# Quantifiers and Operations on Modalities and Contexts

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December 14, 1997

**Abstract**

We can reason about theories, as well as in them. Many natural phenomena can be captured by theories, often by introducing a modality, true just of the theory. Rather than treat these phenomena as the sentences true in this modality, we can treat the entire theory or modality as an object. This point of view was proposed by McCarthy, who proposed calling these reified modalities contexts.

Contexts, viewed as theories or modalities, have structural properties, and associated natural structural operations, such as union and intersection. These structural properties are most naturally captured by an algebra, reminiscent of relation or dynamic algebra , a set of operations that can be applied to contexts to form new contexts.

We introduce an algebra, reminiscent of relation or dynamic algebra, but which acts on modalities, not relations. This algebra allows us to construct new modalities from simpler modalities. We use the natural correspondence between a modality and the underlying accessibility relation. The operations on modalities are induced by operations on the underlying relations.

Thus we can have a larger space of modalities than just those that are explicitly named. We add quantifiers that range over modalities, so that we can state propositions like, "no context with the property $P$ exists", or "all contexts obey $Q$". We give an axiomatization of these quantifiers which we show is complete with respect to a natural semantics.

# 1 Putting Modalities Together

## 1.1 Modalities

Modal operators are widely used to capture intensional aspects of knowledge, belief, temporality and action. They capture the notion that there may be different viewpoints from which a particular phenomena can be viewed. These different vantage points can be *agents beliefs*, *points in time*, *differing stages of knowledge*, or different *contexts*.

A unifying semantics for these notions was proposed by Kripke. The essential idea is that a new syntactic operator, $\Box$, is semantically modeled by a set of possible worlds, $W$, and an accessibility relation on worlds $R$. Each world is given an assignment to propositional atoms[1], and the truth conditions for $\Box\phi$ at a world $w$, are defined as the truth of $\phi$ in all accessible worlds.

A natural extension to this idea is to allow more than one modality. One popular use of this is to have modalities of the beliefs of each member of a set of believers, (Halpern 1995) provides a recent overview of this approach. It is then customary to label each modal operator with a subscript to distinguish them. If $\Box_{Sally}$ denotes the beliefs of Sally, and $\Box_{Harry}$, the beliefs of Harry, then if Sally believes that Harry believes that Sally Liked Harry, we would write, $\Box_{Sally}\Box_{Harry}Likes(Sally, Harry)$.

## 1.2 Making new modalities from old

In this paper, we ask what modalities should exist, given that we already have a certain set of modalities. Thus, we differ from previous work on indexed modalities in that we are interested in making new modalities, not in stating facts that require multiple modalities for their expression.

We can imagine that we have the two modalities $\Box_{Sally}$ and $\Box_{Harry}$ that we considered earlier. It seems obvious that there is a set of beliefs that they share. This set of beliefs seems to be a well behaved modality. Other modalities are the union of the beliefs, the propositions they do not believe, and many others.

One way of posing the question of what other modalities might exist, is to ask "What would a system that quantified over modalities look like?" We first present a simple system that allows us to quantify over modalities, but gives us no way to generate new modalities. We then look at natural ways to generate more modalities, using some notions from relational algebra.

## 1.3 Identity conditions on modalities

As well as a theory of what modalities exist, we also need an identity criterion for modalities. We take the most obvious criterion, *extensionality*. That is, we consider two modalities equal if they agree on all sentences.

$$c = c' \stackrel{def}{=} \text{ for all } \phi.\Box_c\phi \equiv \Box_{c'}\phi$$

We consider what structural properties this decision has on the underlying relations. We show it identifies all *bisimilar* models.

[1] Alternately, each world can have an associated first order structure, or a non-standard semantical structure, such as a three valued model. The choice of a propositional valuation is for simplicity, most of results extend to richer valuations.

### 1.4 Plan of Paper

We first look at Kripke models, which provide the basic model theory of modalities.

We give a sound and complete axiomatization for arbitrary universes of relations. This gives a simple system which allows quantification over modalities, but which does not have any closure conditions on what modalities should exist.

We then suggest three operators that make new modalities from old modalities. Therefore our universe of modalities will include all modalities definable using these operations. We give an axiomatization of these operators that we show is sound and complete for universes that include all *first order definable modalities*. This shows that these three operators are sufficient to define all first order definable modalities.

To prove this theorem we have to characterize all *first order definable modalities*. We consider when the modalities Kripke models define are equivalent. This needs the notion of bisimulation. We then look at what functions on the relations in Kripke models give rise to functions on modalities. We then prove a theorem which characterizes the first order definable functions on relations that give rise to functions on modalities. We show that the syntactic class of first order definable functions on modalities can be captured by our three operations on modalities.

We conclude by looking at previous related work in AI, and suggesting some natural uses for the operators we have introduced.

## 2 Modalities and Relations

We now state the standard definitions of Kripke structures, and review how Kripke models can give a semantics to a normal modality.

### 2.1 Kripke Models

We define propositional modal languages in the usual way.

**Definition 1** *A propositional modal language over a set $P$ of propositional atoms is the smallest set of formulas containing each element of $P$, and $\neg\varphi$, $\varphi \wedge \psi$, $\Box\varphi$, where $\phi$ and $\psi$ are formulas.*

We can define the other propositional connectives, $\vee$, $\rightarrow$, $\equiv$, and the modal operator $\Diamond\phi = \neg\Box\neg\phi$ in the usual way.

**Definition 2** *A Kripke structure for propositional modal logic over a set $P$ of propositional atoms is a triple $\langle W, R, V\rangle$ where $W$ is a non-empty set (the set of* worlds*), $R$ is a binary relation on $W$ and $V$ is a function assigning to every world a set of propositional atoms from $P$.*

**Definition 3** *The satisfaction relation $\models$ for $K = \langle W, R, V\rangle$, a Kripke structure over a set of propositional atoms $P$ is a relation between worlds and modal formulas. We define this relation inductively over the structure of a formula. Thus, for any $w \in W$,*

$$\begin{array}{lcr} w, K \models p & \equiv & p \in V(w) \\ w, K \models \neg\varphi & \equiv & \textit{it is not the case that } w, K \models \varphi \\ w, K \models \varphi \wedge \psi & \equiv & w, K \models \varphi \textit{ and } w, K \models \psi \\ w, K \models \Box\varphi & \equiv & \forall w' \in W \textit{ such that } \; w \, R \, w', \;\; w', K \models \varphi \end{array}$$

**Definition 4** *Let $K = \langle W, R, V \rangle$ be a Kripke structure, then a formula $\varphi$ is* satisfied *in $K$ if and only if $\exists w \in W.\ \ w, K \models \varphi$. A formula $\varphi$ is* valid in *$K$, written $K \models \phi$, if and only if $\forall w \in W.\ \ w, K \models \varphi$. A formula $\varphi$ is* valid *if and only if $\varphi$ is valid in all Kripke structures.*

# 3 Quantificational Modal Logic

We look at quantifying over modalities in a modal logic. For simplicity, we look at the propositional case, and the modal logic $K$, with multiple modalities. We need to extend our language to allow modal variables, which we write as subscripts, as in $\Diamond_x$. We write modal constants as similarly, $\Diamond_c$, for the constant $c$. We do not allow quantification over objects, our language is propositional. We do not allow quantification over propositions, our language is not second order in this respect. Our only quantifiers are over modalities or contexts.

We also have an alternate notation, $ist(c, \phi)$, where $c$ now plays the role of a modal variable or subscript. This allows a more traditional syntax in some respects.

### 3.0.1 Syntax

The symbols of our language $L_{\mathbb{C},\mathbb{P}}$ are, a countable set of propositional letters $\mathbb{P}$, connectives, $\wedge$ and $\neg$, the symbols $\Box$ and $\Diamond$, a countable set of modal constants $\mathbb{C}$, parentheses ) and (, a countable set of modal variables $x_i, y_i, i \in \omega$, and the quantifier $\forall$. We use $\phi \rightarrow \psi$ as an abbreviation for $\neg(\phi \wedge \neg\psi)$ as usual.

The formulas of our language are the smallest set containing

- All propositional letters, $\mathbb{P}$.
- If $\phi$ and $\psi$ are formulas, then $(\phi \wedge \psi)$ and $(\neg\psi)$ are formulas.
- If $\phi$ is a formula, and $i$ is a modal constant, or a modal variable, then $(\Box_i \phi)$ is a formula.
- If $\phi$ is a formula, and $x$ is a modal variable, then $(\forall x.\phi)$ is a formula.

In the above definition, the formulas $\phi$ and $\psi$ are sub-formulas of the resulting formula, furthermore, the property sub-formula is transitive. We say an occurrence of a variable $x$ is bound in a formula $\phi$, if it is contained in a sub-formula of $\phi$, of the form $\forall x.\psi$. We assume variables are renamed to avoid clashes in the usual way. We say an occurrence of a variable $x$ is free in a formula if it is not bound. We call a formula with no free variables a sentence.

## 3.1 Semantics

The semantics of modal logic is based around a set of worlds, and a set of binary relations over worlds. Every modality is associated with a relation, which we call its accessibility relation. We also have a valuation function, that is a function that assigns a set of propositional atoms to each world.

The semantics of a modal quantified logic, over a language $L_{\mathbb{C},\mathbb{P}}$ are as follows.

**Definition 5** *Given a language $L_{\mathbb{C},\mathbb{P}}$ and $W$ a set of worlds, $V$ a valuation function that assigns a subset of the propositional letters $\mathbb{P}$ to each world, $U$ a set of binary relations on worlds,* *and $A$ an assignment function from modal variables and constants to elements of $U$, we define $w, \langle W, V, U, A\rangle \models_\forall \phi$ by induction on the formulas of $L_{\mathbb{C},\mathbb{P}}$ as follows. Let $S = \langle W, V, U, A\rangle$.*

$$
\begin{array}{lcl}
w, S \models_\forall p & \equiv & p \in V(w) \\
w, S \models_\forall \phi \wedge \psi & \equiv & w, S \models_\forall \phi \textit{ and } w, S \models_\forall \psi \\
w, S \models_\forall \neg\phi & \equiv & w, S \not\models_\forall \phi \\
w, S \models_\forall \Box_i \phi & \equiv & \forall w'.A(i)(w, w') \textit{ implies } w', S \models_\forall \phi \\
w, S \models_\forall \forall x.\phi & \equiv & \forall R \in U.w, \langle W, V, U, A[R/x]\rangle \models_\forall \phi
\end{array}
$$

*where $A[R/x]$ is the assignment that is like $A$, except for the possible difference that it assigns the relation $R$ to the variable $x$.*

We have placed no restrictions on the universe of relations $U$. We now ask the question, what relations should be included in the universe $U$.

### 3.2 Just constants

We can follow the lead of first order logic, and consider possible all domains of relations. This is obviously wrong, if we consider relations as something that can be constructed from other relations using natural operations. It does not take account of the fact that relations which are not named by constants might exist.

However, as a starting point, we give an axiomatization of the case where there is no restriction on the universe of relations. We say $\Gamma \models_\forall \phi$, read, $\phi$ is a valid consequence of $\Gamma$, if, for all models $\mathbb{M} = \langle W, V, U, A\rangle$ and worlds $w$ in $W$, verifying each formula in $\Gamma$, it is the case that $w, \mathbb{M} \models_\forall \phi$.

#### 3.2.1 Axiomatics

The following are the axioms and rules of our minimal system, which we call $\forall$. We assume a Gentzen style system for concreteness.

All valid principles of propositional logic

$\vdash \Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$ Modal Distribution

$\dfrac{\vdash \phi}{\vdash \Box_i \phi}$ Necessitation

$\dfrac{\Gamma \vdash \psi}{\Gamma \vdash \forall x.\psi}$
provided that $x$ does not occur free in $\Gamma$

$\dfrac{\Gamma \vdash \forall x.\psi}{\Gamma \vdash \psi[i/x]}$ where $\psi[i/x]$ is the result of freely substituting each free occurrence of $x$ by $i$

We define logical derivation in the standard way, and denote it $\vdash_\forall$. We now show soundness and completeness for the basic case.

**Theorem 6** *If $\phi$ is a sentence in $L_{\mathbb{C},\mathbb{P}}$, then $\vdash_\forall \phi$ implies $\phi$ is valid.*

**Proof:** We show that the axioms are satisfied in all models and that the rules of inference preserve validity.

We have one non-propositional axiom.

- $\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$. We show this is satisfied in all models, by deriving the right-hand side from the definition of the left hand side.

  $w, \langle W, V, U, A\rangle \models_\forall \Box_i(\phi \to \psi) \equiv \forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall (\phi \to \psi)$
  By the definition of entailment of $\to$ this gives,
  $w, \langle W, V, U, A\rangle \models_\forall \Box_i(\phi \to \psi) \equiv \forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \phi$ implies $w', \langle W, V, U, A\rangle \psi)$
  We now assume that $\forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \phi$.
  This gives,$\forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \psi$
  Discharging the assumption that
  $\forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \phi$ gives,
  $w, \langle W, V, U, A\rangle \models_\forall \Box_i(\phi \to \psi) \Rightarrow \forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \phi$ implies $\forall w' A(i)(w, w') \colon w', \langle W, V, U, A\rangle \models_\forall \psi$
  Which by the definition of $\to$ and $\Box$ gives
  $w, \langle W, V, U, A\rangle \models_\forall \Box_i(\phi \to \psi) \Rightarrow w, \langle W, V, U, A\rangle \models_\forall \Box_i\phi \to \Box_i\psi$.

We have two quantifier rules.

- Let the assumption be $\Gamma$. We assume that for all $w \in W$, $M = \langle W, U, V, A\rangle$, $w, M \models \Gamma$ implies $w, M \models_\forall \psi$. As $x$ does not occur in $\Gamma$, the truth of $w, M \models_\forall \Gamma$ does not depend on the assignment of $x$. Therefore, if $w, M \models_\forall \Gamma$, then $w, \langle W, U, V, A[R/x]\rangle \models_\forall \Gamma$ for any $R$. But this is exactly the condition needed.

- We assume, that for all $w \in W$, $M = \langle W, U, V, A\rangle$, $w, M \models_\forall \Gamma$ implies $w, M \models \forall x.\psi$. The definition of satisfaction for $\forall$ gives us that, for all $w \in W$, $M = \langle W, U, V, A\rangle$, $w, M \models_\forall \Gamma$ implies $\forall R \in U.w, M \models_\forall \psi$. But we note that this implies that in particular the interpretation of $i$, can be used, giving. $w, \langle W, U, V, A[i/x]\rangle \models_\forall \psi$. But this is equivalent to $\langle W, U, V, A\rangle \models \psi[i/x]$, for each free occurrence of $x$, as required.

We now show the other rules preserve validity. We assume the assumptions logically entail the premises and derive the conclusion from the assumptions.

- Propositional rules. Immediate.

- **Nec**: If $\phi$ is true without assumptions, then it is true in all world/model pairs. Therefore, it is true in all worlds in a particular model. In that model, we have $\Box_i\phi$, as it is true in all $i$ accessible worlds.

**Theorem 7** *If $\phi$ is a sentence in $L_{\mathbb{C},\mathbb{P}}$, $\mathbb{C}, \mathbb{P}$ countable, and $\Gamma$ is a set of sentences of $L_{\mathbb{C},\mathbb{P}}$, where $\Gamma \models_\forall \phi$, then $\Gamma \vdash_\forall \phi$*

**Proof:** It suffices to show that every consistent set $\Sigma$ of sentences has a model. We take any consistent set of sentences $\Sigma$, and extend it to a maximal consistent set of sentences $\Sigma'$.

We then add a countable number of new modal constants $C' = \{c'_i | i \in \omega\}$, as witnesses for every existential. A countable number suffices as our language $L_{\mathbb{C},\mathbb{P}}$ is countable. This gives the new set of sentences, $\Sigma''$, in the language $L_{\mathbb{C} \cup C',\mathbb{P}}$. We note that this set is maximal consistent, as we chose each constant to be a witness for an existential that was already completely defined.

We now construct a model $M$, from this set of sentences $\Sigma''$. Finally we show that this model $M$ contains a world $w$ that satisfies all of the sentences in $\Sigma$.

We take as our set of worlds a subset of the set of maximal consistent sets of sentences of $L_{\mathbb{C} \cup C',\mathbb{P}}$. Below we take our valuation function $V(w)$, to be $p \in w$. We define the assignment function $A$ below, which assigns a relation on worlds to every constant in $\mathbb{C} \cup C'$. We take as $U$, the relations we assign to the constants $\mathbb{C} \cup C'$. As the satisfaction of sentences does not depend on the interpretation of variables, this suffices to define the satisfaction of sentences on $M$.

We consider finite sequences of modal constants denoted $s$, $s'$ etc. We denote concatenation by $\cdot$ and identify the singleton sequence with its element. We associate the set of sentences $\phi$ such that $\Diamond_c \phi \in \Sigma''$, with the sequence $c$, and if $s = c \cdot s'$, and $\Diamond_c \phi$ is associated with $s'$, then we associate $\phi$ with $s$. We associate the empty sequence with $\Sigma''$.

This gives us a function from finite sequences of constants to consistent sets of sentences, which we denote $I$.

We consider a consistent set of sentences $\Delta$ as a set of maximal consistent sets of sentences, those containing $\Delta$. Thus $I(s)$ returns a set of maximal consistent sets of sentences.

We take as our set of worlds $W$, the union of the range of $I$.

We take as our interpretation $A$ of the constant $c$, the relation,
$\{\langle w, w' \rangle | \exists s. w \in I(s) \wedge w' \in I(c \cdot s)\}$.

We take as our universe of relations the relations associated with constants, $U = \{R | \exists c \in \mathbb{C} \cup C'. A(c) = R\}$.

Finally we need to show that the restriction of this model $M = \langle W, V, U, A \rangle$, to the constants in $\mathbb{C}$ is a model of $\Sigma$.

We show this by induction on the structure of sentences in $\Sigma''$.

We show that every set of sentences $I(s)$, is satisfied in $w, M$ for every $w \in I(s)$. As we are doing our induction on sentences, rather than formulas, we will need to use the fact that we introduced sufficient witnesses so that every existential had a witness. We choose $\Sigma''$ as our world $I(\epsilon)$.

- $\gamma = p$. As $V$ is defined as $\in$ on letters this is immediate.
- $\gamma = \phi \wedge \psi$. We have by assumption that $w, M \models_\forall \phi$ and $w, M \models_\forall \psi$.
- $\gamma = \neg\phi$. We have by assumption that not $w, M \models_\forall \phi$, as $I(s)$ is consistent.
- $\gamma = \Box_c \phi$. The satisfaction conditions for this are that $\forall w'. w A(c) w'$ implies $w' M \models_\forall \phi$. We know that $\forall w \in I(c). w, M \models_\forall \neg\phi$ exactly when $\neg\phi \in I(c)$. Therefore, if there is a world $w'$ such that $w', M \models_\forall \neg\phi$, and $w R_c w'$, then $\Diamond_c \neg\phi$. But this contradicts $\Box_c \phi$, and thus as $\Sigma''$ is consistent, this is not the case. Therefore, there is no $c$ accessible world where $\neg\phi$ is satisfied.
- $\gamma = \forall x. \phi$. The satisfaction conditions for this are that forall $R \in U. \langle W, V, U, A[R/x] \rangle \models_\forall \phi$. As $\Sigma''$ is complete and consistent, we know that it does not contain $\neg\phi[c/x]$ for some $c$, for if it did, we could derive a contradiction. Therefore it contains $\phi[c/x]$ for each $c \in C \cup C'$. But we recall that $U$ is the interpretation of all constants in $C \cup C'$.

### 3.3 All un-nested modal formulas

A natural way to look at the possible universes of modalities is to consider what operations on modalities should exist. A simple way of making modalities is to make a modality from a formula. If $B$ is a modal propositional formula without modalities then we can define $\Diamond_{i:B?}\phi$, as

$$\Diamond_{i:B?}\phi \equiv (\Diamond_i \phi) \vee B \tag{1}$$

Other ways of defining new modalities suggest themselves. Perhaps the most natural of these is to allow the definition of a new modality in terms of old modalities and union.

Thus, given modalities, $\Diamond_1$ and $\Diamond_2$, we can allow ourselves to make new modalities as follows,

$$\Diamond_{1\cup 2}\phi \equiv \Diamond_1\phi \vee \Diamond_2\phi \tag{2}$$

However, it is worthwhile noticing that

$$\Diamond_{1\cap 2}\phi \equiv \Diamond_1\phi \wedge \Diamond_2\phi$$

does not define a new modality, nor does

$$\Diamond_{-1}\phi \equiv \neg\Diamond_1\phi.$$

The second can be shown by noting that $\neg\Diamond_1\neg(p \vee \neg p)$ is equivalent to $\Box_1 p \vee \neg p$, which is valid by necessitation. Therefore $\Diamond_{-1}\neg(p \vee \neg p)$ is true for all modalities $\Diamond_1$, but we have that $\neg\Diamond_{-1}\neg(p\vee\neg p)$, a contradiction, showing that the above definition renders the system inconsistent.

The definition of $\Diamond_{1\cap 2}$ does not define a modality unless $\Diamond_1\phi \rightarrow \Diamond_2\phi$ or vice-versa. In this case it yields the stronger. If there is a $\phi$ such that $\Diamond_1\phi \wedge \Diamond_2\neg\phi$, then we have $\Diamond_{1\cap 2}\phi \vee \neg\phi$ but not $\Diamond_{1\cap 2}\phi$ or $\Diamond_{1\cap 2}\neg\phi$, which is inconsistent with modal distribution.

### 3.4 Nested Beliefs

Another natural operation to add is to allow nested modalities in the defining formula. Thus we could allow the definition of a modality corresponding to Sally's opinion of Harry's beliefs.

$$\Diamond_{x;y}\phi \equiv \Diamond_y\Diamond_x\phi \tag{3}$$

This system corresponds to the universe of relations being closed under union, union with the worlds that satisfy a property, and relational composition. The composition of two binary relations $R$ and $S$, is defined as , $\forall x\ y.R;S(x,y) \equiv \exists z.R(x,z) \wedge S(z,y)$.

We call the derivability relation with these new axioms 1, 2 and 3 $\vdash_{\forall}$.

We can define a new consequence relation $\models_{\forall}$. This adds the obvious clauses for $\Diamond_{i:p?}$, $\Diamond_{x\cup y}$ and $\Diamond_{x;y}$.

$$\begin{array}{lcl}
w, S \models_{\forall} \Diamond_{i:p?} & \equiv & w, S \models_{\forall}\ \Diamond_i\phi \vee p \\
w, S \models_{\forall} \Diamond_{x\cup y} & \equiv & w, S \models_{\forall} \Diamond_x\phi \text{ and } w, S \models_{\forall} \Diamond_y\phi \\
w, S \models_{\forall} \Diamond_{x;y} & \equiv & w, S \models_{\forall} \Diamond_y\Diamond_x\phi
\end{array}$$

We say $\models_{\forall} \phi$, read, $\phi$ is a valid under $\models_{\forall}$, if, for each world $w$ in $W$, and for all models $\mathbb{M} = \langle W, V, U, A\rangle$, $w, \mathbb{M} \models_{\forall} \phi$, where the set $U$ is closed under all first order definable functions $f$ from relations to relations, such that the induced map $g(\langle W, R, V\rangle) = \langle W, f(R), V\rangle$ is a function on modalities.

**Theorem 8** *If $\phi$ is a sentence in $L_{\mathbb{C},\mathbb{P}}$, then $\vdash_{\forall} \phi$ implies $\phi$ is valid under $\models_{\forall}$.*

**Proof:** We need to check the new axioms. The axioms involving modal sentences can be seen to be true by considering the truth conditions. ∎

**Theorem 9** *If $\phi$ is a sentence in $L_{\mathbb{C},P}$, $\mathbb{C}, P$ countable, and $\models_{\forall} \phi$, then $\vdash_{\forall} \phi$*

The proof relies on some technical material about what constitutes a function on modalities that is presented in the next section. In particular we need to show that every first order definable function on modalities is representable by our three operations, union, composition and $i\colon B?$.

**Proof:** We add sufficient constants, with definitions, so that we have a constant corresponding to every every point in the free algebra of the introduced operations. We need to show that each of these constants is correctly interpreted. That is we need to show that the function $I$ we defined correctly models sequences containing the newly defined modal terms. We show this by showing that we can find a relation that models the modality.

We add these relations by induction on the terms $i$.

- $i = j\colon B?$ This is assigned the relation, $I(j) \cup \{w' | B \in w'\}$.
- $i = j \cup k$ This is assigned $I(i) = I(j) \cup I(k)$.
- $i = j; k$ This is assigned $I(i) = I(k); I(k)$, the relational composition of $I(k)$ and $I(j)$.

Now that we have interpreted all the new constants, we need to check that they are interpreted correctly. This follows immediately from their construction.

We then need to show that this structure we have defined is closed under first order definable functions on modalities.

We use a theorem 13 that we prove in the next section. Let $f$ be any function $f$ from binary relations over $W$ to binary relation over $W$, that maps any two Kripke structures $\langle W, R, V\rangle$, $\langle W, R', V\rangle$ with the same theory, (i.e. $\langle W, R, V\rangle \models \phi$ if and only if $\langle W, R', V\rangle \models \phi$) to two Kripke structures with the same theory, ( i.e. $\langle W, f(R), V\rangle \models \phi$ iff $\langle W, f(R'), V\rangle \models \phi$). Then $f(R)$ is modally definable, in the sense that there is modal sentence $\psi(q)$, where each occurrence of $q$ is in the scope of an even number of negations, such that $\langle W, f(R), V\rangle \models \Diamond\phi$ iff $\langle W, f(R), V\rangle \models \psi(\phi)$ if and only if $f$ is first order definable.

We now use the fact that the operations $\cup$, $c\colon B?$ and $c; c'$ allow us to construct a modality $i$ such that $\Diamond_i q$ is equivalent to any modal sentence $\Diamond_c \phi(q)$ where $\neg$ scopes $q$ an even number of times, and where $\wedge$ does not occur outside a modality that contains $q$.

By construction all modalities constructed from $\cup$, $c\colon B?$ and $c; c'$ are in our structure, thus our structure is closed under all first order definable functions $f$.

The rest of the proof follows as before. ∎

## 3.5 Bisimulation

We now introduce a notion, *bisimulation*, that we will use to define equality of modalities. In order to relate the syntactic question of when two modalities are true of the same formulas to the semantic question of a relation between accessibility relations we consider *bisimulation*.

Van Benthem showed the close connection between bisimulation (or more precisely *p-morphisms*) and definability in modal logics (1983). Using the classical transformation of modal formulas to first order formulas, he showed that a first order formula is preserved under bisimulation exactly when it is equivalent to the translation of a modal formula. We use this result to determine when the two Kripke models are extensionally equal. A key part of checking that a function on relations lifts to a function on modalities is reconising that equals are sent to equals.

**Definition 10** *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two Kripke structures over $P$, then $K$ is bisimilar to $K'$, $K \leftrightarroweq K'$, if there is a relation $S \subseteq W \times W'$ such that*

- *$\forall w \in W\ \exists w' \in W'$ such that $w\ S\ w'$, and*
- *$\forall w' \in W'\ \exists w \in W$ such that $w\ S\ w'$, and*
- *$x\ S\ y$ implies $V(x) = V'(y)$, and*
- *$x\ S\ y$ implies $\forall x' \in W.\ x\, R\, x' \rightarrow (\exists y' \in W'.\ \ y\, R'\, y' \wedge x'\ S\ y')$, and*
- *$x\ S\ y$ implies $\forall y' \in W'.\ y\, R'\, y' \rightarrow (\exists x' \in W.\ \ x\, R\, x' \wedge x'\ S\ y')$.*

This notion of bisimulation captures exactly that two models agree on all modal sentences.

**Theorem 11 ((Benthem 1983))** *For any bisimulation between two Kripke models, $M$, $N$, if $M \leftrightarroweq N$ then $M \models \phi$ if and only if $N \models \phi$ for all modal formulas $\phi$.*

We use this theorem in the next section as our identity condition on first order relations. We need the identity condition on relations that correspond to extensionally equal modalities, as we wish to consider what functions on modalities exists, in terms of the functions on relations.

Thus Kripke structures agree on all modal formulas if and only if they are bisimilar. Thus any function on from Kripke structures to Kripke structures is a function on modalities if and only if it maps bisimilar structures to bisimilar structures.

### 3.6 First order translations

The propositional modal language we have introduced may be embedded into first order predicate logic, by the following *standard translation*, where $[y/x]\varphi$ is the result of replacing all free occurrences of $x$ by $y$ in $\varphi$.

$$\begin{array}{lcl}
(p)^* & = & Px \\
(\neg\varphi)^* & = & \neg(\varphi)^* \\
(\varphi \wedge \psi)^* & = & (\varphi)^* \wedge (\psi)^* \\
(\Box\varphi)^* & = & \forall y(R(x,y) \rightarrow [y/x](\varphi)^*) \text{ for some new variable } y
\end{array}$$

In fact, only two variables are needed. We then have the following theorem.

**Theorem 12 ((Benthem 1983))** *A first order formula is (equivalent to) a translation of a modal formula if and only if it is invariant for bisimulation.*

In the next section we use this theorem to show that our new operators can define all first order modalities.

## 3.7 Definable Relations

We now ask what definable functions $f$ on relations give rise to functions on modalities $g(\langle W, R, V\rangle) = \langle W, f(R), V\rangle)$. If we define a new Kripke model, by generating a new binary relation on worlds, when does this definition map Kripke models that have the same theory to Kripke models with them same theory?

The answer to this can be derived from the previous theorem—the only formulas that are invariant for bisimulation are those that are equivalent to modal formulas. In this case it suffices to be able to find a formula $\psi[q]$ such that $\langle W, R', V\rangle \models \Diamond\phi$ if and only if $\langle W, R, V\rangle \models \psi[\phi]$.

**Theorem 13** *Let $R$ be a binary relation on a set of worlds $W$, let $v$ range over valuations on a set of propositional letters $p_1, \ldots, p_n$, and let $P_1, \ldots, P_m$ be unary predicates on $W$, such that $(p_i)^* = P_i(x)$.*

*For all $S \subset W \times W$,*

$$\forall v.\langle W, S, v\rangle \leftrightarroweq \langle W, R, v\rangle \textit{ implies } \forall v.\langle W, \phi(R), v\rangle \leftrightarroweq \langle W, \phi(S), v\rangle,$$

*if and only if, there is a modal formula $\psi$, in the modal language with letters $p_1, \ldots, p_n, q$, such that*

$$(\psi(q))^* \equiv \exists y.\phi(R)(x, y) \wedge Q(y)$$

*where $q$ is a new propositional letter, with $(q)^* = Q(x)$.*

*Further, $\psi$ can be chosen so that every occurrence of $q$ in $\psi(q)$ is scoped by an even number of negations, and $\wedge$ does not occur outside any modalities that bind $q$.*

**Proof:** $\Leftarrow$ We are given an equivalence, and we need to show that a definition $\phi$ preserves bisimulation. To show that $\langle W, \phi(R), V\rangle \leftrightarroweq \langle W, \phi(S), V\rangle$ it suffices to show that they agree on all modal formulas. We prove this by induction on formulas. As we have proved the equivalence for arbitrary $q$ we can use it for arbitrary formulas $\chi$. This gives us truth conditions for statements of the form $\langle W, \phi(R), V\rangle \models \Diamond\chi$ in terms of $\langle W, R, V\rangle \models \psi(< \chi)$. Thus we can reduce the truth of all formulas in $\langle W, \phi(R), V\rangle$ to the truth of formulas in $\langle W, R, V\rangle$. We can similarly reduce the truth of all formulas in $\langle W, \phi(S), V\rangle$ to the truth of the same formulas in $\langle W, S, V\rangle$. As by assumption $\langle W, R, V\rangle \leftrightarroweq \langle W, S, V\rangle$ then these models agree on all formulas. Therefore the new models agree, and are thus bisimilar, as required.

$\Rightarrow$ We are given that $\phi$ preserves bisimulations and need to show that there exists a formula $\psi$. As $\phi$ preserves bisimulation, $\exists y.\phi(R)(x, y) \wedge Q(y)$ is preserved under bisimulation, and is thus equivalent to a modal formula, as required.

To see that $q$ is scoped by an even number of negations in $\psi$ is suffices to note that $Q$ occurs only positively in the right hand side, which is equivalent to $(\psi[q])^*$, so $q$ must not occur negatively in an essential way.

Finally using the fact that new modalities can not be defined by conjoining modalities, that we showed earlier, no modality that binds $q$ can be in the scope of a conjunction. ∎.

Thus we have shown that the first order definitions of functions from Kripke models to Kripke models that are functions on modalities are exactly those that are definable using our three operations $x; y$, $x \cup y$ and $x \colon B?$.

# 4 Other Work

## 4.1 McCarthy's Approach

McCarthy (1993) introduced the notion of a context as a formal object. He argued that contexts should objects in a theory. This led him to introduce the modality, $ist(c, \phi)$, which states that a formula $\phi$ is true in a context $c$.

McCarthy saw contexts as capturing the unstated features of an utterance. He thought these features could be made more explicit by showing how contexts were related. His proposal was to characterize these relations by writing sentences that relate one context to another, lifting axioms. If $\phi$ being true in a context $c$ implied that $\psi$ was true in a context $c'$, he would write, $ist(c, \phi) \rightarrow ist(c', \psi)$.

While McCarthy was developing the idea of contexts a student of his, Guha (1991) applied the ideas to Cyc (Lenat 1995).

Giunchiglia (1993) introduced the notion of a multi-language system. Based on intuitions from proof theory, this attempts captures how patterns of reasoning in different languages can be integrated. Giunchiglia et al. (1993) address issues of non-omniscient belief, while our contexts are all logically closed. Clearly there is a clear relationship between how patterns of reasoning are related, and theories are related.

## 4.2 Quoting Approaches

Montague (1963) showed that modal logic cannot be mimiced by a quoting machinery if the underlying language is sufficiently powerful. Attardi and Simi (1995) have developed a system that avoids these problems. They call their syntactic modalities, *viewpoints*.

## 4.3 Modal Approaches

Montague (1970) considers introducing modalities as objects. Naturally, his modalities are non-extensional. This led him away from considering structural properties, as his interests lay in the application of these ideas to natural language.

Ohlbach, while looking at more efficient ways of reasoning in modal logics (Ohlbach 1988), considered representing modalities in a first order language by reifying the worlds, and explicitly introducing the accessibility relation. This led him to consider *context logics* (Ohlbach 1989). His interest lay mainly in the extra expressive power this gave over standard modalities.

Shoham (1991) considered a class of modal operators each of which characterized a context. Buvač Buvač and Mason (1995) have looked at modal logics with these operators and the additional assumption that each operator has complete information about how the other operators behave. That is, their systems assume, $\Box_1\Box_2\phi \vee \Box_1\neg\Box_2\phi$. This axiom does not seem justified, even for the applications they consider. There is no reason why a database should have complete information about the contents of other databases.

Buvač (Buvač 1994) has investigated decidability, while (Massacci 1996) has considered complexity for a certain fragment.

(Buvač 1996) considers a simple notion of quantification over contexts in , which corresponds to our first completeness theorem, with the assumption $\Box_1\Box_2\phi \vee \Box_1\neg\Box_2\phi$, and a 5 like assumption that multiple modalities may always be reduced to a single modality.

Similar systems to this (Gabbay and Nossum 1997) based on fibring one logic over another (Gabbay 1996) have been proposed. (Nossum 1997) gives some decidability results. (Akman and Surav 1996) provides a survey of formal approaches to contextual reasoning.

## 5 Conclusion and Applications

We have presented a sound and complete model theory for a quantificational logic of contexts. We include operations that allow us to take the union of contexts and union of a context with a given statement. These operations are very natural in combining knowledge sources, and considering the knowledge in common of different sources.

The most natural operations on theories or modalities are the union of two modalities. We often want to combine theories.

The product of two modalities, created by relational composition naturally models the beliefs that one agent has about another. New modalities of this sort are natural when we consider multiple agents.

The notion of $B?$ or adding the $B$ worlds to what is considered possible has clear applicability in adding facts to a given theory. Let the theory true of $\Box_c$ be axiomatized by $\Gamma$, then the theory of $\Box_{c:B?}$ is axiomatized by $\Gamma \wedge \neg B$. This operator allows us to add facts to a theory.

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