A relational model of movement

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Abstract

This paper presents a relational model of movement based on the notion of mobile moving about with constant speed. Three basic relations between mobiles are considered: (1) "Sometime or other, the mobiles x and u meet up somewhere", (2) "The position of the encounter between x and u and the position of the encounter between y and v are equal" and (3) "The moment of the encounter between x and u precedes the moment of the encounter between y and v".

1 Introduction

Considered as an axiomatic theory applied to the formalization of space, geometry has enjoyed an uncommon destiny [4] [6]. As an axiomatic theory applied to the formalization of time, the logic of time has experienced a comparable fate [1] [6]. The reason for the success of the axiomatic method within the context of the formalization of space and time certainly lies in the fact that our perception of space and time inevitably leads us to think out the relative positions of the objects that occupy space and the relative moments of the events that fill time.

One cannot deny that the notions of position and moment suffice to define the movement of any mobile. It is nevertheless true that the relational models of movement are scarce. The reason probably lies in the fact that movement is more often considered as a function from time into space. The most typical relational models of movement are the ones of Carnap [2] and Robb [3] [5].

Carnap considers a relational structure (E, C, T) where E is a nonempty set of slices of particles, C is the binary relation of coincidence between these slices and T is the binary relation of local time order such that $x\ C\ u$ iff "The slices x and u are assigned to the same spacetime point" and $x\ T\ u$ iff "The spacetime points assigned to x and u occur in

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this temporal order". Carnap sets out to show that the relations C and T suffice to express not only the topological structure of temporal order, but that of the spatial order as well [2].

Robb associates to the spacetime geometry of Minkowski a relational structure of the form $(P, <)$ where the elements of P are the spacetime points of Minkowski and $<$ is the binary relation of causal precedence between these points such that $x < u$ iff "The spacetime point u lies inside or on the future light cone based at the spacetime point x". Robb sets out to axiomatize the spacetime geometry of Minkowski and gives twenty-one postulates having the spacetime point and the binary relation $<$ as their only primitive notions [3] [5].

Yet, the individuals of the relational models of Carnap and Robb are not mobiles moving about through space and time but slices of particles or spacetime points of Minkowski — which amounts to the same thing. For Carnap and Robb, the notion of mobile — defined either as a maximal linearly ordered set of slices of particles or as a line of the spacetime geometry of Minkowski — stems from the notions of space and time. Strictly speaking, the relational models of Carnap and Robb do not tell us about movement, they talk about space and time. Would it be possible for us to define a relational model of movement having the mobile as its only primitive notion ?

To simplify this job, our mobiles will move about with constant speed. Therefore, the binary relation "Sometime or other, the mobiles x and u meet up somewhere" occurs immediately to us. Now, let us look at our mobiles moving about with constant speed and let us suppose that each encounter produces a spot and a flash. Consequently, we will be able of deciding whether "The position of the encounter between x and u and the position of the encounter between y and v are equal" or not. Similarly, we will be able of deciding whether "The moment of the encounter between x and u precedes the moment of the encounter between y and v" or not.

Whereas the choice of the relations between mobiles is arbitrary, we have to interpret them realistically in a spacetime environment. That is the reason for which this paper presents the spacetime frame — a relational model of movement based on the notions of position, moment and spacetime line — and the frame of mobiles — a relational model of movement based on the notion of mobile.

The spacetime frame is a structure of the form $\underline{S} = (S, T, L, \leq_T, in)$ where S is a non-empty set of positions, T is a non-empty set of moments, L is a non-empty set of spacetime lines, $\leq_T$ is a binary relation on T such that $i \leq_T j$ iff "The moment i precedes the moment j" and in is a ternary relation on S, T and L such that $\langle A, i\rangle \ in \ x$ iff "The spacetime point $\langle A, i\rangle$ belongs to the spacetime line x". The set of the spacetime frames is considered as a category which arrows are the homomorphisms between spacetime frames.

The frame of mobiles is a structure of the form $\underline{M} = (M, R, \cong_S, \preceq_T)$ where M is a non-empty set of mobiles, R is a binary relation on M such that $x \ R \ u$ iff "Sometime or other, the mobiles x and u meet up somewhere", $\cong_S$ is a binary relation on $G(R)$ — the graph of R — such that $(x, u) \cong_S (y, v)$ iff "The position of the encounter between x and u and the position of the encounter between y and v are equal" and $\preceq_T$ is a binary relation on $G(R)$ such that $(x, u) \preceq_T (y, v)$ iff "The moment of the encounter between x and u precedes the moment of the encounter between y and v". The set of the frames of mobiles is considered as a category which arrows are the homomorphisms between frames of mobiles.

This paper is organized in the following way. Section 2 presents the spacetime frame. Section 3 presents the frame of mobiles. Sections 4.1, 4.2 and 5 are devoted to the proof that the category of the frames of mobiles and the category of the spacetime frames are isomorphic.

2 Spacetime frame

This section presents the spacetime frame. A *spacetime frame* is a structure of the form $\underline{S} = (S, T, L, \leq_T, in)$ where:

- S is a non-empty set of *positions*.
- T is a non-empty set of *moments*.
- L is a non-empty set of *spacetime lines*.
- $\leq_T$ is a binary relation on T — $\leq_T$ is a binary relation of *precedence* between the moments of T: $i \leq_T j$ iff "The moment i precedes the moment j".
- in is a ternary relation on S, T and L — in is a binary relation of *incidence* between the spacetime points of $S \times T$ and the spacetime lines of L: $\langle A, i\rangle$ in x iff "The spacetime point $\langle A, i\rangle$ is incident with the spacetime line x".

Let $on_{\underline{S}}$ be the binary relation on $S \times T$ and L defined by $\langle A, i\rangle\ on_{\underline{S}}\ x$ iff there is $u \in L$ such that $x \neq u$, $\langle A, i\rangle\ in\ x$ and $\langle A, i\rangle\ in\ u$. The relations $\leq_T$ and $on_{\underline{S}}$ satisfy the following conditions:

- $\leq_T$ is reflexive, antisymmetrical, transitive and total.
- $(S \times T, L, on_{\underline{S}})$ is a geometry of incidence, that is to say:
  - For every $x \in L$, there is $\langle A, i\rangle \in S \times T$ such that $\langle A, i\rangle\ on_{\underline{S}}\ x$.
  - For every $\langle A, i\rangle, \langle B, j\rangle \in S \times T$, there is $x \in L$ such that $\langle A, i\rangle\ on_{\underline{S}}\ x$ and $\langle B, j\rangle\ on_{\underline{S}}\ x$.
  - For every $\langle A, i\rangle, \langle B, j\rangle \in S \times T$ and for every $x, y \in L$, if $\langle A, i\rangle\ on_{\underline{S}}\ x$, $\langle B, j\rangle\ on_{\underline{S}}\ x$, $\langle A, i\rangle\ on_{\underline{S}}\ y$ and $\langle B, j\rangle\ on_{\underline{S}}\ y$ then either $\langle A, i\rangle = \langle B, j\rangle$ or $x = y$.

The set of the spacetime frames is considered as a category which arrows are the homomorphisms between spacetime frames. More precisely, let $\underline{S} = (S, T, L, \leq_T, in)$ and $\underline{S}' = (S', T', L', \leq'_T, in')$ be spacetime frames. An *homomorphism from $\underline{S}$ into $\underline{S}'$* is a mapping g from S, T and L into S', T' and L' such that:

- For every $i, j \in T$, $i \leq_T j$ only if $g(i) \leq'_T g(j)$.
- For every $A \in S$, for every $i \in T$ and for every $x \in L$, $\langle A, i\rangle\ in\ x$ only if $\langle g(A), g(i)\rangle\ in'\ g(x)$.

An *isomorphism from $\underline{S}$ into $\underline{S}'$* is a bijective mapping g from S, T and L into S', T' and L' such that:

- For every $i, j \in T$, $i \leq_T j$ iff $g(i) \leq'_T g(j)$.
- For every $A \in S$, for every $i \in T$ and for every $x \in L$, $\langle A, i \rangle$ *in* x iff $\langle g(A), g(i) \rangle$ *in'* $g(x)$.

3 Frame of mobiles

This section presents the frame of mobiles. A *frame of mobiles* is a structure of the form $\underline{M} = (M, R, \cong_S, \preceq_T)$ where:

- M is a non-empty set of *mobiles*.
- R is a binary relation on M — R is a binary relation of *encounter* between the mobiles of M: $x\ R\ u$ iff "Sometime or other, the mobiles x and u meet up somewhere".
- $\cong_S$ is a binary relation on $G(R)$ — the graph of R — $\cong_S$ is a binary relation of *colocality* between the encounters of $G(R)$: $(x, u) \cong_S (y, v)$ iff "The position of the encounter between x and u and the position of the encounter between y and v are equal".
- $\preceq_T$ is a binary relation on $G(R)$ — $\preceq_T$ is a binary relation of *precedence* between the encounters of $G(R)$: $(x, u) \preceq_T (y, v)$ iff "The moment of the encounter between x and u precedes the moment of the encounter between y and v".

Let $\cong_T$ be the binary relation on $G(R)$ defined by $\alpha \cong_T \beta$ iff $\alpha \preceq_T \beta$ and $\beta \preceq_T \alpha$. Let $on_{\underline{M}}$ be the binary relation on $G(R)_{|\cong_S} \times G(R)_{|\cong_T}$ and M defined by $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ x$ iff there is $u \in M$ such that $x\ R\ u$ and $a \cong_S (x, u) \cong_T \alpha$. The relations R, $\cong_S$, $\preceq_T$ and $on_{\underline{M}}$ satisfy the following conditions:

- R is serial, irreflexive and symmetrical.
- $\cong_S$ is reflexive, symmetrical, transitive and such that, for every $(x, u) \in G(R)$, $(x, u) \cong_S (u, x)$.
- $\preceq_T$ is reflexive, transitive, total and such that, for every $(x, u) \in G(R)$, $(x, u) \cong_T (u, x)$.
- For every $x, u \in M$ and for every $\langle \cong_S (a), \cong_T (\alpha) \rangle \in G(R)_{|\cong_S} \times G(R)_{|\cong_T}$, if $x \neq u$, $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ x$ and $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ u$ then $x\ R\ u$ and $a \cong_S (x, u) \cong_T \alpha$.
- $(G(R)_{|\cong_S} \times G(R)_{|\cong_T}, M, on_{\underline{M}})$ is a geometry of incidence, that is to say:
  - For every $x \in M$, there is $\langle \cong_S (a), \cong_T (\alpha) \rangle \in G(R)_{|\cong_S} \times G(R)_{|\cong_T}$ such that $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ x$.
  - For every $\langle \cong_S (a), \cong_T (\alpha) \rangle, \langle \cong_S (b), \cong_T (\beta) \rangle \in G(R)_{|\cong_S} \times G(R)_{|\cong_T}$, there is $x \in M$ such that $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ x$ and $\langle \cong_S (b), \cong_T (\beta) \rangle\ on_{\underline{M}}\ x$.
  - For every $\langle \cong_S (a), \cong_T (\alpha) \rangle, \langle \cong_S (b), \cong_T (\beta) \rangle \in G(R)_{|\cong_S} \times G(R)_{|\cong_T}$ and for every $x, y \in M$, if $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ x$, $\langle \cong_S (b), \cong_T (\beta) \rangle\ on_{\underline{M}}\ x$, $\langle \cong_S (a), \cong_T (\alpha) \rangle\ on_{\underline{M}}\ y$ and $\langle \cong_S (b), \cong_T (\beta) \rangle\ on_{\underline{M}}\ y$ then either $\langle \cong_S (a), \cong_T (\alpha) \rangle = \langle \cong_S (b), \cong_T (\beta) \rangle$ or $x = y$.

The set of the frames of mobiles is considered as a category which arrows are the homomorphisms between frames of mobiles. More precisely, let $\underline{M} = (M, R, \cong_S, \preceq_T)$ and $\underline{M}' = (M', R', \cong'_S, \preceq'_T)$ be frames of mobiles. An *homomorphism from $\underline{M}$ into $\underline{M}'$* is a mapping f from M into M' such that:

- For every $x, u \in M$, $x \; R \; u$ only if $f(x) \; R' \; f(u)$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \cong_S (y, v)$ only if $(f(x), f(u)) \cong'_S (f(y), f(v))$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \preceq_T (y, v)$ only if $(f(x), f(u)) \preceq'_T (f(y), f(v))$.

An *isomorphism from $\underline{M}$ into $\underline{M}'$* is a bijective mapping f from M into M' such that:

- For every $x, u \in M$, $x \; R \; u$ iff $f(x) \; R' \; f(u)$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \cong_S (y, v)$ iff $(f(x), f(u)) \cong'_S (f(y), f(v))$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \preceq_T (y, v)$ iff $(f(x), f(u)) \preceq'_T (f(y), f(v))$.

4 Spacetime frames and frames of mobiles

4.1 From spacetime frames into frames of mobiles

To every spacetime frame is associated a frame of mobiles in the following way. Let $\underline{S} = (S, T, L, \leq_T, in)$ be a spacetime frame. Let $\Phi(\underline{S}) = (M, R, \cong_S, \preceq_T)$ where:

- $M = L$.
- For every $x, u \in M$, $x \; R \; u$ iff there is $A \in S$ and there is $i \in T$ such that $x \neq u$, $\langle A, i \rangle \; in \; x$ and $\langle A, i \rangle \; in \; u$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \cong_S (y, v)$ iff there is $A \in S$ and there is $i, j \in T$ such that:
  - $\langle A, i \rangle \; in \; x$ and $\langle A, i \rangle \; in \; u$.
  - $\langle A, j \rangle \; in \; y$ and $\langle A, j \rangle \; in \; v$.
- For every $(x, u), (y, v) \in G(R)$, $(x, u) \preceq_T (y, v)$ iff there is $A, B \in S$ and there is $i, j \in T$ such that:
  - $\langle A, i \rangle \; in \; x$ and $\langle A, i \rangle \; in \; u$.
  - $\langle B, j \rangle \; in \; y$ and $\langle B, j \rangle \; in \; v$.
  - $i \leq_T j$.

Direct calculations would lead to the conclusion that:

Lemma 1 *Let $\underline{S}$ be a spacetime frame. $\Phi(\underline{S})$ is a frame of mobiles.*

To every homomorphism between spacetime frames is associated an homomorphism between frames of mobiles in the following way. Let $\underline{S} = (S, T, L, \leq_T, in)$ and $\underline{S}' = (S', T', L', \leq'_T, in')$ be spacetime frames. Let $\underline{M} = \Phi(\underline{S}) = (M, R, \cong_S, \preceq_T)$ and $\underline{M}' = \Phi(\underline{S}') = (M', R', \cong'_S, \preceq'_T)$. Let g be an homomorphism from $\underline{S}$ into $\underline{S}'$. Let $\Phi(g)$ be the mapping from M into M' defined in the following way:

- For every $x \in M$, $\Phi(g)(x) = g(x)$.

Direct calculations would lead to the conclusion that:

Lemma 2 *Let $\underline{S}$ and $\underline{S}'$ be spacetime frames. Let $\underline{M} = \Phi(\underline{S})$ and $\underline{M}' = \Phi(\underline{S}')$. Let g be an homomorphism from $\underline{S}$ into $\underline{S}'$. $\Phi(g)$ is an homomorphism from $\underline{M}$ into $\underline{M}'$.*

Lemma 3 *Let $\underline{S}$, $\underline{S}'$ and $\underline{S}''$ be spacetime frames. Let $\underline{M} = \Phi(\underline{S})$, $\underline{M}' = \Phi(\underline{S}')$ and $\underline{M}'' = \Phi(\underline{S}'')$. Let g be an homomorphism from $\underline{S}$ into $\underline{S}'$. Let g' be an homomorphism from $\underline{S}'$ into $\underline{S}''$. $\Phi(g \circ g') = \Phi(g) \circ \Phi(g')$.*

Lemma 4 *Let $\underline{S}$ be a spacetime frame. Let $\underline{M} = \Phi(\underline{S})$. $\Phi(id_{\underline{S}}) = id_{\underline{M}}$.*

Therefore:

Theorem 1 *Φ is a functor from the category of the spacetime frames into the category of the frames of mobiles.*

4.2 From frames of mobiles into spacetime frames

To every frame of mobiles is associated a spacetime frame in the following way. Let $\underline{M} = (M, R, \cong_S, \preceq_T)$ be a frame of mobiles. Let $\Sigma(\underline{M}) = (S, T, L, \leq_T, in)$ where:

- $S = G(R)_{|\cong_S}$.
- $T = G(R)_{|\cong_T}$.
- $L = M$.
- For every $\cong_T(\alpha), \cong_T(\beta) \in T$, $\cong_T(\alpha) \leq_T \cong_T(\beta)$ iff $\alpha \preceq_T \beta$.
- For every $\cong_S(a) \in S$, for every $\cong_T(\alpha) \in T$ and for every $x \in L$, $\langle \cong_S(a), \cong_T(\alpha) \rangle$ *in* x iff $\langle \cong_S(a), \cong_T(\alpha) \rangle$ $on_{\underline{M}}$ x.

Direct calculations would lead to the conclusion that:

Lemma 5 *Let $\underline{M}$ be a frame of mobiles. $\Sigma(\underline{M})$ is a spacetime frame.*

To every homomorphism between frames of mobiles is associated an homomorphism between spacetime frames in the following way. Let $\underline{M} = (M, R, \cong_S, \preceq_T)$ and $\underline{M}' = (M', R', \cong'_S, \preceq'_T)$ be frames of mobiles. Let $\underline{S} = \Sigma(\underline{M}) = (S, T, L, \leq_T, in)$ and $\underline{S}' = \Sigma(\underline{M}') = (S', T', L', \leq'_T, in')$. Let f be an homomorphism from $\underline{M}$ into $\underline{M}'$. Let $\Sigma(f)$ be the mapping from S, T and L into S', T' and L' defined in the following way:

- For every $\cong_S(x, u) \in S$, $\Sigma(f)(\cong_S(x, u)) = \cong'_S(f(x), f(u))$.
- For every $\cong_T(x, u) \in T$, $\Sigma(f)(\cong_T(x, u)) = \cong'_T(f(x), f(u))$.
- For every $x \in L$, $\Sigma(f)(x) = f(x)$.

Direct calculations would lead to the conclusion that:

Lemma 6 *Let $\underline{M}$ and $\underline{M}'$ be frames of mobiles. Let $\underline{S} = \Sigma(\underline{M})$ and $\underline{S}' = \Sigma(\underline{M}')$. Let f be an homomorphism from $\underline{M}$ into $\underline{M}'$. $\Sigma(f)$ is an homomorphism from $\underline{S}$ into $\underline{S}'$.*

Lemma 7 *Let $\underline{M}$, $\underline{M}'$ and $\underline{M}''$ be frames of mobiles. Let $\underline{S} = \Sigma(\underline{M})$, $\underline{S}' = \Sigma(\underline{M}')$ and $\underline{S}'' = \Sigma(\underline{M}'')$. Let f be an homomorphism from $\underline{M}$ into $\underline{M}'$. Let f' be an homomorphism from $\underline{M}'$ into $\underline{M}''$. $\Sigma(f \circ f') = \Sigma(f) \circ \Sigma(f')$.*

Lemma 8 *Let $\underline{M}$ be a frame of mobiles. Let $\underline{S} = \Sigma(\underline{M})$. $\Sigma(id_{\underline{M}}) = id_{\underline{S}}$.*

Consequently:

Theorem 2 *Σ is a functor from the category of the frames of mobiles into the category of the spacetime frames.*

5 Isomorphism

This section is devoted to the proof that:

- For every spacetime frame $\underline{S}$, $\Sigma(\Phi(\underline{S}))$ and $\underline{S}$ are isomorphic.
- For every frame of mobiles $\underline{M}$, $\Phi(\Sigma(\underline{M})) = \underline{M}$.

Let $\underline{S} = (S, T, L, \leq_T, in)$, $\underline{M} = \Phi(\underline{S}) = (M, R, \cong_S, \preceq_T)$ and $\underline{S}' = \Sigma(\underline{M}) = (S', T', L', \leq'_T, in')$. For every $\langle A, i\rangle \in S \times T$, there is $x, u \in L$ such that $x \neq u$, $\langle A, i\rangle$ *in* x and $\langle A, i\rangle$ *in* u. Let g be the mapping from S, T and L into S', T' and L' defined in the following way:

- For every $A \in S$, $g(A) = \cong_S (x, u)$ where $x \neq u$, $\langle A, i\rangle$ *in* x and $\langle A, i\rangle$ *in* u.
- For every $i \in T$, $g(i) = \cong_T (x, u)$ where $x \neq u$, $\langle A, i\rangle$ *in* x and $\langle A, i\rangle$ *in* u.
- For every $x \in L$, $g(x) = x$.

Direct calculations would lead to the conclusion that g is an isomorphism from $\underline{S}$ into $\underline{S}'$. Therefore:

Lemma 9 *Let $\underline{S}$ be a spacetime frame. $\Sigma(\Phi(\underline{S}))$ and $\underline{S}$ are isomorphic.*

Moreover, direct calculations would lead to the conclusion that:

Lemma 10 *Let $\underline{M}$ be a frame of mobiles. $\Phi(\Sigma(\underline{M})) = \underline{M}$.*

Therefore, the frame of mobiles provides a realistic answer to the issue of the formalization of movement through a relational model having the mobile as its only primitive notion:

Theorem 3 *The category of the spacetime frames and the category of the frames of mobiles are isomorphic.*

6 Conclusion

The formalization of movement through a relational model having the mobile as its only primitive notion could take the following new turns:

- Consider either extensive rather than punctual mobiles or changing rather than constant speeds.

- Work out a modal logic of movement which semantic basis is the frame of mobiles.
- Mechanize reasoning in the context of the frame of mobiles.

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