\mathtt{s\ a})$	$B_4^D = \mathcal{D}_{\mathcal{V}}(B_4)$
B_5	$(\mathtt{d\ a}), (\mathtt{b\ s\ f})$	Lemma 2.11
B_5^D	$(\mathtt{b\ d}), (\mathtt{s\ f\ a})$	$B_5^D = \mathcal{D}_{\mathcal{V}}(B_5)$
B_6	$(\mathtt{b\ f}), (\mathtt{s\ a})$	Lemma 2.12
C_0	$(\mathtt{s}), (\mathtt{b\ f})$	Lemma 2.13
C_0^D	$(\mathtt{f}), (\mathtt{s\ a})$	$C_0^D = \mathcal{D}_{\mathcal{V}}(C_0)$
D_0	$(\mathtt{s\ f}), (\mathtt{b\ d\ a})$	Lemma 2.14
D_1	$(\mathtt{s\ f}), (\mathtt{b\ a})$	Corollary 2.15
D_2	$(\mathtt{s\ f}), (\mathtt{b\ d})$	Corollary 2.15
D_3	$(\mathtt{s\ f}), (\mathtt{d\ a})$	Corollary 2.15
D_4	$(\mathtt{s\ f}), (\mathtt{d})$	Corollary 2.15

Table 4: NP-hard subclasses of $\mathcal{V}$.

4 Conclusion

We have studied computational properties of the point-interval algebra. All of the 2^{32} possible subclasses are classified with respect to whether their corresponding satisfiability problem is in P or NP-complete. The classification reveals that there are exactly five maximal subclasses having a polynomial-time satisfiability problem.

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Appendix

This appendix collects the complexity results needed for the proof of Theorem 3.5. The proofs of membership in P can be found in part A and the NP-hardness results in part B.

1 Polynomial-Time Problems

Proving that $\mathcal{V}$-SAT($\mathcal{V}^{23}$) $\in$ P is straightforward.

Proposition 1.1 Deciding satisfiability of a set of PA formulae is a polynomial-time problem.

Proof: See [Vilain *et al.*, 1989]. □

Lemma 1.2 $\mathcal{V}$-SAT($\mathcal{V}^{23}$) is in P.

Proof: Follows immediately from the definition of $\mathcal{V}^{23}$ and the previous proposition. □

Before we can show that the other algebras in Table 3 have polynomial-time satisfiability problems, we need an auxiliary definition.

Definition 1.3 Let $S \subseteq \mathbb{R}$ be finite and denote the absolute value of x with abs(x). The *minimal distance in* S, MD(S), is defined as

$$\min\{\mathrm{abs}(x-y) \mid x, y \in S \text{ and } x \neq y\}.$$

Observe that $|S| \geq 2$ in order to make MD(S) defined. For all such S, MD(S) > 0. The definition of minimal distance can be extended to $\mathcal{V}$-interpretations in the following way: Let $\Im$ be a $\mathcal{V}$-interpretation that assigns values to a set of points $\mathcal{P}$ and a set of intervals $\mathcal{I}$. Let

$$\mathrm{MD}(\Im) = \mathrm{MD}(\{\Im(p) \mid p \in \mathcal{P}\} \cup \{\Im(I^-), \Im(I^+) \mid I \in \mathcal{I}\}).$$

Lemma 1.4 $\mathcal{V}$-SAT($\mathcal{V}^{20}_{\mathsf{s}}$) is in P.

Proof: By Lemma 2.3, it is sufficient to show that $\mathcal{V}$-SAT($v^{20}_{\mathtt{s}}$) is in P. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($v^{20}_{\mathtt{s}}$). Define the function $\sigma : v^{20}_{\mathtt{s}} \to \{\leq, =, \neq, \top\}$ as follows: $\sigma((\mathtt{s})) =$ "$=$", $\sigma((\mathtt{b\ s})) =$ "$\leq$", $\sigma((\mathtt{b\ a})) =$ "$\neq$", $\sigma((\mathtt{b\ s\ d\ a})) =$ "$\top$" and $\sigma((\mathtt{b\ s\ f\ a})) =$ "$\top$".

Let $\Theta' = \{x_p \sigma(R) y_I | pRI \in \Theta\}$. In the sequel, y_I will be considered as the starting point of the interval I. By Proposition 1.1, it is a polynomial-time problem to decide PA-satisfiability of Θ'. We show that Θ is $\mathcal{V}$-satisfiable iff Θ' is PA-satisfiable.
if: Let $\Im'$ be a PA-model of Θ'. Let $\varepsilon = \mathrm{MD}(\Im')/2$. We construct an $\mathcal{V}$-interpretation $\Im$ of Θ as follows:

- For each variable x_p in Θ' let $\Im(p) = \Im'(x_p)$.
- For each variable y_I in Θ' let $\Im(I^-) = \Im'(y_I)$ and $\Im(I^+) = \Im'(y_I) + \varepsilon$

We show that $\Im$ is an $\mathcal{V}$-model of Θ. Let pRI be an arbitrary formula in Θ. We have five different cases.

1. $R = (\mathtt{s})$. Then $\Im'(x_p)\sigma((\mathtt{s}))\Im'(y_I) \Leftrightarrow \Im'(x_p) = \Im'(y_I) \Leftrightarrow \Im(p) = \Im(I^-)$. Consequently, $p(\mathtt{s})I$ under $\Im$.
2. $R = (\mathtt{b\ s})$. Then $\Im'(x_p)\sigma((\mathtt{b\ s}))\Im'(y_I) \Leftrightarrow \Im'(x_p) \leq \Im'(y_I) \Leftrightarrow \Im(p) \leq \Im(I^-)$. Consequently, $p(\mathtt{b\ s})I$ under $\Im$.
3. $R = (\mathtt{b\ a})$. Then $\Im'(x_p)\sigma((\mathtt{b\ a}))\Im'(y_I) \Leftrightarrow \Im'(x_p) \neq \Im'(y_I) \Leftrightarrow \Im(p) \neq \Im(I^-)$. If $\Im(p) < \Im(I^-)$ then $p(\mathtt{b})I$ under $\Im$. If $\Im(p) > \Im(I^-)$ then $\Im(p) > \Im(I^+)$ since $I^+ = I^- + \varepsilon$. In this case $p(\mathtt{a})I$ under $\Im$. Consequently, $p(\mathtt{b\ a})I$ under $\Im$.
4. $R = (\mathtt{b\ s\ f\ a})$. Assume $p(\mathtt{d})I$ under $\Im$. Then $\Im(I^-) < \Im(p) < \Im(I^+) \Leftrightarrow \Im(I^-) < \Im(p) < \Im(I^-) + \varepsilon \Leftrightarrow \Im'(y_I) < \Im'(x_p) < \Im'(y_I) + \varepsilon$ which contradicts the choice of ε. Hence, $p(\mathtt{b\ s\ f\ a})I$ under $\Im$.
5. $R = (\mathtt{b\ s\ d\ a})$. This case is analogous to the previous case.

only-if: Let $\Im$ be an $\mathcal{V}$-model of Θ. We construct a PA-interpretation $\Im'$ of Θ' as follows:

- For each point p in Θ let $\Im'(x_p) = \Im(p)$.
- For each interval I in Θ let $\Im'(y_I) = \Im(I^-)$.

Next, we show that $\Im'$ is a PA-model of Θ'. Let $x_p R y_I$ be an arbitrary formula in Θ'. We have four different cases:

1. $R =$ "$=$". Assume to the contrary that $\Im'(x_p) \neq \Im'(y_I)$. Then $\Im(p) \neq \Im(I^-)$ and $p(\texttt{b d f a})I$ under $\Im$. But $x_p = y_I \in \Theta'$ iff $p(\texttt{s})I \in \Theta$ which leads to a contradiction. Hence, $x_p = y_I$ under $\Im'$.

2. $R =$ "$\neq$". This case is analogous to the previous case.

3. $R =$ "$\leq$". Assume $\Im'(x_p) > \Im'(y_I)$. Then $\Im(p) > \Im(I^-)$ and $p(\texttt{d f a})I$ under $\Im$. But $x_p \leq y_I \in \Theta'$ iff $p(\texttt{b s})I \in \Theta$ which leads to a contradiction. Hence, $x_p \leq y_I$ under $\Im'$.

4. $R =$ "$\top$". This relation holds trivially in every PA-model of Θ'. □

Corollary 1.5 $\mathcal{V}$-SAT$(\mathcal{V}^{20}_{\texttt{f}})$ is in P.

Proof: $\mathcal{V}^{20}_{\texttt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{20}_{\texttt{s}})$. □

Lemma 1.6 $\mathcal{V}$-SAT$(\mathcal{V}^{17}_{\texttt{s}})$ is in P.

Proof: Let Θ be an arbitrary instance of $\mathcal{V}$-SAT$(\mathcal{V}^{17}_{\texttt{s}})$. If a formula of the form $p \perp I$ is in Θ then Θ is not satisfiable. Otherwise, consider the following $\mathcal{V}$-interpretation: $\Im(p) = 0$ for every point p and $\Im(I^-) = 0$ and $\Im(I^+) = 1$ for every interval I. Let pRI be an arbitrary formula in Θ. By the definition of $\mathcal{V}^{17}_{\texttt{s}}$, $\texttt{s} \in R$. Obviously, $\Im$ satisfies pRI. Since it is a polynomial-time problem to check whether $p \perp I \in \Theta$ or not, the lemma follows. □

Corollary 1.7 $\mathcal{V}$-SAT$(\mathcal{V}^{17}_{\texttt{f}})$ is in P.

Proof: $\mathcal{V}^{17}_{\texttt{s}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{17}_{\texttt{f}})$. □

2 NP-Hardness Results

This section provides NP-hardness proofs for the subclasses of $\mathcal{V}$ presented in Table 4. The reductions are mostly made from different subalgebras of Allen's interval algebra. Consequently, we begin this section by recapitulating some results concerning Allen's algebra. To make the proofs of NP-hardness less cumbersome, we will employ a technique which we refer to as *model transformations*; the definitions and results needed are collected in Section B.2.

Basic relation	Symbol	Example	Endpoint relations
X before Y	$\prec$	xxx	$X^- < Y^-$, $X^- < Y^+$,
Y after X	$\succ$	yyy	$X^+ < Y^-$, $X^+ < Y^+$
X meets Y	m	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y met-by X	$\mathsf{m}^{\smile}$	yyyy	$X^+ = Y^-$, $X^+ < Y^+$
X overlaps Y	o	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y overlapped-by X	$\mathsf{o}^{\smile}$	yyyy	$X^+ > Y^-$, $X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y includes X	$\mathsf{d}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$, $X^- < Y^+$,
Y started by X	$\mathsf{s}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X finishes Y	f	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y finished-by X	$\mathsf{f}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^- < Y^+$
X equals Y	$\equiv$	xxxx	$X^- = Y^-$, $X^- < Y^+$,
		yyyy	$X^+ > Y^-$, $X^+ = Y^+$

Table 5: The thirteen basic relations of the $\mathcal{A}$-algebra. The endpoint relations $X^- < X^+$ and $Y^- < Y^+$ that are valid for all relations have been omitted.

2.1 Allen's Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals.* An interval X is represented as an ordered pair $\langle X^-, X^+\rangle$ of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations.* Their exact definitions can be found in Table 5. Such disjunctions are represented as sets of basic relations. The algebra is provided with the operations of *converse, intersection and composition* on intervals. The definitions of these operations can be found in [Allen, 1983]. By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. The reasoning problem we will consider is the problem of *satisfiability* ($ISAT$) of a set of interval variables with relations between them, *i.e.* deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals hold. Such an assignment is said to be a *$\mathcal{A}$-model* for the interval variables and relations. For $\mathcal{A}$, we have the following result.

Theorem 2.1 Define the sets $\mathcal{N}_3$ and $\mathcal{F}$ as follows:

$$\mathcal{N}_3 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^\smile)\}$$

and

$$\mathcal{F} = \{(\equiv\ \mathsf{s}\ \mathsf{s}^\smile), (\prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{f}^\smile)\}.$$

$ISAT(\mathcal{N}_3)$ and $ISAT(\mathcal{F})$ are NP-hard problems.

Proof: The result for $\mathcal{N}_3$ can be found in [Drakengren and Jonsson, 1997b].

To prove that $ISAT(\mathcal{F})$ is NP-hard, we will use a closure operation for Allen's algebra which was defined by Nebel and Bürckert [Nebel and Bürckert, 1995]. Assume $\mathcal{S} \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{S})$ the $\mathcal{A}$-*closure* of $\mathcal{S}$ under converse, intersection and composition, *i.e.* the least subset of $\mathcal{A}$ containing $\mathcal{S}$ closed under the three operations. Nebel and Bürckert [Nebel and Bürckert, 1995] have shown that $\mathcal{A}$-SAT($\mathcal{S}$) is NP-hard if and only if $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is. Hence, to show NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$), we can study $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{F})$) instead of $\mathcal{A}$-SAT($\mathcal{F}$).

Now, consider the following set of relations:

$$\Delta_0 = \{(\equiv\ \mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile), (\prec\ \succ)\}.$$

Golumbic and Shamir [Golumbic and Shamir, 1993] have shown that $\mathcal{A}$-SAT(Δ_0) is NP-complete. It can be verified that Δ_0 is a subset of $\mathcal{C}_{\mathcal{A}}(\mathcal{F})$ and NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$) follows. $\Box$

In the previous lemma, $\mathcal{C}_{\mathcal{A}}$ was computed by the utility `aclose` [Nebel and Bürckert, 1993].

2.2 Model Transformations

One of our main vehicles for showing NP-hardness of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem.

Definition 2.2 A *model transformation* is a mapping on $\mathcal{V}$-interpretations.

This definition is very general. To make it applicable in practice, we need a way of describing model transformations.

Definition 2.3 Let T be a model transformation. A function $f_T : \mathbf{B} \to 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{V}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and pbJ under $\Im$ then $pf_T(b)J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$.

Lemma 2.4 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{V}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{V}$ be such that $\mathcal{V}$-SAT($\mathcal{R}$) is NP-hard and $r_k \subseteq r'_k$, $1 \leq k \leq n$. If there exists a model transformation T with a description f_T such that $f_T(r'_k) \subseteq r_k$ for every $1 \leq k \leq n$ then $\mathcal{V}$-SAT($\mathcal{R}'$) is NP-hard.

Proof: Arbitrarily choose an instance Θ of $\mathcal{V}$-SAT($\mathcal{R}$) and let $\Theta' = \{pr'I \mid prI \in \Theta\}$. Obviously, this is a polynomial-time transformation and Θ' is an instance of $\mathcal{V}$-SAT($\mathcal{R}'$). We show that Θ is satisfiable iff Θ' is satisfiable.
only-if: Let $\Im$ be a model of Θ. Recall that $r_k \subseteq r'_k$, $1 \leq k \leq n$. Hence, $\Im$ is a model of Θ' since every formula $pr'I \in \Theta'$ is weaker than the corresponding formula $prI \in \Theta$.
if: Let $\Im'$ be a model of Θ'. We show that $T(\Im')$ is a model of Θ. Arbitrarily choose a formula prI in Θ. By the construction of Θ', there exists a formula $pr'I \in \Theta'$. Thus, we have $pr'I$ under $\Im'$ which implies $pf_T(r')I$ under $T(\Im')$ since f_T is a description of T. Furthermore, $f_T(r') \subseteq r$ so prI holds under $T(\Im')$. Since prI was arbitrarily chosen, $T(\Im')$ is a model of Θ. □

Definition 2.5 Let $\Im$ be an arbitrary $\mathcal{V}$-interpretation and let $\varepsilon = \mathrm{MD}(\Im)/2$. Define the model transformation $T_{o,o'}$ for $o, o' \in \{+, -, 0\}$ as

1. for each point p let $T_{o,o'}(\Im)(p) = p$; and
2. for each interval I let $T_{o,o'}(\Im)(I^-) = I^- o \varepsilon$ if $o \in \{+, -\}$ and $T_{o,o'}(\Im)(I^-) = I^-$ otherwise; and
3. for each interval I let $T_{o,o'}(\Im)(I^+) = I^+ \ o' \ \varepsilon$ if $o' \in \{+, -\}$ and $T_{o,o'}(\Im)(I^+) = I^+$ otherwise.

Descriptions of the model transformations in the previous definition can be found in Table 6.

2.3 NP-Hard Subclasses of $\mathcal{V}$

Lemma 2.6 $\mathcal{V}$-SAT(A_0) is NP-hard.

	b	s	d	f	a
$T_{-,-}$	b	d	d	a	a
$T_{-,0}$	b	d	d	f	a
$T_{-,+}$	b	d	d	d	a
$T_{0,-}$	b	s	d	a	a
$T_{0,0}$	b	s	d	f	a
$T_{0,+}$	b	s	d	d	a
$T_{+,-}$	b	b	d	a	a
$T_{+,0}$	b	b	d	f	a
$T_{+,+}$	b	b	d	d	a

Table 6: Descriptions of some model transformations.

Proof: Reduction from $\mathcal{A}$-SAT$(\mathcal{N}_3)$ which is NP-hard by Lemma 2.1. Let I and J be two intervals. We will show how to express $I(\mathtt{o}\ \mathtt{o}^{\smile})J$ and $I(\prec\ \succ)J$ in the point-interval algebra by only using the point-interval relations in A_0, *i.e.*, $(\mathtt{d})$ and $(\mathtt{b}\ \mathtt{a})$. By doing so, we have shown NP-hardness of $\mathcal{V}$-SAT(A_0).

Assume we want to relate the intervals I and J with the relation $(\mathtt{o}\ \mathtt{o}^{\smile})$. Observe that $I(\mathtt{o}\ \mathtt{o}^{\smile})J$ holds iff there exist three real numbers a, b, c with the following properties:

$I^- < a < I^+$ but $a < J^-$ or $a > J^+$

$I^- < b < I^+$ and $J^- < b < J^+$

$J^- < c < J^+$ but $c < I^-$ or $c > I^+$

Obviously, we can relate I and J with $(\mathtt{o}\ \mathtt{o}^{\smile})$ by introducing three fresh points a, b, c and the following six point-interval formulae:

$$\begin{array}{ll} a(\mathtt{d})I & a(\mathtt{b}\ \mathtt{a})J \\ b(\mathtt{d})I & b(\mathtt{d})J \\ c(\mathtt{d})J & c(\mathtt{b}\ \mathtt{a})I. \end{array}$$

In order to relate two intervals I, J with the relation $(\prec\ \succ)$, we introduce two fresh intervals K, L and relate them with $(\mathtt{o}\ \mathtt{o}^{\smile})$ by using three fresh points a, b and c as above. Observe that either $\Im(a) < \Im(b) < \Im(c)$ or $\Im(c) < \Im(b) < \Im(a)$ in any model $\Im$.

Now, relate a, b, c to I, J as follows:

$$\begin{array}{ll} a(\mathtt{d})I & a(\mathtt{b}\ \mathtt{a})J \\ b(\mathtt{b}\ \mathtt{a})I & b(\mathtt{b}\ \mathtt{a})J \\ c(\mathtt{b}\ \mathtt{a})I & c(\mathtt{d})J. \end{array}$$

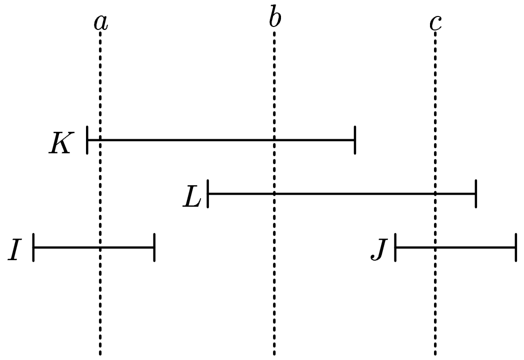

Figure 1: The construction of $I(\prec\ \succ)J$ in the proof of Lemma 2.6. The figure shows the situation when $\Im(I^+) < \Im(b) < \Im(J^-)$; the other case is analogous.

Let $\Im$ be a model satisfying these restrictions. It follows that either $\Im(I^+) < \Im(b) < \Im(J^-)$ or $\Im(J^+) < \Im(b) < \Im(I^-)$ and, consequently, $I(\prec\ \succ)J$. An explaination of this construction can be found in Figure 1. □

Lemma 2.7 $\mathcal{V}$-SAT(A_i), $1 \le i \le 8$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(A_0). Use the following model transformations

$$\begin{array}{ll} i=1: T_{+,0} & i=2: T_{0,-} \\ i=3: T_{-,0} & i=4: T_{-,-} \\ i=5: T_{0,+} & i=6: T_{+,+} \\ i=7: T_{-,+} & i=8: T_{+,-} \end{array}$$

and apply Lemma 2.4. □

Lemma 2.8 $\mathcal{V}$-SAT(B_0) is NP-hard.

Proof sketch: The proof of Lemma 2.6 goes through if the relation (`b a`) is replaced by (`b f`). □

Lemma 2.9 $\mathcal{V}$-SAT(B_i), $1 \le i \le 2$, is NP-hard.

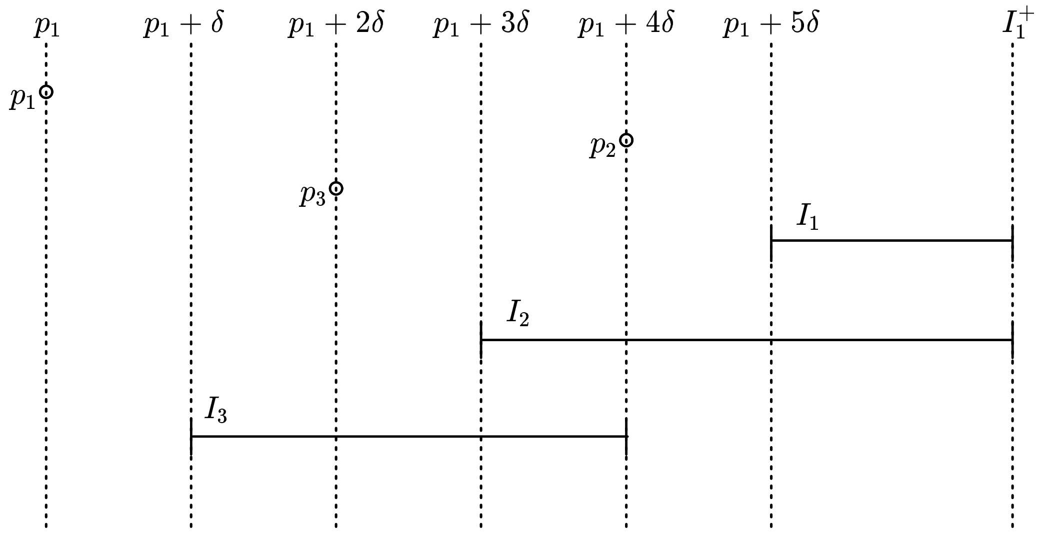

Figure 2: The construction used in Lemma 2.8.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_0). Use the following model transformations

$$i = 1 : T_{+,0} \qquad i = 2 : T_{-,0}$$

and apply Lemma 2.4. □

Lemma 2.10 $\mathcal{V}$-SAT(B_3) is NP-hard.

Proof: By Lemma 2.8, $\mathcal{V}$-SAT(B_0) is NP-hard. Let $E = \{(\mathtt{b}), (\mathtt{d\ a}), (\mathtt{b\ f})\}$. Then $(\mathtt{b\ f}) \otimes (\mathtt{d\ a}) \otimes (\mathtt{b}) = (\mathtt{b\ s\ d})$ and $(\mathtt{b\ s\ d}) \cap (\mathtt{d\ a}) = (\mathtt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem. We show how to construct an instance Θ' of the $\mathcal{V}$-SAT(B_3) problem that is satisfiable iff Θ is satisfiable.

We begin by showing how to relate a point p_1 and an interval I_1 such as $p_1(\mathtt{b})I_1$ by only using the relations in B_3. We introduce two fresh points p_2 and p_3 together with two fresh intervals I_2 and I_3. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\mathtt{b\ f})I_1 & (2)\ p_1(\mathtt{b\ f})I_2 & (3)\ p_1(\mathtt{b\ f})I_3 \\ (4)\ p_2(\mathtt{b\ f})I_1 & (5)\ p_2(\mathtt{d\ a})I_2 & (6)\ p_2(\mathtt{b\ f})I_3 \\ (7)\ p_3(\mathtt{b\ f})I_1 & (8)\ p_3(\mathtt{b\ f})I_2 & (9)\ p_3(\mathtt{d\ a})I_3. \end{array}$$

We denote this set of formulae with Ω. Assume that $\Im$ is a $\mathcal{V}$-model of Ω. For the sake of brevity, we identify the points and intervals with their values when interpreted by $\Im$. Hence, instead of writing $\Im(p_1) < \Im(I_1^-)$, we simply write $p_1 < I_1^-$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\delta = (I_1^- - p_1)/5$. Consider the following assignment of values:

$$\begin{array}{lll} I_3^- = p_1 + \delta & p_3 = p_1 + 2\delta & I_2^- = p_1 + 3\delta \\ p_2 = p_1 + 4\delta & I_3^+ = p_1 + 4\delta & I_2^+ = I_1^+. \end{array}$$

This assignment is depicted in Figure 2. Obviously, the assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. By formula (4), we can see that $p_2 < I_1^-$ or $p_2 = I_1^+$. By assumption, $p_1 = I_1^+$. Hence, either $p_2 < I_1^-$ or $p_2 = p_1$. If $p_2 = p_1$ then formula (2) is equivalent to $p_2(\texttt{b f})I_2$ which clearly contradicts formula (5). Thus, $p_2 < I_1^-$ and $p_2(\texttt{b})I_1$. By analogous reasoning one can see that $p_3 < I_1^-$ and $p_3(\texttt{b})I_1$.

Next, observe that formulae (2) and (3) imply $p_1 \leq I_2^+$ and $p_1 \leq I_3^+$. Furthermore, $p_2 < I_1^-$ and $p_3 < I_1^-$ which implies $p_2 < I_1^+$ and $p_3 < I_1^+$. By our initial assumption $p_1 = I_1^+$ we get

A: $p_2 < I_1^+ = p_1 \leq I_3^+$

B: $p_3 < I_1^+ = p_1 \leq I_2^+$

Consequently, $p_2 < I_3^+$ and $p_3 < I_2^+$. Observe that $p_2(\texttt{b f})I_3$ and $p_3(\texttt{b f})I_2$ by formulae (6) and (8), respectively. Hence, $p_2 < I_3^-$ and $p_3 < I_2^-$.

We have to study four cases, depending on how the $\mathcal{V}$-model assigns values to the variables in formula (5) and (9).

1. $I_2^- < p_2 < I_2^+$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < p_2$ which leads to a contradiction.

2. $I_2^+ < p_2$ and $I_3^+ < p_3$. Then $p_2 < I_3^- < I_3^+ < p_3 < I_2^- < I_2^+ < p_2$ which is a contradiction.

3. $I_2^+ < p_2$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < I_2^+ < p_2$. Contradiction.

4. $I_2^- < p_2 < I_2^+$ and $I_3^+ < p_3$. This case is analogous to the previous case.

Consequently, every $\mathcal{V}$-model of Ω satisfies $p_1 < I_1^-$.

We have thus shown how to express the relation $(\mathtt{b})$ by only using $(\mathtt{d\ a})$ and $(\mathtt{b\ f})$. Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_3) problem. NP-hardness of $\mathcal{V}$-SAT(B_3) follows immediately.

□

Lemma 2.11 $\mathcal{V}$-SAT(B_i), $4 \le i \le 5$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_3). Use the following model transformations

$$i = 4 : T_{-,0} \qquad i = 5 : T_{+,0}$$

and apply Lemma 2.4. □

Lemma 2.12 $\mathcal{V}$-SAT(B_6) is NP-hard.

Proof: Let $E = \{(\mathtt{b}), (\mathtt{b\ f}), (\mathtt{s\ a})\}$. Then $(\mathtt{b\ f}) \otimes (\mathtt{s\ a}) \otimes (\mathtt{b}) = (\mathtt{b\ s\ d})$, $(\mathtt{s\ a}) \otimes (\mathtt{b}) \otimes (\mathtt{s\ a}) = (\mathtt{d\ f\ a})$ and $(\mathtt{b\ s\ d}) \cap (\mathtt{d\ f\ a}) = (\mathtt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT(E). We show how to construct an instance Θ' of $\mathcal{V}$-SAT(B_6) that is satisfiable iff Θ is satisfiable.

We begin showing how to relate a point p_1 and an interval I_1 such as $p_1(\mathtt{b})I_1$ by only using the relations in B_6. We introduce two fresh points p_2 and p_3 together with a fresh interval I_2. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\mathtt{b\ f})I_1 & (2)\ p_1(\mathtt{b\ f})I_2 & (3)\ p_2(\mathtt{b\ f})I_1 \\ (4)\ p_2(\mathtt{s\ a})I_2 & (5)\ p_3(\mathtt{s\ a})I_1 & (6)\ p_3(\mathtt{b\ f})I_2 \end{array}$$

We denote this set of formulae with Ω. Let $\Im$ be a $\mathcal{V}$-model of Ω. As in the proof of Lemma 2.10, we identify the points and intervals with their value when interpreted by $\Im$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\Delta = (I_1^+ - I_1^-)/3$. Consider the following assignment of values:

$$p_3 = I_1^-$$

$$p_2 = I_1^+$$

$$I_2^- = I_1^+$$

$$I_2^+ = I_2^- + 1$$

It is not hard to see that this assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. We consider two cases:

Case 1: $p_1(\texttt{f})I_2$. Then $p_1 = I_1^+ = I_2^+$. Hence, if $p_2(\texttt{f})I_1$ then $p_2(\texttt{f})I_2$ which contradicts formula (4). Consequently, $p_2(\texttt{b})I_1$ and, by formula (4), $p_2(\texttt{s})I_2$. Obviously, $I_2^- < I_1^- < I_2^+$.

By formula (5), $p_3(\texttt{s a})I_1$. Assume $p_3(\texttt{s})I_1$. Then $p_3(\texttt{d})I_2$ which contradicts formula (6). Assume to the contrary that $p_3(\texttt{a})I_1$. But this implies $p_3(\texttt{a})I_2$ which also contradicts formula (6).

Case 2: $p_1(\texttt{b})I_2$. Since $p_1(\texttt{f})I_1$, I_1 and I_2 are disjoint intervals and I_1 is strictly before I_2. By formula (3), $p_2(\texttt{b f})I_1$. If $p_2(\texttt{b})I_1$ then $p_2(\texttt{b})I_2$ which contradicts formula (4). If $p_2(\texttt{f})I_1$ then $p_2(\texttt{b})I_2$ which, again, contradicts formula (4).

We have thus shown how to express the relation $(\texttt{b})$ by only using $(\texttt{b f})$ and $(\texttt{s a})$. Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_6) problem. NP-hardness of $\mathcal{V}$-SAT(B_6) follows immediately. $\square$

Lemma 2.13 $\mathcal{V}$-SAT(C_0) is NP-hard.

Proof: Reduction from $\mathcal{A}$-SAT$(\mathcal{F})$ which is NP-hard by Lemma 2.1. Let Θ be an instance of $\mathcal{A}$-SAT$(\mathcal{F})$. We construct a set Θ' as follows.

1. For each formula of the type $I(\prec\ \texttt{d}^\smile\ \texttt{o m m}^\smile\ \texttt{f}^\smile)J$ in Θ, introduce a new point $p_{I,J}$ and let $p_{I,J}(\texttt{s})I$ and $p_{I,J}(\texttt{b f})J$ in Θ';

2. For each formula of the type $I(\equiv\ \texttt{s s}^\smile)J$ in Θ, introduce a new point $q_{I,J}$ and let $q_{I,J}(\texttt{s})I$ and $q_{I,J}(\texttt{s})J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_0) problem. It is a routine verification to show that Θ is satisfiable iff Θ' is satisfiable. $\square$

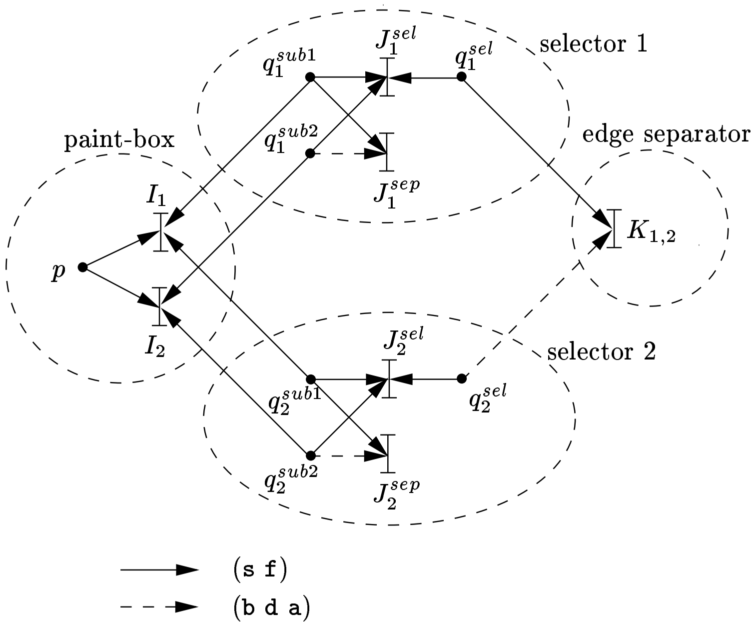

Figure 3: An example of the construction in the proof of Theorem 2.14 for a connected two-vertex graph.

Theorem 2.14 $\mathcal{V}$-SAT(D_0) is NP-hard.

Proof: Proof by reduction from GRAPH 3-COLOURABILITY, which is NP-complete [Garey and Johnson, 1979]. Let $G = \langle V, E \rangle$ be an undirected graph. Construct a corresponding set of point-interval formulae as follows (the construction is illustrated graphically in Figure 3 for the connected two-vertex graph $\langle \{v_1, v_2\}, \{\{v_1, v_2\}\} \rangle$).

First, we construct a *paint-box* defining our three "colours". It consists of a point p and two intervals I_1 and I_2, plus the two relations $p(\texttt{s f})I_1$ and $p(\texttt{s f})I_2$. Because of these constraints, the two intervals must have at least one end-point in common, so there is a total of two or three interval endpoints, which constitute our available colours. Hence, each model has a palette with two or three colours. We may, thus, occasionally get models corresponding to 2-colourings of G, but this is unimportant since every 2-colouring is a 3-colouring. Of course, the actual denotations of these colours differ between models, but this is also unimportant since they denote two or three different colours in each and every model.

Next, for each vertex $v_i \in V$ we introduce a *selector* consisting of two intervals J_i^{sel}, J_i^{sep} and three points q_i^{sub1}, q_i^{sub2} and q_i^{sel}. The interval J_i^{sep} acts as a *separator* for the points q_i^{sub1} and q_i^{sub2}. This means that by introducing the relations $q_i^{sub1}(\mathtt{s\ f})J_i^{sep}$ and $q_i^{sub2}(\mathtt{b\ d\ a})J_i^{sep}$ these two points are forced not to coincide. Further, both these points are related to the interval J_i^{sel} via the relations $q_i^{sub1}(\mathtt{s\ f})J_i^{sel}$ and $q_i^{sub2}(\mathtt{s\ f})J_i^{sel}$, which forces each endpoint of this interval to coincide with either of the two points. Then the point q_i^{sel} is related to interval J_i^{sel} as $q_i^{sel}(\mathtt{s\ f})J_i^{sel}$, which forces it to coincide with either of the two points q_i^{sub1} and q_i^{sub2}. Finally, this whole gadget is connected to the paint-box via the relations $q_i^{sub1}(\mathtt{s\ f})I_1$ and $q_i^{sub2}(\mathtt{s\ f})I_2$. Hence, these two points select a subpalette of exactly two of the available colours, and the point q_i^{sel} then further selects one of these two colours as the colour of vertex v_i.

Finally, for each edge $\{v_i, v_j\} \in E$ we introduce an interval $K_{i,j}$ acting as a separator for the points q_i^{sel} and q_j^{sel}, in the same way as the separator interval within the selectors. More precisely, the relations $q_i^{sel}(\mathtt{s\ f})K_{i,j}$ and $q_j^{sel}(\mathtt{b\ d\ a})K_{i,j}$ are introduced. Obviously, these two formulae can be simultaneously satisfied iff vertices v_i and v_j have different colours, since the two points q_i^0 and q_j^0 must not coincide.

It is obvious that G is 3-colourable iff the set of point-interval formulae just constructed is $\mathcal{V}$-satisfiable, so NP-hardness of the algebra follows. $\square$

Corollary 2.15 $\mathcal{V}$-SAT(D_i), $1 \leq i \leq 4$, is NP-hard.

Proof: Use the same construction as in the proof of Theorem 2.14. We note that relations of the type $(\mathtt{b\ d\ a})$ are used only in separators, forcing two points apart. Consider a separator interval I for two points p and q which are related as $p(\mathtt{s\ f})I$ and $q\ r\ I$, where $r = (\mathtt{b\ d\ a})$ in the original construction. We now consider how r can be constrained. First suppose that $p = q$. In this case at most one of the two relations must be satisfied, which holds as long as r does not contain either s or f. Conversely, suppose $p \neq q$. In this case, there must exist a choice of endpoints for I such that both formulae are satisfied. It is obviously a sufficient criterion that r contains either $\mathtt{d}$ or both $\mathtt{a}$ and $\mathtt{b}$ (either of the latter alone is not sufficient, since other constraints in the model may dictate whether $p < q$ or $q < p$). It follows that r can be chosen as either of the relations $(\mathtt{b\ a})$, $(\mathtt{b\ d})$, $(\mathtt{d\ a})$ and $(\mathtt{d})$, which proves the corollary. $\square$