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Protocol of the discussion during April, 1998 on

Ontologies of Time

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From: Jixin Ma on 1.4.1998

What follows is our response to the arguments about the ontology of time from Pat Hayes, Ray Reiter, and John McCarthy.

Response to John

The example of car accelerating demonstrates the need of time points for time ontology.

A similar example is throwing a ball up into the air. The motion of the ball can be modelled by a quantity space of three elements: going-up, stationary, and going-down. Intuitively, there are intervals for going up and going down. However, there is no interval, no matter how small, over which the ball is neither going up nor going down. The property of being stationary is naturally associated with a point, rather than any interval (including Allen and Hayes' moment), a "landmark" point which separates two other intervals.

Response to Ray

> *During all the years that the debate has raged about time points vs intervals, we devotees of the sitcalc have never seen it as an issue.*

It has already been an issue of the sitcalc. For instance, in Pinto and Reiter's 1995 paper, Reasoning about Time in the Situation Calculus, [j-amai-14-251], the concept of situation is extended to have a time span (an interval?) which is characterised by a starting time and an ending time (two points?). During the time span of a situation no fluents change truth value. Also, an action with duration is modelled with two instantaneous actions (start-action and end-action). So, all the debates about time points vs intervals apply here, and the Dividing Instant Problem still arises. In fact, all Ray's arguments show **again**, with his revised formulation of the sitcalc to accommodate temporal reasoning, either there is no way of expressing the claim that a proposition (like Light is on) is true or false at some time points, or one has to artificially take the unjustifiable semi-open interval solution. This is not surprise at all and is what one would expect, since the problem is actually there for the underlying time theory itself, and therefore would be there for any formalism which wants to support explicit time representation.

Response to Pat

> *...For example, the 1990 AIJ critique of Allen's account by Galton (wrongly) assumes that Allen's intervals are sets of points on the real line.*

After re-reading Galton's paper [j-aij-42-159], as we understand, Galton's arguments are in general based on the assumption that Allen's intervals are primitive, rather than sets of points on the real line. In fact, the main revision Galton proposes to Allen's theory is a diversification of the temporal ontology to include both intervals and points. That is, in Galton's revised theory, intervals are still taken as primitive. Having pointed out this, however, as shown in Ma, Knight and Petrides' 1994 paper [j-cj-37-114], Galton's determination to define points in terms of the "meeting places" of intervals does not, as he claims, axiomatise points on the same footing as intervals, and hence that some problems still remain in these revisions.

> *there does seem to be a simple, basic, account which can be extended in various ways to produce all the other alternatives, and this core theory is the one I was referring to.*

Does this core theory refer that one in which "intervals are uniquely defined by their endpoints (which are also the points they fit between) and two intervals meet just when the endpoint of the first is the startpoint of the second"? Or Allen's one? - It seems the former one.

Anyway, yes. There **does** seem to be such a simple, basic core theory. For general treatments, in Ma and Knight's CJ 94 paper [j-cj-37-114], a time theory is proposed (as an extention to Allen and Hayes' interval-based one) which takes both intervals and points as primitive on the same footing - neither intervals have to be constructed out of points, nor points have to be created as the places where intervals meet each other, or as some limiting construction of intervals. The temporal order is simply characterised in terms of a single relation "Meets" between intervals/points. Some advantages of this time theory are:

(1) It retains Allen's appealing characteristics of treating intervals as primitive which overcomes the Dividing Instant Problem.

(2) It includes time points into the temporal ontology and therefore makes it possible to express some instantaneous phenomenon, and adequate and convenient for reasoning correctly about continuous change.

(3) It is so basic that it can be specified in various ways to subsume others. For instance, one may simply take the set of points as empty to get Allen's interval time theory, or specify each interval, say T, as $\langle$*T-left*, *T-right*$\rangle$ where *T-left* $\leq$ *T-right*, to get that one Pat prefers.

> *The way I prefer, myself, is to say that propositions hold only during intervals, so that it is simply ill-formed to assert a proposition of a single point; but to allow the possibility of pointlike intervals, of the form $\langle t, t\rangle$, to be the temporal durations of propositions which are (intuitively) thought of as happening at a single 'point'. Or, put another way, some intervals may consist of just a single point, and some points may completely fill an interval. These pointlike intervals are the way that (this version of) the theory encodes the times when instantaneous truths hold. This doesnt require us to say that* every *point fills an interval, notice: since 'interval' is a basic predicate, it*

> *is perfectly consistent to say* $\neg interval(\langle t,t\rangle)$*; this would entail, for example, that nothing changed at that particular time. But it allows us to consider the proposition that a tossed ball's vertical velocity is zero, and assert that it is true at a single 'point', ie formally, that its interval of truth was pointlike. And since it is easy to characterise pointlike in the theory:*
>
> ```
> ((pointlike i) iff ((begin i) = (end i)))
> ```

Yes, it's true. **And**, it seems that, all these can be reached equivalently by simply taking pointlike interval $\langle t,t\rangle$ as identical with point t in the case where both intervals and points are included in the time ontology.

> *one can, for example, say something like*
>
> ```
> ((illuminated i) or (dark i)))  implies (not (pointlike i))
> ```
>
> *so that the light is neither on nor off* **at** *the switching point. In this theory, every proposition has a 'reference interval' during which it is true, and a proposition might not be true of subintervals of its reference interval. (Though some propositions might be. This kind of distinction has often been made in the linguistic literature. Note however that this intuition is basically incompatible with the idea that an interval is identical to the set of the points it contains.)*

This can be distinguished by applying *Holds_In* and *Holds_On* (that is Allen's Holds, see Galton's paper [j-aij-42-159]. Actually, as shown in Ma and Knight's 1996 paper [j-cj-37-114], to characterise the intuitive relationship between *Holds_On* and *Holds_In* , in the case where intervals are allowed, some extra axiom is needed.

Also, it seems that, in Pat's formulation, for expressing that interval $\langle a,b\rangle$ is a subinterval of interval $\langle c,d\rangle$, one would have $c \le a \le b \le d$. In this case, we get that pointlike intervals $\langle a,a\rangle$ and $\langle b,b\rangle$ (or equivalently, points a and b) **fall in** interval $\langle c,b\rangle$.

> *We certainly need something corresponding to 'points', I agree. I meant only that the formal theory can be crafted in the way Ive outlined above, or alternatively by identifying the pointlike intervals with their endpoints, and allowing a proposition to hold at a single point. This is in many ways more intuitively transparent but it is formally a bit more awkward, since pointlike isnt definable any more, and one has to put in special axioms forbidding points to meet each other. The 'reference interval' of a proposition could now be a single point in the theory. This is essentially the theory that Allen and I described in our 1985 paper* [c-ijcai-85-528]*, though it takes a little work to see it.*

On the one hand, many cases suggest the need of allowing a proposition to hold at a single point. For instance, see the example of throwing a ball up into the air described earlier in the response to John McCarthy.

On the other hand, allowing a proposition to holds at a single point doesn't necessarily make pointlike un-definable. It depends on if one would impose some extra constraints, such as

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((illuminated i) or (dark i)))  implies (not (pointlike i))
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as introduced by Pat for the light switching example, which actually leads to the assertion that the light is neither on nor off AT the switching point.

Actually, in the later version of Allen and Hayes's theory that appears in 1989 [j-ci-5-225], an awkward axiom is proposed to forbid moments to meet each other. It is interesting to note that, although moments are quite like points (moments are non-decomposable), they still have positive duration (they are not pointlike). Moments are included in Allen and Hayes' time ontology, while points are not. One of the reasons that such an axiom is awkward is that it doesn't catch the intuition in common-sense usage of time. In fact, in many applications, one would like to take some quantity as the basic unit of time. E.g., we may take a second as the basic unit. In other words, seconds are treated as moments - they cannot be decomposed into smaller units. However, for a given second, we may still want to express the next one, that is, a second can meet another second, although they are both non-decomposable.

True, and indeed the Allen relations only have their usual transitivity properties when applied to intervals which are nonpointlike and forward-oriented. Of course both these are properties expressible in the theory, so that the Allen transitivity relationships can be stated there, suitably qualified. (When the alternative extension axioms are added, the qualifications become tautologous.)

...BTW, the claim that "meets" and "before" being exclusive is "intuitive" depends on how one's intuition is formed. Part of what I learned by having to construct alternative formalisations is that intuition is very malleable. Having gotten used to pointlike intervals, I dont find this exclusivity condition at all intuitive.

If the exclusivity condition is not intuitive at all, as Pat's claims, then why is it proposed for Allen's relations applied to intervals? Is it simply because they can be easily defined as exclusive when applied to nonpointlike intervals? Why not when applied to pointlike ones? (Simply because it cannot be conveniently defined?). Actually, in the case where both intervals and points are treated as primitive on the same footing, it is straightforward to extend Allen's 13 exclusive temporal relations between intervals to govern both intervals and points, while without losing the property of exclusivity. Vilain's [c-aaai-82-197] and Ma and Knight's [j-cj-37-114] systems are two examples.

But these cases only make sense if one thinks of interval and points in the usual mathematical way, which is exactly what Im suggesting we don't need to do. We can get almost everything we need just from the ordering structure: we don't need to get all tied up in distinguishing cases which can only be formally stated by using all the machinery of real analysis.

The cases make sense **not** only if one thinks of intervals and points in the usual mathematical way. In fact, all the three cases are demonstrated under the assumption that both intervals and points are treated as primitive, rather than in the usual mathematical way.

Again I was careless in using the word "knowledge", sorry. I should have said: in order to answer the question whether the light is on or

> *off, one has to specify the interval with respect to which this question is posed. On this view, the truth or otherwise of a proposition is only meaningful with respect to certain intervals. I dont mean that the facts are determined by knowing more about the details of the interval, but that the question is a different question when asked about one interval than when asked about another, and for some intervals in may be simply meaningless. Is the light on or off at (exactly) 3.00 pm? The only way to answer this is to find a suitable non-pointlike interval of light or darkness completely surrounding 3.00 pm, because 'being on' is the kind of proposition that requires a nonpointlike reference interval.*

But it seems that there are also some other kind of proposition to which one cannot assign any nonpointlike reference interval. For instance, in the throwing ball up into the air example, proposition "the ball is stationary" can only be true at points, and for any point we cannot find any non-pointlike interval (completely) surrounding it over which the ball is stationary.

> *This has nothing to do with whether an interval is open or closed: in fact, there is no such distinction in this theory. It only arises in a much more complicated extension which includes set theory and an extensionality axiom for intervals.*

In the time theory where both intervals and points are taken as primitive, we can (if we like) talk about the open and closed nature of intervals with some **knowledge** being available. This kind of knowledge can be given in terms of the Meets relation, rather than some "much more complicated extension which includes set theory and an extensionality axiom for intervals". In fact, we can define that:

> *interval I is left-open at point P iff* $Meets(P, I)$
>
> *interval I is right-open at point P iff* $Meets(I, P)$
>
> *interval I is left-closed at point P iff there is an interval I' such that* $Meets(I', I) \wedge Meets(I', P)$
>
> *interval I is right-closed at point P iff there is an interval I' such that* $Meets(I, I') \wedge Meets(P, I')$

That's all, and it seems quite intuitive. For instance, with the knowledge $MEETS(P, I)$ which says that interval I is immediately after point P, one can intuitively reach that point P is on the left of interval I and P is not a part of I, and there is no other time element standing between P and I. Therefore, we say interval I is left-open at point P. Similarly, with knowledge $MEETS(I, I') \wedge MEETS(P, I')$, one can reach that point P is a part and the "finishing" part of interval I, that is interval I is "right"-closed at P.

It is important to note that the above definition about the open and closed nature of intervals is given in terms of only the knowledge of the *Meets* relation. However, if one would like to specify intervals as point-based ones, such a definition will be in agreement with the conventional definition about the open and closed intervals.

Jixin

References:

[c-aaai-82-197] Marc Vilain. *A System for Reasoning about Time.* Proc. AAAI National Conference on Artificial Intelligence, 1982, pp. 197-201.

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[j-aij-42-159] Anthony Galton. *A critical examination of Allen's theory of action and time.* Artificial Intelligence Journal, vol. 42 (1990), pp. 159-188.

[j-amai-14-251] Javier Pinto and Ray Reiter. *Reasoning about Time in the Situation Calculus.* Annals of Mathematics and Artificial Intelligence, vol. 14 (1995), pp. 251-268.

[j-ci-5-225] James F. Allen and Patrick J. Hayes. *Moments and points in an interval-based temporal logic.* Computational Intelligence, vol. 5, pp. 225-238.

[j-cj-37-114] Jixin Ma and Brian Knight. *A General Temporal Theory.* Computer Journal, vol. 37 (1994), pp. 114-123.

From: Pat Hayes on 1.4.1998

John McCarthy wrote:

> *McCarthy and Hayes 1969 paper* [n-mi-4-463] *used time as a fluent on situations, i.e. time(s). ...*

My chief motivation for this was the fact that the mapping from situations to times is many-one, since the sitcalc can distinguish different situations with the same clocktime. The differences between how precisely times are known, etc., can be handled by making the set of "times" obey different axioms. For example, in R-sitcalc the appropriate kind of time for a situation would be an interval of clocktimes, presumably; something like 'from 4.12 to 5.15'.

> *My previous message gave a reason for including time points in a theory of events and actions. The theory could be founded so as to regard them as degenerate intervals, but I don't see any advantage in that, although I suppose the idea stems from the fact that people and robots can't measure time precisely.*

Actually no. The motivation for introducing 'pointlike intervals' was just to maintain a certain expressive neatness, where all propositions, even instantaneous ones, are asserted w.r.t. a reference interval.

That idea - that intervals are approximations to points, and the length of an interval represents a degree of ignorance about the location of a point - gives a rather different ontology. In that case, for example, it doesnt really make sense to be able to refer to the precise endpoints or meeting-points of intervals (since if one can, then absolute precision about timepoints comes for free.) The Allen set of thirteen relations reduces to just six (before, overlap, inside, and inverses) since those that require endpoints to be exactly identified (meets, starts, ends, equal, endby, startby, meetby) are undefinable (except in an infinite limit.) This is the theory called 'approximate-point' in my time catalog. There arent any points in this theory, of course, though they could be defined if one added enough mathematical machinery to be able to talk about limits of infinite sequences.

Pat Hayes

References:

[n-mi-4-463] John McCarthy and Pat Hayes. *Some Philosophical Problems from the Standpoint of Artificial Intelligence.* Machine Intelligence, vol. 4 (1969), pp. 463-502. Online at:
http://www-formal.stanford.edu/jmc/mcchay69.html.

From: Pat Hayes on 13.4.1998

Answer to Jixin's contribution to this discussion on 1.4:

There is really little point in arguing about which theories are more 'intuitive' unless one is more precise about what one's intuitions are. There are two fundamental problems with arguments like this. First, intuitions are malleable, and one can get used to various ways of thinking about time (and no doubt many other topics) so that they seem 'intuitive'. Second, our untutored intuition seems to be quite able to work with different pictures of time which are in fact incompatible with one another. Jixin's own intuition, for example, seems to agree with McCarthy's that time is continuous, and yet also finds the idea of contiguous atomic 'moments' (intervals with no interior points) quite acceptable. But you can't have it both ways: if moments can meet each other, then there might not be a point where the speed is exactly 60mph, or the ball is exactly at the top of the trajectory. If time itself is discrete, then the idea of continuous change is meaningless. Appealing to a kind of raw intuition to decide what axioms 'feel' right lands one in contradictions. (That was one motivation for the axiom in Allen's and my theory, which Jixin found "awkward", that moments could not meet. The other was wanting to be able to treat moments as being pointlike. That was a mistake, I'll happily concede.)

Jixin wrote:

> *What follows is our response to the arguments about the ontology of time from Pat Hayes, Ray Reiter, and John McCarthy.*
>
> *Response To John:*
>
> *The example of car accelerating demonstrates the need of time points for time ontology.*
>
> *A similar example is throwing a ball up into the air. The motion of the ball can be modelled by a quantity space of three elements: going-up, stationary, and going-down. Intuitively, there are intervals for going up and going down. However, there is no interval, no matter how small, over which the ball is neither going up nor going down. The property of being stationary is naturally associated with a point, rather than any interval (including Allen and Hayes' moment), a "landmark" point which separates two other intervals.*

Yes, I agree. However, notice that there is a coherent frame of mind which would deny this. According to this intuition, which is similar to Newton's old idea of the infinitesimal, one would say that there are no points, but some intervals are so small that they can be treated like points at a sufficiently larger scale. In this perspective, it would be false to claim that there was *no* interval at which the velocity is zero; rather, one would say that the interval was infinitesimal. (If you want to deny the reasonableness of this perspective, first reflect on the fact that it is nearer to physical reality than any model based on the real line.)

Response To Ray: (—)

I agree with Jixin here.

Response To Pat:

> ...For example, the 1990 AIJ critique of Allen's account by Galton (wrongly) assumes that Allen's intervals are sets of points on the real line.

After re-reading Galton's paper [j-aij-42-159]*, as we understand, Galton's arguments are in general based on the assumption that Allen's intervals are primitive, rather than sets of points on the real line. In fact, the main revision Galton proposes to Allen's theory is a diversification of the temporal ontology to include both intervals and points. That is, in Galton's revised theory, intervals are still taken as primitive.*

Galton's intuitions are clearly based on thinking of intervals as sets of points. He takes it as simply obvious, for example, that there is a distinction between open and closed intervals.

Having pointed out this, however, as shown in Ma, Knight and Petrides' 1994 paper [j-cj-37-114]*, Galton's determination to define points in terms of the "meeting places" of intervals does not, as he claims, axiomatise points on the same footing as intervals, and hence that some problems still remain in these revisions.*

> there does seem to be a simple, basic, account which can be extended in various ways to produce all the other alternatives, and this core theory is the one I was referring to.

Does this core theory refer that one in which "intervals are uniquely defined by their endpoints (which are also the points they fit between) and two intervals meet just when the endpoint of the first is the startpoint of the second"? Or Allen's one? - It seems the former one.

Anyway, yes. There **does** *seem to be such a simple, basic core theory. For general treatments, in Ma and Knight's CJ94 paper* [j-cj-37-114]*, a time theory is proposed (as an extention to Allen and Hayes' interval-based one) which takes both intervals and points as primitive on the same footing - neither intervals have to be constructed out of points, nor points have to be created as the places where intervals meet each other, or as some limiting construction of intervals. The temporal order is simply characterised in terms of a single relation "Meets" between intervals/points.*

This theory seems to be similar to that outlined in my 1990 paper with Allen, [j-ci-5-225] (and given at greater length in a U of Rochester tech report of the same date.) But there is little point in bickering about who said what first, as almost all this discussion (including for example Allens 13 relations) can be found in publications written in the last century, if one looks hard enough. All AI work in this area (including my own) is like children playing in a sandbox. The theories and idea themselves are a much more interesting topic.

One technical point, about 'primitive'. One of the things I realised when working with James on this stuff was that if ones axioms about points were minimally adequate it was trivial to define interval in terms of points; and one can also define points in terms of intervals, although that construction is less obvious. (I was immensely pleased with it until being told that it was well-known in algebra, and first described by A. N. Whitehead around 1910.) Moreover, these definitions are mutually transparent, in the sense that if one starts with points, defines intervals, then redefines points, one gets an isomorphic model; and vice versa. So to argue about which of points or intervals are 'primitive' seems rather pointless. We need them both in our ontology. If one likes conceptual sparseness, one can make either one rest on the other as a foundation; or one can declare that they are both 'primitive'. It makes no real difference to anything.

> *Some advantages of this time theory are: (1) It retains Allen's appealing characteristics of treating intervals as primitive which overcomes the Dividing Instant Problem.*

See above. But in any case this doesnt overcome the problem. Allen's treatment allows lights to just come on, but it doesnt provide anywhere for the ball to be motionless.

> *(2) It includes time points into the temporal ontology and therefore makes it possible to express some instantaneous phenomenon, and adequate and convenient for reasoning correctly about continuous change. (3) It is so basic that it can be specified in various ways to subsume others. For instance, one may simply take the set of points as empty to get Allen's interval time theory, or specify each interval, say T, as $\langle$T-left, T-right$\rangle$ where T-left $\leq$ T-right, to get that one Pat prefers.*

Not quite right. In my simple theory, *T-left* isnt before *T-right*, it equals it.

....

> *Yes, it's true.* **And**, *it seems that, all these can be reached equivalently by simply taking pointlike interval $\langle t, t\rangle$ as identical with point t in the case where both intervals and points are included in the time ontology.*

Yes, that is another alternative way to formalise things.

> one can, for example, say something like
>
> ```
> ((illuminated i) or (dark i)))  implies
>     (not (pointlike i))
> ```
>
> so that the light is neither on nor off at the switching point. In this theory, every proposition has a 'reference interval' during which it is true, and a proposition might not be true of subintervals of its reference interval. (Though some propositions might be. This kind of distinction has often been made in the linguistic literature. Note however that this intuition is basically incompatible with the idea that an interval is identical to the set of the points it contains.)

> *This can be distinguished by applying Holds_In and Holds_On (that is Allen's Holds, see Galton's 1990 paper* [j-aij-42-159]*).*

Yes, exactly, although there is no need to use this formal strategy, as I explain in the time catalog section 1. Briefly, HoldsIn P i is true just when i is a subinterval of a reference interval j where HoldsOn P j . Again, it is largely an aesthetic judgement, but I find Galton's *hold-on* vs. *hold-in* distinction awkward and unintuitive. (It suggests that there are two different 'ways to be true'.)

> *....Actually, as shown in Ma and Knight's 1996 paper* [j-cj-37-114]*, to characterise the intuitive relationship between Holds-On and Holds-In, in the case where intervals are allowed, some extra axiom is needed.*

I will check this paper to see what you mean in detail, thanks.

> *Also, it seems that, in Pat's formulation, for expressing that interval $\langle a,b\rangle$ is a subinterval of interval $\langle c,d\rangle$, one would have $c \leq a \leq b \leq d$. In this case, we get that pointlike intervals $\langle a,a\rangle$ and $\langle b,b\rangle$ (or equivalently, points a and b)* **fall in** *interval $\langle c,b\rangle$.*

Yes; but note that if the theory uses reference intervals, that fact that P holds for an interval I doesnt imply that it holds for every point (still less every interval) **in** I. So this is quite consistent:

- Ball is rising for interval $\langle a,b\rangle$
- Ball is falling for $\langle b,c\rangle$
- Ball is stationary for $\langle b,b\rangle$

You can consistently add that rising and falling are true for all nonpointlike subintervals and every properly contained subinterval of the reference interval.

> *On the one hand, many cases suggest the need of allowing a proposition to holds at a single point. For instance, see the example of throwing a ball up into the air described earlier in the response to John MaCarthy.*
>
> *On the other hand, allowing a proposition to holds at a single point doesn't necessarily make pointlike un-definable. It depends on if one would impose some extra constraints, such as*

```
((illuminated i) or (dark i)))  implies
     (not (pointlike i))
```

> *as introduced by Pat for the light switching example, which actually leads to the assertion that the light is neither on nor off AT the switching point.*

In my theory it leads to the conclusion that $\langle a,a\rangle$ does not exist (or, is not a reference interval for 'illumination'), where a is the switching point. All the distinctions between kinds of point - ones where something is true and ones where something is switching - can be cast into a typology of intervals. (This example illustrates why I like to distinguish between the point a - which undoubtedly exists, is where the intervals meet, has a clock time, etc. - and the interval $\langle a,a\rangle$, which, if it existed, might be an embarrassment.)

> *Actually, in the later version of Allen and Hayes's theory that appears in 1989* [j-ci-5-225], *an awkward axiom is proposed to forbid moments to meet each other. It is interesting to note that, although moments are quite like points (moments are non-decomposable), they still have positive duration (they are not pointlike). Moments are included in Allen and Hayes' time ontology, while points are not. One of the reason that such an axiom is awkward is that it doesn't catch the intuition in common-sense using of time.*

I agree. This was awkward in our old paper, and stemmed from our reluctance to accept the idea of intervals which could meet themselves. I'm now reconciled to that idea: in fact, it seems inevitable, much as the existence of zero seems inevitable once one allows subtraction.

> > But these cases only make sense if one thinks of interval and points in the usual mathematical way, which is exactly what Im suggesting we don't need to do. We can get almost everything we need just from the ordering structure: we don't need to get all tied up in distinguishing cases which can only be formally stated by using all the machinery of real analysis.

> *The cases make sense* **not** *only if one thinks of intervals and points in the usual mathematical way. In fact, all the three cases are demonstrated under the assumption that both intervals and points are treated as primitive, rather than in the usual mathematical way.*

Well, it depends on what axioms one assumes! Perhaps I have been speaking too carelessly about the 'usual mathematical way'. Heres my intuition: the standard account of the continuum seems *forced* to resolve the dividing point problem by deciding which interval contains the point, distinguishing open from closed intervals, because it *identifies* an interval with a set of points. (So if both intervals 'contain' the point, the intervals must intersect.) One can take points as basic or intervals as basic or both as primitive; that's irrelevant, but the crucial step is that (set-of-points = interval) identification. Thats exactly what I want to avoid. My point is only that if we abandon that idea (which is only needed for the formal development of analysis within set theory, a rather arcane matter for us), then there is a way to formalise time (using both intervals and points as primitive, if you like) which neatly avoids the problem.

> > ... Is the light on or off at (exactly) 3.00 pm? The only way to answer this is to find a suitable non-pointlike interval of light or darkness completely surrounding 3.00 pm, because 'being on' is the kind of proposition that requires a nonpointlike reference interval.

> *But it seems that there are also some other kind of proposition to which one cannot assign any nonpointlike reference interval. For instance, in the throwing ball up into the air example, proposition "the ball is stationary" can only be true at points, and for any point we cannot find any non-pointlike interval (completely) surrounding it over which the ball is stationary.*

Yes, exactly. Some properties can accept pointlike reference intervals, some can't. Like Galton's distinction between 'at rest' and 'motionless' (former requires nonpointlike, latter doesnt.) Thats the point. Notice the distinctions are now all about intervals. They arent between different ways of being true, but are bread-and-butter distinctions between intervals, expressible within the theory. The machinery of truth wrt an interval is the same in both cases (and in others, eg 'intermittently true' and other exotic variations.)

> This has nothing to do with whether an interval is open or closed: in fact, there is no such distinction in this theory. It only arises in a much more complicated extension which includes set theory and an extensionality axiom for intervals.

In the time theory where both intervals and points are taken as primitive, we can (if we like) talk about the open and closed nature of intervals with some **knowledge** *being available. This kind of knowledge can be given in terms of the Meets relation, rather than some "much more complicated extension which includes set theory and an extensionality axiom for intervals". In fact, we can define that:*

- *interval I is left-open at point P iff $Meets(P, I)$*
- *interval I is right-open at point P iff $Meets(I, P)$*
- *interval I is left-closed at point P iff there is an interval I' such that $Meets(I', I) \wedge Meets(I', P)$*
- *interval I is right-closed at point P iff there is an interval I' such that $Meets(I, I') \wedge Meets(P, I')$*

That's all, and it seems quite intuitive......

It is important to note that the above definition about the open and closed nature of intervals is given in terms of only the knowledge of the Meets relation. However, if one would like to specify intervals as point-based ones, such a definition will be in agreement with the conventional definition about the open and closed intervals.

That certainly seems to be an elegant device. (Though the definitions have nothing to do with knowledge; all Jixin is saying is that the definitions of open and closed can be given in terms of *MEETS*. As Allen and I showed in our old paper, the entire theory can be reduced to *MEETS*.) However, in order to be nontrivial, it must be that points 'separate' meetings, ie if $meets(I, P)$ and $meets(P, J)$ then $\neg meets(I, J)$, right? For if not, all left-open intervals are also left-closed, etc. This seems to make 'points' similar to our old 'moments': in fact, if Jixin's theory predicts $meets(P, Q) \rightarrow P = Q$ for points P and Q, then I'll lay odds it is isomorphic to our moments theory. One of the main observations in our paper was that with the no-meets axiom, one can map moments to points with no change to the theorems provable.

On the other hand, if the theory allows distinct points to *MEET*, I'd be interested to know how it is able to map smoothly to a conventional account of the continuum, since that is provably impossible. One-point closed intervals exist everywhere on the real line, but no two of them are adjacent. Atomic adjacent times (whatever we call them) are pretty much a definition of discrete time models, and are incompatible with density, let alone continuity.

Pat Hayes

PS. Maybe the most useful thing would be to put all these axiomatic theories into some common place with a common syntax - we could use vanilla-KIF - so people can compare and contrast them. I dont have enough, er, time to offer to do this right now, im afraid, but will cooperate with anyone who will volunteer.

References:

[j-aij-42-159] Anthony Galton. *A critical examination of Allen's theory of action and time.* Artificial Intelligence Journal, vol. 42 (1990), pp. 159-188.

[j-ci-5-225] James F. Allen and Patrick J. Hayes. *Moments and points in an interval-based temporal logic.* Computational Intelligence, vol. 5, pp. 225-238.

[j-cj-37-114] Jixin Ma and Brian Knight. *A General Temporal Theory.* Computer Journal, vol. 37 (1994), pp. 114-123.

From: Jixin Ma on 15.4.1998

Pat wrote:

> *Jixin's own intuition, for example, seems to agree with McCarthy's that time is continuous, and yet also finds the idea of contiguous atomic 'moments' (intervals with no interior points) quite acceptable. But you can't have it both ways: if moments can meet each other, then there might not be a point where the speed is exactly 60mph, or the ball is exactly at the top of the trajectory. If time itself is discrete, then the idea of continuous change is meaningless. Appealing to a kind of raw intuition to decide what axioms 'feel' right lands one in contradictions. (That was one motivation for the axiom in Allen's and my theory, which Jixin found "awkward", that moments could not meet. The other was wanting to be able to treat moments as being pointlike. That was a mistake, I'll happily concede.)*

First of all, as Pat noted (see below), our theory is similar (in fact, it is specially stated to be an extension/revision) to that of Allen and Hayes ([c-ijcai-85-528], [j-ci-5-225]). As an extension, it allows not only intervals and moments, but also points (however, it dose not necessarily have to include points or moments). As a revision, Allen and Hayes' constraint that moments could not meet each other is replaced by "a point cannot meet another point", that is there must be an interval (or a moment) standing between any two points (if these two point are explicitly expressed). However, basically, the theory doesn't commit the time structure as (left-, right-, left-and-right) linear, (left-, right-, left-and-right) branching, continuous or discrete, etc., although all these can be characterised by means of some corresponding extra axioms.

Yes, in this time theory, atomic moments (AND points) are acceptable. However, they are just ACCEPTABLE, but not necessarily to be everywhere over the time. The theory only claims that a time element is either an interval (or specially, a moment) or a point. If one insists on using moments/points somewhere over the time, they can be explicitly expressed there. For somewhere else over the time, it may be the case that each time element is a decomposable interval. That is, there may be no moments/points at all. It is also consistent to have a time structure where each time element is either a decomposable interval or a point, or even a time structure where each time element is a decomposable interval. In fact, generally speaking, the basic time structure may be neither dense nor discrete anywhere, or may be continuous over some parts and discrete over other parts. This depends on what you want to express and what extra axiom you would impose.

Pat wrote

> *Galton's intuitions are clearly based on thinking of intervals as sets of points. He takes it as simply obvious, for example, that there is a distinction between open and closed intervals.*

I still cannot find where Galton claimed this. Actually, before I answered Pat's last contribution (Newsletter ENRAC 27.3 (98030), I had re-read Galton's paper and phoned Galton to confirm about this. Yes, a distinction between open and closed intervals did appear in Galton's paper, but I found it was only used to demonstrate the corresponding problem in thinking of intervals as sets of points, and hence as the reason for him (and for James and Pat?) to take intervals as primitive. Anyway, this issue doesn't really affect this discussion.

Pat wrote

> > Some advantages of this time theory are: (1) It retains Allen's appealing characteristics of treating intervals as primitive which overcomes the Dividing Instant Problem.
>
> *See above. But in any case this doesn't overcome the problem. Allen's treatment allows lights to just come on, but it doesnt provide anywhere for the ball to be motionless.*

So, that is why we need to extend Allen and Hayes' treatment by allowing time points as primitive as well. On the one hand, treating time intervals as primitive avoids the question of whether intervals are open or closed; on the other hand, allowing time points (as primitive) provides means for expressing instantaneous phenomena such as "the ball is motionless at a point".

Pat wrote

> > (2) It includes time points into the temporal ontology and therefore makes it possible to express some instantaneous phenomenon, and adequate and convenient for reasoning correctly about continuous change. (3) It is so basic that it can be specified in various ways to subsume others. For instance, one may simply take the set of points as empty to get Allen's interval time theory, or specify each interval, say T, as $\langle$*T-left*, *T-right*$\rangle$ where *T-left* $\leq$ *T-right*, to get that one Pat prefers.
>
> *Not quite right. In my simple theory, T-left isnt before T-right, it equals it.*

But, the relation *T-left* $\leq$ *T-right* does already include *T-left* $=$ *T-right*, that is *T-left* equals *T-right*. In the case *T-left* $<$ *T-right*, T is an interval; in the case *T-left* $=$ *T-right*, T is a pointlike interval.

Pat wrote

> *Yes, exactly, although there is no need to use this formal strategy, as I explain in the time catalog section 1. Briefly,* ⌜Holds-In P i⌝ *is true just when i is a subinterval of a reference interval j where* ⌜Holds-On P j⌝.
>
> *Again, it is largely an aesthetic judgement, but I find Galton's hold-on vs. hold-in distinction awkward and unintuitive. (It suggests that there are two different 'ways to be true'.)*

Yes. And in Ma et al's paper [j-cj-37-114], some examples are given to show the problem with Galton's distinction between *hold-on* and *hold-in*. Also, it is claimed that the fundamental reason for the problem is that Galton wanted to characterise the fact that a proposition holds for an interval in terms of that the proposition holds for every point **within** the interval.

Pat wrote

> *Well, it depends on what axioms one assumes! Perhaps I have been speaking too carelessly about the 'usual mathematical way'. Heres my intuition: the standard account of the continuum seems* forced *to resolve the dividing point problem by deciding which interval contains the point, distinguishing open from closed intervals, because it* identifies *an interval with a set of points. (So if both intervals 'contain' the point, the intervals must intersect.) One can take points as basic or intervals as basic or both as primitive; that's irrelevant, but the crucial step is that (set-of-points = interval) identification. Thats exactly what I want to avoid. My point is only that if we abandon that idea (which is only needed for the formal development of analysis within set theory, a rather arcane matter for us), then there is a way to formalise time (using both intervals and points as primitive, if you like) which neatly avoids the problem.*

It is **also** exactly what the approach of treating both intervals and points as primitive wants to avoid - it avoids to "identify" an interval with a set of points. **In addition**, such an approach allows expressions of all the three cases shown in my former discussion (Newsletter ENRAC 13.3 (98027)), without thinking of intervals and points in the usual mathematical way, or "identify an interval with a set of points"

Pat wrote

> In the time theory where both intervals and points are taken as primitive, we can (if we like) talk about the open and closed nature of intervals with some **knowledge** being available. This kind knowledge can be given in terms of the Meets relation, rather than some "much more complicated extension which includes set theory and an extensionality axiom for intervals". In fact, we can define that:

- interval I is left-open at point P iff $Meets(P, I)$
- interval I is right-open at point P iff $Meets(I, P)$
- interval I is left-closed at point P iff there is an interval I' such that $Meets(I', I) \wedge Meets(I', P)$
- interval I is right-closed at point P iff there is an interval I' such that $Meets(I, I') \wedge Meets(P, I')$...

That certainly seems to be an elegant device. (Though the definitions have nothing to do with knowledge; all Jixin is saying is that the definitions of open and closed can be given in terms of MEETS. As Allen and I showed in our old paper, the entire theory can be reduced to MEETS.) However, in order to be nontrivial, it must be that points 'separate' meetings, ie if $meets(I, P)$ and $meets(P, J)$ then $\neg meets(I, J)$, right? For if not, all left-open intervals are also left-closed, etc. This seems to make 'points' similar to our old 'moments': in fact, if Jixin's theory predicts $meets(P, Q) \rightarrow P = Q$ for points P and Q, then I'll lay odds it is isomorphic to our moments theory. One of the main observations in our paper was that with the no-meets axiom, one can map moments to points with no change to the theorems provable.

On the other hand, if the theory allows distinct points to MEET, I'd be interested to know how it is able to map smoothly to a conventional account of the continuum, since that is provably impossible. One-point closed intervals exist everywhere on the real line, but no two of them are adjacent. Atomic adjacent times (whatever we call them) are pretty much a definition of discrete time models, and are incompatible with density, let alone continuity.

Yes. If $Meets(I, P) \wedge Meets(P, J)$ then $\neg Meets(I, J)$. Actually, in our theory, If $Meets(I, P) \wedge Meets(P, J)$ then $Before(I, J)$, where $Before(I, J) \rightarrow \neg Meets(I, J)$, since all the 13 relations are exclusive to each other. Therefore, it is impossible for an interval to be both left-open and left-closed.

Also, in our theory, $Meets(P, Q)$ implies that at least one of P and Q is an interval (or a moment) - they cannot be both points.

In addition, it is important to note that the constraint that a point cannot meet another point makes it is possible to establish a consistency checker for temporal database systems (see Knight and Ma's 1992 paper [j-aicom-5-75]).

Yes. As claimed earlier in this dicussion and actually pointed out in our published paper, our theory is in fact an extension to that of Allen and Hayes. Our points are quite like Allen and Hayes' moments - they cannot meet each other. However, on the one hand, points are fundamentally different from moments - points have no duration while moments do have, no matter how small they are. Therefore, it is more convenient to use points than moments in modelling some instantaneous phenomenon, especially in the case where duration reasoning is involved. On the other hand, if moments are simply mapped to points, how to express the **real** moments, i.e., non-decomposable intervals with positives duration (like the "seconds" example given in my last discussion)?

Jixin

References:

[c-ijcai-85-528] James Allen and Pat Hayes. *A Common-Sense Theory of Time.* Proc. International Joint Conference on Artificial Intelligence, 1985, pp. 528-531.

[j-aicom-5-75] Brian Knight and Jixin Ma. *A General Temporal Model Supporting Duration Reasoning.* AI Communications, vol. 5 (1992), pp. 75-84.

[j-ci-5-225] James F. Allen and Patrick J. Hayes. *Moments and points in an interval-based temporal logic.* Computational Intelligence, vol. 5, pp. 225-238.

[j-cj-37-114] Jixin Ma and Brian Knight. *A General Temporal Theory.* Computer Journal, vol. 37 (1994), pp. 114-123.

From: Sergio Brandano on 18.4.1998

Salve.

The following are some fragments from the current discussion:
From Pat Hayes - ENRAC 14.3.1998

> *instantaneous intervals completely. It is also quite consistent to have arbitrary amounts of density, discreteness, etc.; for example, one can say that time is continuous except in a certain class of 'momentary' intervals whose ends are distinct but have no interior points.*

From Jixin Ma - ENRAC 15.4.1998

> *time element is a decomposable interval. In fact, generally speaking, the basic time structure may be neither dense nor discrete anywhere, or may be continuous over some parts and discrete over other parts.*

Pat and Jixin, what do you mean when you write "continuous"?

Here in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking about (the founding notion of) continuity. I really find it difficult to believe that you like to make an exception in this sense, also because the hat here is "formal (temporal) reasoning". It also seems to me that any temporal structure must necessarily fail to be persuasive if on one hand it includes the notion of continuity and on the other it refuses it; how can time be continuous ... with some exception? Either it is continuous, or it is not! That is, either the Basic Time Structure assumes the axiom of completeness, or it does not!!

In fact, in this discussion I have not yet seen any explanation why an alternative notion of continuous structure is needed at all? I am not asking you to argue about your own notion, I just ask you to give a convincing argument on the need of a notion which is an alternative to the classical one, such as: "the problem P of temporal reasoning about actions and change can not be solved adopting the axiom of completeness", or "the axiom of completeness is too strong an assumption for our purposes; axiom A is better suited, because...".

Sergio

From: Jixin Ma on 21.4.1998

To Pat

As an addition to my response (ENRAC 15.4 (98035)) to Pat's suggestion of "simply map Allen and Hayes' moments to Ma and Knight's points":

The constraint that "moments cannot meet each other" will lead to the conclusion that we can have neither a completely discrete nor a completely dense system which contains both moments and decomposable intervals. However, if we revise Allen and Hayes' system to include both points and intervals (including moments), and impose the "not-meet-each-other" constraint on points only, rather than on moments, this objection does not apply.

To Sergio

> *In fact, in this discussion I have not yet seen any explanation why an alternative notion of continuous structure is needed at all? I am not asking you to argue about your own notion, I just ask you to give a convincing argument on the need of a notion which is an alternative to the classical one, such as: "the problem P of temporal reasoning about actions and change can not be solved adopting the axiom of completeness", or "the axiom of completeness is too strong an assumption for our purposes; axiom A is better suited, because...".*

First of all, what do you mean "the classical one"? (the classical continuous time structure)? Does it refer to the classical physical model of time, where the structure is a set of points which is isomorphic to the real line?

> *in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking' about (the founding notion of) continuity.*

At the ontological level, the notion of continuous time vs. discrete time is closely related to questions "Is the set of time elements dense or not?", and "Are there really time atoms?". For a point-based model, the continuity is usually characterised as "Between any two points, there is a third"; while for an interval-based model (like that of Allen), it is characterised as "Every interval can be decomposed into two adjacent sub-intervals". In addition, as for a model which takes both intervals and points as primitive, one may characterise two levels of density. At the weak level, it is only required that each interval can be divided into two adjacent sub-intervals. At the strong level, it is required that there is always a point within any interval. It is easy to infer that the latter can imply the former.

As for general treatments, the Basic Time Structure **does not** have to impose the axiom of density or discreteness (Similar arguments apply to issues such as linear/non-linear, bounded/un-bounded). Therefore, the time structure as a whole may be continuous or discrete, or neither continuous nor discrete.

Now, "why an alternative notion of continuous structure is needed at all"? It has been noted that temporal knowledge in the domain of artifical intelligence, including "temporal reasoning about actions and change", is usually imcomplete, and using time intervals in many cases is more convenient and more in-keeping with common sense of temporal concepts than to use the classical abstraction of points. In fact, the notion of time intervals (or periods) has been introduced for a long time in the literature. In addition, in order to overcome/bypass the annoying question of whether various intervals are open or closed, various approaches have been proposed. An example is Allen's interval-based time theory. As for these time theories, the old (classical?) notion of continuity no longer simply applies.

Jixin

From: Pat Hayes on 21.4.1998

Sergio Brandano wrote:

> *Here in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking about (the founding notion of) continuity. I really find it difficult to believe that you like to make an exception in this sense, also because the hat here is "formal (temporal) reasoning". It also seems to me that any temporal structure must necessarily fail to be persuasive if on one hand it includes the notion of continuity and on the other it refuses it; how can time be continuous ... with some exception? Either it is continuous, or it is not! That is, either the Basic Time Structure assumes the axiom of completeness, or it does not!!*

Why cannot time be continuous in some places but discontinuous at others? There is no mathematical objection to such a structure, and it has been argued that a continuum punctuated by a sparse collection of points of discontinuity might be a plausible mathematical picture of time which seems to 'flow smoothly' except when things happen suddenly. (Similar arguments can be made for describing spatial boundaries, by the way; and elementary physics makes similar assumptions, where velocity is supposed to change smoothly except when 'impact' occurs.)

You talk about a 'founding notion' of continuity as being that captured by the axiom of completeness. Here, in my view, you commit a philosophical error (especially in Pisa!) There are *intuitions* about continuity which one can try to capture in various formal ways, but there is no 'founding notion' of continuity other than those intuitions. In the late 19th century, famous mathematicians objected strongly to the view of the continuum as consisting of a set of points, for example. This modern perspective, now taught in high schools, is a modern invention, not a 'founding' notion. It is more recent than the gasoline engine, yet people have had intuitions about smoothness, instantaneity and continuity for eons. (Whether or not one agrees with me on this admittedly controversial point, it seems unwise to *identify* a mathematical property such as continuity with any kind of axiom until one has verified that no other axiom will do as well; and as I am sure Sergio knows, there are many alternative ways to axiomatize continuity.)

In my view, axioms are tools which we can manipulate at will; they are not set in stone or somehow inevitable. Different formal accounts of time might be appropriate for different purposes or to capture different intuitions. (I agree with Jixin that it is useful to seek a common 'core' theory which can be extended in various ways to describe various possible more complex temporal structures; and that this theory will have to be rather weak.)

> *In fact, in this discussion I have not yet seen any explanation why an alternative notion of continuous structure is needed at all? I am not*

> *asking you to argue about your own notion, I just ask you to give a convincing argument on the need of a notion which is an alternative to the classical one, such as: "the problem P of temporal reasoning about actions and change can not be solved adopting the axiom of completeness", or "the axiom of completeness is too strong an assumption for our purposes; axiom A is better suited, because...".*

The 'dividing point' problem which gave rise to this discussion would do. According to the modern account of the continuum, this point must exist, and since all intervals consist of points, the light is therefore either on or off at it. But it seems more natural, as well as formally simpler, to just say that the question is meaningless; perhaps (though this is no longer my own preference) because that point doesnt exist.

A more mundane example is given by temporal databases, which usually assume in their basic ontology that time is discrete: for example, they routinely describe times as integers representing the number of milliseconds since the birth of Christ. (Of course, one can always insist that these are to be understood as being embedded in a continuum, but then what use is an axiom whose sole purpose is to insist that times exist which have no name and about which nothing can be asserted, other than that they exist?)

Pat Hayes

From: John McCarthy on 22.4.1998

Pat Hayes wrote

> *In my view, axioms are tools which we can manipulate at will; they are not set in stone or somehow inevitable. Different formal accounts of time might be appropriate for different purposes or to capture different intuitions. (I agree with Jixin that it is useful to seek a common 'core' theory which can be extended in various ways to describe various possible more complex temporal structures; and that this theory will have to be rather weak.)*

and

> *A more mundane example is given by temporal databases, which usually assume in their basic ontology that time is discrete: for example, they routinely describe times as integers representing the number of milliseconds since the birth of Christ. (Of course, one can always insist that these are to be understood as being embedded in a continuum, but then what use is an axiom whose sole purpose is to insist that times exist which have no name and about which nothing can be asserted, other than that they exist?)*

If axioms are guaranteed to be used only in a particular program or set of programs, they need be no stronger than necessary.

As to the rhetorical "what use", suppose the theory is to tolerate the elaboration that two successive events, shooting Pat and his falling to the ground, occurred between successive ticks of the clock. If you guarantee that no such elaborations will be required or that you are willing to do major surgery on your theory should elaboration be required, then you are ok with a weak theory even if it is unextendable.

From: Sergio Brandano on 22.4.1998

In reply to Pat and Jixin.

I apologize for the length of this message, although it mainly consists of quoted text. As "skin perception", it seems to me my critique hits the target. The arguments of reply I received, in fact, are not as convincing as they were supposed to be. The details follow.

To Jixin

> *First of all, what do you mean "the classical one"? (the classical continuous time structure)? Does it refer to the classical physical model of time, where the structure is a set of points which is isomorphic to the real line?*

I can just quote myself ...

> Here in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking about (the founding notion of) continuity.

> *At the ontological level, the notion of continuous time vs. discrete time is closely related to questions "Is the set of time elements dense or not?", and " Are there really time atoms?".*

The word "continuity", even at the ontological level, can not be read as "continuous with some exception".

> *For a point-based model, the continuity is usually characterized as "Between any two points, there is a third"; while for an interval-based model (like that of Allen), it is characterized as "Every interval can be decomposed into two adjacent sub-intervals".*

The axiom of completeness states:

Let be A and B non empty subsets of S such that $a \leq b$ for all $a \in A$ and $b \in B$. Then there exists $\xi \in S$ such that $a \leq \xi \leq b$ for all $a \in A$ and $b \in B$.

Now, the set S, that is your domain, may consist as well either of time-points or time-intervals; S holds real numbers on the former case, intervals from the real line on the latter case.

> *As for general treatments, the Basic Time Structure* **does not** *have to impose the axiom of density or discreteness (Similar arquements apply to issues such as linear/non-linear, bounded/un-bounded). Therefore, the time structure as a whole may be continuous or discrete, or neither continuous nor discrete.*

I agree with your premise: the Basic Time Structure does not have to impose the choice, in fact it leaves you free in that sense. As soon as you make the choice, then you obtain either a continuous structure or a discrete structure, just depending on this choice. I do not agree, instead, with your conclusion. If I leave you the freedom to choose, it does not mean the Structure is neither continuous nor discrete; it simply means you still have to make the choice.

> *Now, "why an alternative notion of continuous structure is needed at all"? It has been noted that temporal knowledge in the domain of aritifical intelligence, including "temporal reasoning about actions and change", is usually imcomplete, and using time intervals in many cases is more* `convenient` *and more in-keeping with common sense of temporal concepts than to use the classical abstraction of points. In fact, the notion of time intervals (or periods) has been introduced for a long time in the literature. In addition, in order to overcome/bypass the annoying question of if intervals are open or closed, various approaches have been proposed. An example is Allen's interval-based time theory. As for these time theories, the old (classical?) notion of continuity no longer simply applies.*

My question referred to what is *needed* rather than *convenient*. I understand it may be convenient, in some cases, to use intervals, but this is not pertinent with my criticism, which still holds.

Let me ask you a more stringent question.

Premise: It is evident that if you assume the axiom of completeness, the domain S can just be continuous, while if you do not assume the axiom of completeness then S is necessarily discrete.

Question: Suppose that you define your neither continuous nor discrete Temporal Structure. What is your domain S? What is your replacement for the axiom of completeness? Does this structure (provably) solve for at least one problem what can not be (provably) solved via the axiom of completeness? Can you give an example?

To Pat

> *Why cannot time be continuous in some places but discontinuous at others?*

Places? If we shall understand time like physicians understands the space, then things become considerably simpler: time can just be discrete, since space itself is fully discrete (the observable one). The use of real lines, in physics, is just a theoretical convenience. If by "place" we shall mean, instead, some point in a lattice (and we shall provide a convenient reference system for the branching-time case), then the case may hold. In part my question was, in fact, to insert in the actual discussion explicit and convincing arguments about the case (examples, counterexamples and axioms, are welcome).

> *There is no mathematical objection to such a structure, and it has been*

If a Temporal Structure exists in this sense, may I have a look at its domain (that is at the S domain, as stated above)? Can you post the reference together with the explicit case? But the primary question still remains whether is it needed at all.

> *You talk about a 'founding notion' of continuity as being that captured by the axiom of completeness. Here, in my view, you commit a philosophical error (especially in Pisa!) There are* intuitions *about continuity which one can try to capture in various formal ways, but there is no 'founding notion' of continuity other than those intuitions. ...*

What properly formalizes the notion of continuity is the axiom of completeness. Alternative notions are equivalent, until we speak about continuous domains. The point was whether one can have a continuous domain (that is the S I stated above) ... with exceptions.

Concerning the intuition, let me remind that the student who discovered the square root of 2 was killed (down the cliff), and no one was allowed to speak about ... "the fault of the god" for long time. Humans' common sense, to me, is something we shall not call too much.

> *century, famous mathematicians objected strongly to the view of the continuum as consisting of a set of points, for example. This modern perspective, now taught in high schools, is a modern invention, not a 'founding' notion. It is more recent than the gasoline engine, yet people have had intuitions about smoothness, instantaneity and continuity for eons. (Whether or not one agrees with me on this admittedly controversial point, it seems unwise to* identify *a mathematical property such as continuity with any kind of axiom until one has verified that no other axiom will do as well; and as I am sure Sergio knows, there are many alternative ways to axiomatize continuity.)*

... unwise ?

If another axiom exists, which *does as well*, then it is surely equivalent to the axiom of completeness, just because *it does as well.* Alternative notions are clearly equivalent, until we speak about continuous domains. The point here, instead, was whether one can have a continuous domain with exceptions, that is the claim I originally criticized.

> *In my view, axioms are tools which we can manipulate at will; they are not set in stone or somehow inevitable. Different formal accounts of time might*

The point is what would you like to make out of it. If you do something I can do with a simpler approach and without arousing criticism, then my approach will have much more impact than yours, I think you may agree at least on this point.

> *be appropriate for different purposes or to capture different intuitions. (I agree with Jixin that it is useful to seek a common 'core' theory which can be extended in various ways to describe various possible more complex temporal structures; and that this theory will have to be rather weak.)*

Concerning the core theory that you and Jixin are willing to obtain, I already developed a Basic Time Structure which may be of interest. It is as simple as I managed to design it, without un-useful complications. The structure works well in my case. You are welcome to read and comment my contribution, which may be found in my ETAI's reference.

Finally, concerning your examples:

> *The 'dividing point' problem which gave rise to this discussion would do. According to the modern account of the continuum, this point must exist, and since all intervals consist of points, the light is therefore either on or off at it. But it seems more natural, as well as formally simpler, to just say that the question is meaningless; perhaps (though this is no longer my own preference) because that point doesn't exist.*

Why is it a convincing argument?

> *A more mundane example is given by temporal databases, which usually assume in their basic ontology that time is discrete: for example, they routinely describe times as integers representing the number of milliseconds since the birth of Christ. (Of course, one can always insist that these are to be understood as being embedded in a continuum, but then what use is an axiom whose sole purpose is to insist that times exist which have no name and about which nothing can be asserted, other than that they exist?)*

Why is it a convincing argument ?
Best Regards Sergio

From: Erik Sandewall on 22.4.1998

Pat,
In answer to Sergio, you wrote

> *Why cannot time be continuous in some places but discontinuous in others?*

(Jixin answered along similar lines). I have no problems accepting that a *function of* time may be piecewise continuous, or that it may be undefined for some points along the time axis. However, it seems to me that there are several problems with saying that *time itself* is piecewise continuous (btw - do you mean piecewise dense?).

The first problem is with respect to motivation. For what reasons would Time suddenly skip over potential timepoints? If the reason is, as you wrote, that

> *The 'dividing point' problem which gave rise to this discussion would do. According to the modern account of the continuum, this point must exist, and since all intervals consist of points, the light is therefore either on or off at it. But it seems more natural, as well as formally simpler, to just say that the question is meaningless; perhaps (though this is no longer my own preference) because that point doesn't exist.*

then exactly what events in the world would be allowed to contribute to the continuity faults? Does the next time I hit a key on my keyboard qualify? And what about the midpoint halfway between two continuity faults, is it also a continuity fault, recursively?

The other problem is with respect to the axiomatizations. Since your article "A catalog of temporal theories" characterizes the various theories through axiomatizations, I thought I'd go back to that article and check how you had done this formally. However I was not able to find it; the closest I got was the denseness axiom on page 15. If the intuitive notion is that time itself is continuous in some places but not in others, wouldn't it be natural to start with an axiomatization of continuous time (such as the real numbers) and then to proceed from there? For example, a domain of piecewise continuous time could be represented as a twotuple $\langle R, D\rangle$ where R is the real numbers and D is a "small" subset of it; the intention being that $R - D$ is the modified time domain in question. The notions of non-standard intervals could then be constructed as the natural next step.

Maybe I'm missing something - are constructs of this kind subsumed by the axioms in your report, or can they be inferred as theorems? Or why is this not the natural way of doing things?

Erik

From: Jixin Ma on 23.4.1998

To Sergio,

> First of all, what do you mean "the classical one"? (the classical continuous time structure)? Does it refer to the classical physical model of time, where the structure is a set of points which is isomorphic to the real line?

I can just quote myself ...

> Here in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking about (the founding notion of) continuity...

Concerning the core theory that you and Jixin are willing to obtain, I already developed a Basic Time Structure which may be of interest. It is as simple as I managed to design it, without un-useful complications. The structure works well in my case. you are welcome to read and comment my contribution, which may be found in my ETAI's reference.

So, you didn't refer "the classical one" to "the Basic Time Structure" you developed, did you? If No, why **did** you develop it? What is **your** convincing argument(s) on the need of such a structure? Is it also an alternative to the classical one? (Sorry, I am here using the similar question raised by youself to ask you, though I don't have to, see below). If Yes, I shouldn't ask this question.

> At the ontological level, the notion of continuous time vs. discrete time is closely related to questions "Is the set of time elements dense or not?", and " Are there really time atoms?".

The word "continuity", even at the ontological level, can not be read as "continuous with some exception".

What I actually said is very clear as you quoted above. Does it imply that "the word continuity can be read as continuous with some exception"? In fact, even when Pat talked about "continuous with some exception", he didn't really mean that it is as same as the word "continuity". What he means, as I understand, is just that, with the exception of time moments, each time interval can be decomposed into (at least two) sub-intervals.

The axiom of completeness states:

Let be A and B non empty subsets of S such that $a \leq b$ for all $a \in A$ and $b \in B$. Then exists $\xi \in S$ such that $a \leq \xi \leq b$ for all $a \in A$ and $b \in B$.

Now, the set S, that is your domain, may consists as well either of time-points or time-intervals; S holds real numbers on the former case, intervals from the real line on the latter case.

Firstly, you said here, "the former case" and "the latter case". Can these two cases be mixed together? In other words, can the domain contain both time-points and time-intervals. I suppose it should. Otherwise, you will meet some problem in satisfying the so-called completeness axiom (see below).

Secondly, you take time-points as real numbers, and intervals "from" the real line. Are your intervals sets of real numbers limited by their end-points (real numbers)? If no, what are they? If yes, have you considered the dividing instant problem? This problem would be more obvious with your time structure when you try to impose the axiom of completeness (see below).

Thirdly, if the domain S consists of time-intervals, you need to re-define (or revise, or, at least, explain) the relation $\leq$ between elements of the domain S. After you have done this properly (You didn't show how to do it, you just claimed that the domain "may" contain either time-points or time-intervals), you have to show, for the case that interval a in A is immediately before interval b in B (that is, there is no other time elements between a and b) what is the required ξ such that $a \leq \xi \leq b$. Obviously, ξ cannot be an interval (non-pointlike), otherwise, it will overlap with a and b. Therefore, if you can define what it is, it has to be a point (This is why I said earlier in the above that if your domain contains intervals, it needs to contain points as well). Now, you meet the dividing instant problem, as I expected.

By the way, may I take this as one of the "un-useful" complications with your time structure?

> As for general treatments, the Basic Time Structure **does not** have to impose the axiom of density or discreteness (Similar arquements apply to issues such as linear/non-linear, bounded/un-bounded). Therefore, the time structure as a whole may be continuous or discrete, or neither continuous nor discrete.

> *I agree with your premise: the Basic Time Structure does not have to impose the choice, in fact it leaves you free in that sense. As soon as you make the choice, then you obtain either a continuous structure or a discrete structure, just depending on this choice. I do not agree, instead, with your conclusion. If I leave you the freedom to choose, it does not mean the Structure is neither continuous nor discrete; it simply means you still have to make the choice.*

If you don't impose the continuous axiom (!!! as argued by Pat, it does *not* have to be the so-called axiom of completeness !!!) or discrete axiom, the structure **can** be neither continuous nor discrete. I think it is very easy to form a structure which satifies the basic axiomatisation, but does not satisfy the continuous requirement, and does not satisfy the discrete requirement. In fact, you can write down any extra constraint as long as it is consistent with the basic theory.

> Now, "why an alternative notion of continuous structure is needed at all"? It has been noted that temporal knowledge in the domain of artifical intelligence, including "temporal reasoning about actions and change", is usually imcomplete, and using time intervals in many cases is more convenient and more in-keeping with common sense of temporal concepts than to use the classical abstraction of

> points. In fact, the notion of time intervals (or periods) has been introduced for a long time in the literature. In addition, in order to overcome/bypass the annoying question of if intervals are open or closed, various approaches have been proposed. An example is Allen's interval-based time theory. As for these time theories, the old (classical?) notion of continuity no longer simply applies.

> *My question referred to what is* needed *rather than* convenient. *I understand it may be convenient, in some cases, to use intervals, but this is not pertinent with my criticism, which still holds.*

So, you think intervals are *not needed*? Anyway, our arguements about the convenience of using intervals are based on the belief of the need of them.

> *Let me ask you a more stringent question.*
>
> *Premise: It is evident that if you assume the axiom of completeness, the domain S can just be continuous, while if you do not assume the axiom of completeness then S is necessarily discrete.*

Wrong! Even if you do not asssume the axiom of completeness, it is still not necessarily discrete.

> *Question: Suppose that you define your neither continuous nor discrete Temporal Structure. What is your domain S? What is your replacement for the axiom of completeness? Does this structure (provably) solve for at least one problem what can not be (provably) solved via the axiom of completeness? Can you give an example?*

The domain is just a collection of time elements each of which is either an interval (in a particular case, a moment) or a point.

The basic core theory doesn't commit itself to whether the time structure is continuous or discrete. So, if you would like one which is neither continuous nor discrete, you don't need the axiom of completeness. Why do I need a replacement for it, anyway, if it is not supposed?

Extra axioms regarding dense/discrete, linear/non-linear, bounded/non-bounded time structure, etc. can be given (e.g., see Ma and Knight's 1994 paper [j-cj-37-114]). Specially, the characterisation of continuity does not have to be in the form of axiom of completeness. In addition, as shown above, in the case where time intervals are addressed, it becomes very complicated (if not impossible) to simply apply such an axiom.

As for example you would like to see, the DIP is a typical one, as I have shown in the above.

Also, in your reply to Pat you wrote:

> *What properly formalizes the notion of continuity is the axiom of completeness. Alternative notions are equivalent, until we speak about continuous domains. The point was whether one can have a continuous domain (that is the S I stated above) ... with exceptions.*
>
> *If another axiom exists, which* does as well, *then it is surely equivalent to the axiom of completeness, just because* it does as well. *Alternative notions are clearly equivalent, until we speak about continuous domains. The point here, instead, was whether one can have a continuous domain with exceptions, that is the claim I originally criticized.*

First, as shown in the above, the axiom of completeness doesn't simply apply to the case when time intervals are involved. Therefore, your claim that "alternative notions of continuity are clearly equivalent" is unjustified, at least it is not **clear**!

Second, the question of "whether one can have a continuous domain with exceptions" depends on how you understand the real meaning. It is important to note that neither Pat nor myself claims that one can have such a structure as you understood and hence described by "a continuous domain with exceptions". Of course, if you have already assumed that the domain as a whole is continuous, then it must be continuous - no exception! This is just like if you impose that "The traffic light was green throughout last week", then, of course, it was green any time during last week, no exception. Similarly, if you impose that "The traffic light was red throughout last week", then it was red any time during last week. Again, no exception. However, if you don't have either of them, why can't one have the case that over the last week, the traffic light was sometimes red, and sometimes green, and even sometimes yellow?

As I said earlier, when Pat talked about "continuous with exceptions", he actually meant that "except at those time moments, the time is continuous", or more specially, "except for time moments, each time interval is decomposable". I don't think he would actually assume, in the first place, the continuity of the *whole domain*, then expect there are some exceptions. Do I understand your meaning rightly, Pat?

Jixin

References:

[j-cj-37-114] Jixin Ma and Brian Knight. *A General Temporal Theory.* Computer Journal, vol. 37 (1994), pp. 114-123.

From: Sergio Brandano on 23.4.1998

Pat Hayes wrote (ENRAC 21.4.1998):

> *Why cannot time be continuous in some places but discontinuous at others? There is no mathematical objection to such a structure, and it has been argued that a continuum punctuated by a sparse collection of points of discontinuity might be a plausible mathematical picture of time which seems to 'flow smoothly' except when things happen suddenly. (Similar arguments can be made for describing spatial boundaries, by the way; and elementary physics makes similar assumptions, where velocity is supposed to change smoothly except when 'impact' occurs.)*

If a given temporal structure includes the solution to the problem of representing "perceived smooth" flux and "perceived fast" flux of time, then that temporal structure is necessarily agent-centric, since different agents may have a different perception of the world. In being agent-centric, this structure can not aim at generality. In fact, if we design an agent-centric temporal structure and the world is inhabited by more than one agent, then we must design a more general structure that reconciles the different views from the different agents. I say "must" because, otherwise, we pre-destine agents to never interact with each other, which would be a major restriction.

Sergio

From: Pat Hayes on 24.4.1998

John McCarthy wrote

> *If axioms are guaranteed to be used only in a particular program or set of programs, they need be no stronger than necessary.*
>
> *As to the rhetorical "what use", suppose the theory is to tolerate the elaboration that two successive events, shooting Pat and his falling to the ground, occurred between successive ticks of the clock. If you guarantee that no such elaborations will be required or that you are willing to do major surgery on your theory should elaboration be required, then you are ok with a weak theory even if it is unextendable.*

in answer to my remarks:

> In my view, axioms are tools which we can manipulate at will; they are not set in stone or somehow inevitable. Different formal accounts of time might be appropriate for different purposes or to capture different intuitions. ...

and

> A more mundane example is given by temporal databases, which usually assume in their basic ontology that time is discrete:...

If I understand what John is saying, then I completely agree with him. (If he intended to disagree with me, then maybe I dont understand his point.) However, note that both Jixin and I are trying to give a theory which is as elaboration-tolerant as possible, without being completely vacuous.

Answers to Sergio Brandano

Sergio seems to be on a different planet, as his responses to both Jixin and I seem to quite miss the point of our debate, and often to be completely free of content.

> *I can just quote myself ...*

Well, you *can*; but to do so is at best unhelpful, and at worst arrogant. If someone fails to understand you and asks for clarification, to simply repeat yourself is obviously unlikely to give them the clarification they need.

> Here in Pisa, we write "continuity" and we read "axiom of completeness", which is what everyone commonly means when speaking about (the founding notion of) continuity.

May I ask in return if "everyone" here is meant to refer to everyone in Pisa, or to a broader community? If the former, my advice is to travel more; if the latter, then you are simply wrong.

> *The word "continuity", even at the ontological level, can not be read as "continuous with some exception".*

There is an entire mathematical theory of punctuated continua, ie spaces which are continuous everywhere except for a non-dense set of points. Such structures even arise naturally from purely continuous phenomana in, for example, catastrophe theory. The formal trick, you see, is to alter the axiom so that instead of reading 'for all points...' it reads 'there exists a set S such that for all points not in S...'. The result is also an axiom, believe it or not.

> For a point-based model, the continuity is usually characterized as "Between any two points, there is a third"; while for an interval-based model (like that of Allen), it is characterized as "Every interval can be decomposed into two adjacent sub-intervals".

> *The axiom of completeness states:*
>
> *Let be A and B non empty subsets of S such that $a \leq b$ for all $a \in A$ and $b \in B$. Then exists $\xi \in S$ such that $a \leq \xi \leq b$ for all $a \in A$ and $b \in B$.*
>
> *Now, the set S, that is your domain, may consists as well either of time-points or time-intervals; S holds real numbers on the former case, intervals from the real line on the latter case.*

You havent said what $\leq$ means for intervals. If it means that the endpoint of a is point-$\leq$ the first point of b, then this axiom seems false; for consider a point p and the set $A = \{\langle p1, p\rangle\}$ for any $p1 < p$ and $B = \{\langle p, p2\rangle\}$ for any $p2 > p$. This satisfies your premise, but there is no interval between any of the intervals in A and any interval in B (unless you allow intervals consisting of a single point.) But in any case, you are here *assuming* that the real line is your intended model. But this axiom doesnt characterize the real line. Its true on the rationals, for one thing, but thats not all. For example, here is a nonstandard model of your axiom: interpret points as pairs $\langle n, q\rangle$ where n is an integer and q is a rational number or the symbol "i", and say that $\langle a, b\rangle \leq \langle c, d\rangle$ just when $a < c \vee (a = c \wedge d = "i") \vee (a = c \wedge b \leq d)$. This amounts to N copies of Q laid end-to-end with points at infinity placed between them. It satisfies your axiom. Im sure that anyone with a little imagination can easily cook up lots more such nonstandard worlds.

> *Premise: It is evident that if you assume the axiom of completeness, the domain S can just be continuous, while if you do not assume the axiom of completeness then S is necessarily discrete.*

If you do not assume the axiom, then S may be discrete, continuous or any mixture. Did you mean to say, if you assume the *negation* of the axiom? But the negation of your axiom simply says that some point of discontinuity exists; it does not impose a discrete structure on the whole of S. It is much more difficult to axiomatise a discrete structure than a dense one; in fact, it cannot be done in first-order logic.

> *Question: Suppose that you define your neither continuous nor discrete Temporal Structure. What is your domain S? What is your replacement for the axiom of completeness?*

See above, but modify the domain to exclude the "i" symbols. This structure (N copies of Q) is dense almost everywhere, but your axiom fails to hold when the sets A and B are infinite in a particular way. There are $2|Q$ subsets in the power set of this domain, and only $N|2$ of them fail your axiom, so by almost any standard it is true 'almost' everywhere. (Another interesting example is got by reversing N and Q, so that one has Q copies of N laid end-to-end. This fails your axiom 'locally', ie when the subsets are only finitely separated, but satisfies it for sufficiently separated sets. It is like a discrete space which changes to a dense one when the scale is reduced sufficiently. See J.F.A.K. van Benthem, *The Logic of Time*, [mb-Benthem-83] for a lovely discussion of such examples.

> Why cannot time be continuous in some places but discontinuous at others?

> *Places? If we shall understand time like (physicists) understands the space, ...*

Yes, that is more or less what I have in mind. Do you propose to formalise a theory of time which is incompatible with physics? (Why??)

> There is no mathematical objection to such a structure, and it has been

> *If a Temporal Structure exists in this sense, may I have a look at its domain (that is at the S domain, as stated above)?*

See above examples and use your imagination.

But this is a trivial challenge. It can be done for any set S with a (strict) ordering $<$. Select a subset P of S, and define a new order relation $\ll$ on $S + P$ as follows: $x \ll y$ iff $(x \in S \wedge y \in P \wedge x = y)$ xor $x < y$. (+ here is disjoint union, xor is exclusive-or). This inserts a 'twin' of each point in P just after it, with no points between them. If S is dense/continuous/whatever, then this new structure is that too everywhere except at points in P. If P is a dense subset of S, this construction effectively makes two copies of the original set in that region with a 'sawtooth' ordering that jumps back and forth between them, inserting a discrete section into the originally dense ordering:

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P            . . . . . . .          .
S ...........|\|\|\|\|\.........|\......
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looks like this when 'straightened out':

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............. .. .. .. .. .. .. .. .. ............... .........
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(BTW, another way to describe this is that each point in the P-subset of S is replaced by a 'two-sided' point.) If S is dense, then your axiom applies everywhere except at points in P.

(Aside to Jixin: this is the intuition behind the idea of replacing moments by points. The endpoints of a moment can be thought of as the *result* of this construction on a smaller set of points, and the construction can be reversed by identifying the endpoints of the moment, ie treating the moment as being pointlike. The result is a timeline with some points identified as being 'interval-like', ie capable of having something true at them. If moments never meet, then all the axioms of the Allen-Hayes theory apply to the S-line iff they applied to the original. This is why your theory and ours are essentially the same.)

Erik Sandewall wrote

> *..... I have no problems accepting that a* function of *time may be piecewise continuous, or that it may be undefined for some points along the time axis. However, it seems to me that there are several problems with saying that* time itself *is piecewise continuous (btw - do you mean piecewise dense?).*

(Yes, most of this discussion is really about density rather than continuity. Ive just let this ride for now.)

> *The first problem is with respect to motivation. For what reasons would Time suddenly skip over potential timepoints? If the reason is, as you wrote, that*
>
> > The 'dividing point' problem which gave rise to this discussion would do. According to the modern account of the continuum, this point must exist, and since all intervals consist of points, the light is therefore either on or off at it. But it seems more natural, as well as formally simpler, to just say that the question is meaningless; perhaps (though this is no longer my own preference) because that point doesn't exist.
>
> *then exactly what events in the world would be allowed to contribute to the continuity faults? Does the next time I hit a key on my keyboard qualify?*

It all depends on whether you find it useful to describe things that way. The idea of intervals simply *meeting* seems to be a very useful way to think about time, and it immediately gives rise to all these problems.

> *And what about the midpoint halfway between two continuity faults, is it also a continuity fault, recursively?*

Not necessarily. (Im not sure quite what your point is here. Must there be a waterfall exactly between two waterfalls?)

> *The other problem is with respect to the axiomatizations. Since your article "A catalog of temporal theories" characterizes the various theories through axiomatizations, I thought I'd go back to that article and check how you had done this formally. However I was not able to find it; the closest I got was the denseness axiom on page 15.*

Throughout the catalog, I give density and discreteness axioms. As I say in the text, you can take your pick; or, if you like, you can say that time is dense sometimes and discrete others, making obvious slight changes to the axioms to make these assertions. The axioms in the theories of the catalog are offered to you like pieces of an erector set. I make no committment to their truth, only that they fit together properly.

As to whether time *really is* discrete or continuous, etc., the only people who can answer questions like that are physicists, not we who merely craft ontologies.

> *If the intuitive notion is that time itself is continuous in some places but not in others, wouldn't it be natural to start with an axiomatization*

of continuous time (such as the real numbers) and then to proceed from there? For example, a domain of piecewise continuous time could be represented as a twotuple $\langle R, D\rangle$ where R is the real numbers and D is a "small" subset of it; the intention being that $R-D$ is the modified time domain in question. The notions of non-standard intervals could then be constructed as the natural next step.

Yes, that is a possible approach. However, (1) the real numbers are already a very complicated domain to axiomatise, requiring such things as set theory and notions of limits, etc..; I was looking for something much more mundane; and (2) as Ive said repeatedly, the real line isnt a very good model of our temporal intuitions, in my view, but comes along with a lot of misleading assumptions which are not necessary for temporal reasoning.

Maybe I'm missing something - are constructs of this kind subsumed by the axioms in your report, or can they be inferred as theorems?

Neither. The time axioms are far too weak to be able to infer anything about real analysis. However, it should be possible to construct models of the time axioms using ordinary mathematical notions like the integers and the reals, and indeed I try to do that for every theory in the catalog. At the very least, this helps guide ones intuitions about just what it is ones axioms really say, instead of what one hopes they ought to say.

Pat

References:

[mb-Benthem-83] Johan van Benthem. *The Logic of Time.* Reidel, 1983.

From: Sergio Brandano on 24.4.1998

In reply to Jixin Ma (ENRAC 23.4.1998)

So, you didn't refer "the classical one" to "the Basic Time Structure" you developed, did you? If No, why **did** *you develop it? What is* **your** *convincing argument(s) on the need of such a structure? Is it also an alternative to the classical one? (Sorry, I am here using the similar question raised by youself to ask you, though I don't have to). If Yes, I shouldn't ask this question.*

By "the classical one" I mean the classical notion of continuity.

By "the basic time structure" I mean a basic (minimal) time structure.

By "the time structure X" I mean the temporal structure we like to deal with. It is obtained from the basic time structure via additional axioms.

You also invited me to be more explicit with respect to the following sentence.

The axiom of completeness states:

Let A and B be non empty subsets of S such that $a \leq b$ for all $a \in A$ and $b \in B$. Then there exists $\xi \in S$ such that $a \leq \xi \leq b$ for all $a \in A$ and $b \in B$.

Now, the set S, that is your domain, may consists as well either of time-points or time-intervals; S holds real numbers on the former case, intervals from the real line on the latter case.

The (temporal) domain S, as I meant, may consist either of time-points xor of time-intervals (exclusive "or").

An interval from the real-line is an ordered set of real numbers limited by its end-points, which are not necessarily included in the set.

Suppose S consists of intervals from the real line. Assume $\langle s1, t1\rangle \in A$ and $\langle s2, t2\rangle \in B$, intervals in S. We say that $\langle s1, t1\rangle \leq \langle s2, t2\rangle$ iff $t1 \leq s2$. The strict order relation $<$ is an abbreviation for $\leq \wedge \neq$.

Suppose now that $\langle s1, t1\rangle < \langle s2, t2\rangle$. The axiom of completeness states the existence of $\xi \in S$ such that $\langle s1, t1\rangle \leq \xi \leq \langle s2, t2\rangle$. I reply in advance to your next question: "Why did you write $\leq$ instead of $<$?". The reply is that $\leq$ means "less or equal", that is ξ may not be equal to $t1$ or $s2$, but it can do so. Note that since ξ belongs to S, then ξ is an interval. This is also meant as a reply to your question about the dividing instant problem.

I could not penetrate the rest of your message.

Best Regards

Sergio